# Lecture 3

# $C^*$-modules and operator spaces, and noncommutative function theory

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## I. Introduction.

There is a profound two-way interaction between $C^*$-modules and operator spaces, which has attracted much interest in recent years.

- $C^*$-module theory fits comfortably into the framework of operator modules, as opposed to Banach modules.

- Every operator space $X$ sits c. isometrically inside its *noncommutative Shilov boundary* $\mathcal{T}(X)$, which is a $C^*$-module or TRO.

- Thus general operator space can be studied using $C^*$-module methods.

- Topics like nuclearity for operator spaces can be studied using $C^*$-module methods (Effros-Ozawa-Ruan, Kye and Ruan, ...).

Recall that a *TRO* is a closed subspace $Z$ of $B(K,H)$ (or of a $C^*$-algebra), with $ZZ^\star Z \subset Z$.

Example: $pA(1-p)$, for a $C^*$-algebra $A$ and a projection $p$ in $A$ (or in $M(A)$).

A (right) $C^*$*-module* $Y$ over a $C^*$-algebra $A$ is defined like a Hilbert space except that it is a right $A$-module, and the inner product is $A$-valued, and satisfies

(1) $\langle y|y\rangle \geq 0$ for all $y \in Y$,

(2) $\langle y|y\rangle = 0$ if and only if $y = 0$,

(3) $\langle y|za\rangle = \langle y|z\rangle a$ for all $y, z \in Y, a \in A$,

(4) $\langle y|z\rangle^* = \langle z|y\rangle$ for all $y, z \in Y$,

(5) $Y$ is complete in the norm $\|y\| = \|\langle y|y\rangle\|^{\frac{1}{2}}$.

Clearly every TRO $Z$ is a right $C^*$-module over the C$^*$-algebra $Z^*Z$, with inner product $\langle x|y\rangle = x^*y$

(Similarly it is a left $C^*$-module over $ZZ^*$)

Say $T : Y \to Z$ is *adjointable* if $\exists\ S\colon Z \to Y$ with

$$\langle\, T(y) \,|\, z \,\rangle \;=\; \langle\, y \,|\, S(z) \,\rangle, \qquad y \in Y, z \in Z.$$

Write $\mathbb{B}(Y,Z)$ for the set of adjointable maps from $Y$ to $Z$, and write $S$ as $T^*$.

Fact: $\mathbb{B}(Y) = \mathbb{B}(Y, Y)$ is a $C^*$-algebra with respect to the usual norm We define $\mathbb{K}(Y)$ to be the closure in $\mathbb{B}(Y)$, of the linear span of the 'rank-one' operators $|z\rangle\langle y|$, for $y \in Y, z \in Z$.

Here $|z\rangle\langle y|$ is the operator which takes an $x \in Y$ to $z\langle y|x\rangle$.

$\mathbb{K}(Y)$ is a closed two-sided ideal in $\mathbb{B}(Y)$.

Linking C*-algebra: $\forall$ $C^*$-module $Y$ over $A$ $\exists$ C$^*$-algebra

$$\mathcal{L}(Y) = \begin{bmatrix} \mathbb{K}(\mathsf{Y}) & Y \\ \bar{Y} & A \end{bmatrix}.$$

And $Y = p\mathcal{L}(Y)(1-p)$ for projection $p = 1 \oplus 0$.

Application: C*-modules = TROs

So C*-modules are operator spaces, and they fit miraculously together.

A few examples of the 'miraculous':

- $\mathbb{B}(Y) = \mathcal{A}_\ell(Y) \subset CB(Y)$ c. isom.
  $B_A(Y) = \mathcal{M}_\ell(Y) \subset CB(Y)$ c. isom.
  Indeed theory of one-sided multipliers and $M$-ideals generalizes the theory of module maps and submodules of $C^*$-modules

- The most important tensor product of C*-modules, is just the 'module Haagerup' tensor product, an op. space construction.

- Quotients of $C^*$-modules are $C^*$-modules, and are op. spaces, and the two structures are compatible!

- $\cdots$

See e.g. Chapter 8 in [B-Le Merdy book]

## II. The noncommutative Shilov boundary

A little more detail...

Quotients in TRO language:

Say subspace $J$ of a TRO $Z$ is a *triple ideal* if $ZZ^*J \subset J$ and $JZ^*Z \subset J$.

$J$ is an *inner ideal* if $JZ^*J \subset J$

For triple ideals, $Z/J$ is a TRO, and op. space, and canonical $q : Z \to Z/J$ is a triple morphism.

Say $(Z, i)$ is a *triple extension* of an op. space $X$ if $i : X \to Z$ is c. isom. into TRO $Z$, and $\nexists$ proper subTRO of $Z$ containing $i(X)$.

Recall from Lecture 2:

**Theorem** (Hamana) For any triple extension $(Z, i)$ of $X$, $\exists$ triple morphism $\theta : Z \to \mathcal{T}(X)$ with $\theta \circ i = j$.

Read this as: $\mathcal{T}(X)$ is the *smallest* TRO containing $X$.

Consequences:

- Object $\mathcal{T}(X)$ with this universal property is 'unique'.

- If $(Z, i)$ is any triple extn. of $X$, can **define** $\mathcal{T}(X) =$ quotient of $Z$ by a triple ideal.

- Can define $\mathcal{T}(X)$ to be any triple extn. $(Z, i)$ with no proper subTRO $J$ for which canon. map $X \to Z/J$ is c. isometric.

- $\mathcal{T}(X)$ is rigid and essential (in sense of Lecture 2).

- Can identify $\mathcal{T}(X) \subset \mathcal{T}(X^{**})$ canonically (B-H-N).

- For $1 \in X \subset C(K)$, $\mathcal{T}(X) = C(\partial X)$, where $\partial X$ is classical Shilov bdy of $X$ in $K \cdots$

**Example.** $X = \ell^1_2$. Claim: $\mathcal{T}(X) = (C(\mathbb{T}), j)$, where $j : (\alpha, \beta) \mapsto \alpha 1 + \beta z$, where $z(e^{i\theta}) = e^{i\theta}$.

Proof: Clearly $i$ is a (complete) isometry, so $X$ is a unital operator space, and $i(X)$ generates $C(\mathbb{T})$ by density of trig polynomials. So this is a triple extension. It is clear that $\nexists$ ideal in $C(\mathbb{T})$, or equivalently no closed subset $E$ of $\mathbb{T}$, s.t. $x \mapsto i(x)_{|E}$ is isometric. So by third bullet above, $\mathcal{T}(X) = C(\mathbb{T})$.

Even quicker: Usual Shilov boundary of $i(X)$ in $C(\mathbb{T})$ is $\mathbb{T}$.

## III. Appln. to structure in operator spaces

I am a bit obsessed by using the above to study operator spaces using C*-module/TRO techniques.

We have already seen some examples of this theme: one-sided $M$-ideals, recovery of forgotten product in an operator algebra, ... .

Discuss a couple more examples:

Application of ‘Morita equivalence’.

In the language of TROs, let $W$ be a WTRO (= TRO with predual), then $W$ is $A$-$B$-bimodule where $A$ is weak closure of $WW^*$ and $B$ is weak closure of $W^*W$.

Stable isomorphism theorem: $C_\infty(W) \cong C_\infty(B)$, and hence $M_\infty(W) \cong M_\infty(B)$ c. isometrically. So $M_\infty(W)$ is a $W^*$-algebra with some product.

Apply the above in the case that $W = \mathcal{T}(X)^{**}$ or $I(X)^{**}$.

So $X \subset I(X) \subset I(X)^{**} = W$, and so

$$M_\infty(X) \subset M_\infty(W)$$

and the latter is a W*-algebra.

How can this be useful?

Eg. Sketch original (not best) proof that $\tau_u$ c. contr. $\Rightarrow$ $u \in \mathcal{M}_\ell(X)$ (Lecture 2)

First prove it for $u : N \to N$, for W*-algebra $N$, i.e. $\tau_u$ contr. $\Rightarrow$ $ux = ax$ for fixed $a \in N$.

Exercise. [Hint: look at $\tau_u(\begin{bmatrix} p \\ p^\perp \end{bmatrix})$ for projn. $p$ and use density of span of projections in $N$ ]

Then show if $\tau_u$ c. contr. for $X$, can extend $u$ to $\tilde{u}$ on $I(X)$ by injectivity.

Still have $\tau_{\tilde{u}}$ c. contr. by rigidity.

Take second dual, get map $w = \tilde{u}^{**}$ on $W = I(X)^{**}$. Then $u' = w_\infty$ is a map on $N = M_\infty(W)$, and check that $\tau_{u'}$ is contr.

Thus by the W*-algebra case, $u'(x) = ax$ for fixed $a \in N$. Then 'restrict' to $X$ to get result.

Key point: we have used a deep fact about $C^*$-modules/Morita equivalence in this proof, to deduce a result about $X$.

## IV. Structure via $\mathcal{T}(X)$: order

I want now to discuss a very recent example ([B-Werner, B-Neal] 2006) of using $\mathcal{T}(X)$ to study structure in $X$, in this case:

*Order in operator spaces*

We think of a 'positive cone' $X_+$ as a structure that a (possibly nonunital, nonselfadjoint) operator space $X$ may have.

If $X$ is contained in a $C^*$-algebra $A$ then
$X_+ = X \cap A_+$

Study this in terms independent of partic. $A$

(Note: we're on totally new ground if $X$ not selfadj)

We use a new, very algebraic approach

Key idea: Since $X \subset$ TRO $\mathcal{T}(X)$, first step is to study order in TROs

This turns out to be very pretty and intricate (C*-)algebraic in nature

Second step: apply first study to $\mathcal{T}(X)$ to deduce results about cones on $X$

The first step:

A cone $\mathfrak{d}$ in a TRO $Z$ is called *natural* if there exists a one-to-one triple morphism $\varphi : Z \to A$, for a C*-algebra $A$, such that $\varphi(\mathfrak{d}) = \varphi(Z) \cap A_{+}$.

We can classify the natural cones on a TRO $Z$, they are in bijective correspondence with the inner ideals in $Z$ which are triple isomorphic to a C*-algebra, and also in bijective correspondence with a class of partial isometries in $Z^{**}$.

To see one implication here:

$Z \subset A$ subTRO of $C^*$-algebra

$Z_+ = Z \cap A_+$ is a natural cone

Look at $J = Z \cap Z^* \cap Z^*Z \cap ZZ^*$

Exercise: This is is a C*-subalgebra of $A$ which is also an inner ideal in $Z$. Moreover, the positive cone $J_+$ of this C*-subalgebra equals $Z_+$.

Recall: Akemann's *open projections*

**Definition** (Akemann, ...) If $B$ is a $C^*$-algebra then an orthogonal **projection** $q \in B^{**}$ is **open** if it is an increasing limit of positive elements in $B$

It is **closed** if $1 - q$ is open.

(= usual topological notions if $B$ commutative)

**Def.** (B-Werner) If $Z$ is a TRO, then a partial isometry $u$ in $Z^{**}$ is called *open* if there is a net $x_t \in Z$ converging weak* to $u$ s.t. $u^*x_t$ increasing ... .

(B-Neal) iff the projn. $\hat{u} = \frac{1}{2}\begin{bmatrix} uu^* & u \\ u^* & u^*u \end{bmatrix}$ is open w.r.t. $L(Z)$

**Theorem** Natural cones in a TRO $Z$ are in bijective correspondence with open partial isometries in $Z^{**}$.

$$u \rightsquigarrow \mathfrak{c}_u = \{z \in Z : u^* z \geq 0, z = u z^* u\}$$

To study such cones, we extend the properties of open projection (i.e. 'noncommutative topology' to partial isometries). Eg. noncommutative Urysohn lemma

This is essentially C*-algebraic in nature...

There is a theory of *compact partial isometries* in the literature (Akemann and Pedersen, Edwards and Ruttiman), and we use many of their ideas.

Any $x \in Z, \|x\| = 1$ has a 'range' and 'base' partial isometry

$$0 \neq u(x) \leq r(x)$$

Here $r(x)$ is the p.i. in the polar decomp. of $x$, and $u(x)$ is the limit of 'odd powers' of form $uu^*uu^* \cdots u^*u$

$r(x)$ is open, $u(x)$ is compact

A cone on an operator space will be called an *operator space cone* if $\exists$ linear c. positive c. isometry from $X$ into a C*-algebra

**Thm.** Every operator space cone on a TRO is contained in a maximal such cone. This cone is a natural cone.

We classify such maximal cones.

This uses Hamana's $\mathcal{T}(X)$ theorem.

**Example.** Let $S^2$ be the unit sphere, and $Z$ the TRO $\{f \in C(S^2) : f(-x) = -f(x)\}$. In this case open selfadjoint tripotents $u$ in $Z''$ correspond precisely to open subsets $U$ of the sphere (called blue), which do not intersect $-U$ (called red). Suppose that $S^2 \setminus (U \cup (-U))$ is colored black.

The above mentioned characterization, in this example, says that that $u$ (and hence the associated ordering of $Z$) is maximal iff the black region is the boundary of the red region (and hence also of the blue region).

Thus, for example, a sphere whose top hemisphere is red and whose bottom hemisphere is blue, with a black equator line, is maximal; but if you thicken the equator to a black band one loses maximality.

Now we understand orderings on TROs
Lets move to orderings on an operator space $X$.

If $\mathfrak{d}$ is an operator space cone on $X$, say $(X, \mathfrak{d})$ is an *ordered operator space*

We give $\mathcal{T}(X)$ a natural cone, namely the one corresponding to the open partial isometry $u = \vee_{x \in \mathfrak{d}} r(j(x))$.

**Theorem** $\mathcal{T}(X)$ with this cone, satisfies the universal property/diagram for $\mathcal{T}(X)$, but with all maps (including the triple morphism $\theta$) completely positive.

**Proposition** If $\mathfrak{d}$ spans $X$ then this ordered $\mathcal{T}(X)$ is a C*-algebra, and $\theta$ is a $*$-homomorphism

**Example** $\nexists$ c. isometric positive map from $\ell^1_n$ into a C*-algebra

Idea: The usual cone on a vNA predual is spanning, so if there did exist such a map then the ordered $\mathcal{T}(X)$ exists and is a C*-algebra.

Actually $= C^*(F_{n-1})$ (Paulsen and Zhang)

Check the cones match

Now $(0,1,0,\cdots,0) \in \ell^1_n$ is positive, but this corresponds to one of the generators of $F_{n-1}$ which is not positive, a contradiction.

**Theorem** Suppose that $(X, \mathfrak{c})$ is an ordered operator space, and let $j : X \to \mathcal{T}(X)$ be as usual. $\exists$ c. positive complete isometry from $X$ into a C*-algebra, if and only if $\mathfrak{c} \subset j^{-1}(\mathfrak{d}_u)$, where $u$ is an open p.i. in $\mathcal{T}(X)''$.

If these hold and if $\mathfrak{c}$ densely spans $X$, then $\mathfrak{c} = j^{-1}(\mathfrak{d}_u)$ for some open $u$ iff $\mathfrak{c}$ is a maximal op. space cone on $X$.

We also construct a unitization of such ordered operator spaces.

**Final remark.** If $X$ is selfadjoint the theory is slightly easier and better. Some proofs are different.

## V. Noncommutative function theory

**Generalizing 'classical' theory of function algebras/spaces**

... i.e. **go noncommutative**

One way to begin is to take your favorite book on function spaces or uniform algebras ... and start replacing $C(K)$ by C*-algebras, topological arguments by C*-algebra theory, integrals and measure arguments with vNA (or work in $A^{**}$)

Why should we do this?

- Solve problems
- Import powerful ideas and tools
- Develop (even basic) theory
- Often leads in surprising directions
- Its often really cute!!
- ... .

Give one example: **peak sets**

Thesis of my student Hay, and B-Hay-Neal

More on this topic in IWOTA talk

**Peak sets and noncommutative peak interpolation:**

Recall: if $A \subset C(K)$ then a closed set $E \subset K$ is called a **peak set** if if there exists $f \in A$ such that $f|_E = 1$ and $|f(x)| < 1$ for all $x \notin E$.

An intersection of peak sets is called a $p$-**set**

When one looks at the first few results of this theory, the techniques look 1) as if they should 'go noncommutative', 2) will give completely new and interesting NC results/questions...

Peak sets are interesting for several reasons.

Eg: ‘relative’ Urysohn lemma’s and interpoln.

Urysohn’s lemma is the ‘best’ result in general topology

Allows ‘separation’ of two disjoint closed subsets $E$ and $F$ of $K$ ‘by’ a function $f \in C(K)$

We want to ‘separate’ such subsets ‘by’ a function from $A \subset C(K)$

More generally: peak sets connect to **peak interpolation**:

Given a continuous function defined on $E$, when is it the restriction of a function in $A$? By a function in $A$ which is e.g. small on $F$? Or which is 'dominated' by a given 'control function'.

To make such concepts go 'noncommutative' we need: 'noncommutative closed subsets', noncommutative notion of 'peaking', etc

Common ‘noncommutative’ trick: replace sets by (orthogonal projections)

**Definition** (Akemann, ...) If $B$ is a $C^*$-algebra then an orthogonal **projection** $q \in B^{**}$ is **open** if it is an increasing limit of positive elements in $B$

It is **closed** if $1 - q$ is open.

(= usual topological notions if $B$ commutative)

$\exists$ Urysohn lemma, etc (Akemann, ...).

**Definition** (Hay [PhD Thesis]) If $A$ is a unital subalgebra of a $C^*$-algebra $B$, and if $a \in$ Ball$(A)$ then a closed projection $q \in B^{**}$ is a *peak projection* if some $a \in$ Ball$(A)$ 'is 1 on $q$, and on $1 - q$ we have $|a| < 1$'.

That is, $aq = q = qa$, and $(1-q)a^*a(1-q) < 1$.

**Propn.** such $q = u(a) = \text{wlim}_n \, a^n$

A $p$-*projection* is an intersection of peak projections.

Many of the 'first' results about $p$-sets are *Urysohn lemma's relative to* $A$ (Bishop, Glicksberg, Gamelin, Jarosz, ...)

Because it uses operator space duality, and fits with the theme of this series of lectures, lets look in some detail at the NC version of the very first and key lemma in the classical literature

(Not in IWOTA talk)

From Hays thesis:

**Lemma** Let $X, Y$ be possibly incomplete operator spaces. Suppose $T : X \to Y$ is a one-to-one and surjective completely bounded map such that $T^*$ is a complete isometry. Then $T$ is a complete isometry.

**Proof** Follow proof of Banach space variant (Dunford and Schwartz), but using operator space duality. $\square$

**Theorem** Let $X$ be a closed subspace of a $C^*$-algebra $B$. Let $p \in B^{**}$ be an open projection such that $pX^{\perp\perp} \subset X^{\perp\perp}$. Let $I = \{x \in X : px = x\}$, and $q = p^\perp$. Then $qX$ is completely isometric to $X/I$ via the map $x + I \mapsto qx$.

**Sketch of proof.** If $T$ is this map, by the Lemma we need to show $T^*$ c. isometric. That $T^*$ is isometric boils down to showing

$$\|\varphi + (qX)^{\perp}\| \leq \|\varphi(q\cdot) + X^{\perp}\|, \qquad \varphi \in (qZ)^*.$$

This is not hard to show using the hypothesis, and taking a net in $B$ increasing to $q$.

Similar argument at matrix level. $\square$

What does this property mean:

Classical version of last theorem: If $X \subset C(K)$, and closed $E \subset K$ s.t. ... , then for $f \in X$ with $|f| \leq 1$ on $E$, and given $\epsilon > 0$, there is a $g \in X$ with $g = f$ on $E$, and $|g| \leq 1 + \epsilon$ on $K$.

Replacing $X$ by $Xp^{-1}$ for a strictly positive function $p \in C(K)$, and applying the above gives:

**Corollary** (Same assumptions). For strictly positive function $p$ and for $f \in X$ with $|f| \leq p$ on $E$, and given $\epsilon > 0$, there is a $g \in X$ with $g = f$ on $E$, and $|g| \leq p + \epsilon$ on $K$.

Build the interpolation theory from here...

‘NC peak interpolation’ results:

**Proposition** (Hay) Let $A$ be a unital subspace of $B$. Suppose $q \in A^{**}$ is a closed projection. Let $p$ be a strictly positive element in $B$ and let $a \in A$ such that $a^*qa \leq p$. Then, for every $\epsilon > 0$, there exists $b \in A$ satisfying $qb = qa$, such that $b^*b \leq p + \epsilon$.

**Theorem** (B, Hay, Neal) Let $A$ be a unital-subalgebra of $C^*$-algebra $B$ and let $q \in B^{**}$ be a closed projection. Then $q$ is a $p$-projection for $A$ iff for any open projection $u \geq q$, and any $\epsilon > 0$, there exists an $a \in \text{Ball}(A)$ with $aq = q$ and $\|a(1-u)\| < \epsilon$ and $\|(1-u)a\| < \epsilon$.

**Definition.** An **approximate $p$-projection** is a closed projection which lies in $A^{\perp\perp}$.

Still open: approximate $p$-projections are just the $p$-projections.

**Theorem** (Hay) If $A$ is a unital subalgebra of $B$ and $q \in B^{**}$ is a closed projection, then TFAE:

(i) $q$ is an approximate $p$-projection,

(ii) given $\epsilon > 0$, for each open $u \geq q$, there exists $a \in A$ such that $\|a\| \leq 1 + \epsilon$, $qa = q$ and $\|a(1-u)\| \leq \epsilon$, and

(iii) given $\epsilon > 0$, for every strictly positive $p \in B$ with $p \geq q$, there exists $a \in A$ such that $qa = q$ and $a^*a \leq p + \epsilon$.

Applications given in IWOTA talk!!!