Math 3321

Introduction

University of Houston

Lecture 01

Outline

Basic Terminology

Many fields of science and engineering involve studying processes which change with time (typically denoted by t).

Examples:

1. The rate of decay of a radioactive material at time t is proportional to the amount of material present at time t. In mathematical notation this gives:

$$\frac{dy}{dt} = ky, \text{ with } k < 0$$

Basic Terminology

2. If x is the position of a object in simple harmonic motion at time t then the motion is given by:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

3. Let $u(x,t)$ denote the temperature at the postion x at time t along a uniform rod. The equation governing the diffusion of heat is given by:

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$$

Basic Terminology

Each of the equations in the last three examples is a differential equation.

Definition

A *differential equation* is an equation that contains an unknown function together with one or more of its derivatives.

More examples:

4. $y' = \dfrac{xy + y}{x^2 - y^2}$
5. $x^2 \dfrac{d^3y}{dx^3} + 4x \dfrac{dy}{dx} + 6y = \sin(x)$
6. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial y} = \cos(xy)$

Basic Terminology

Definitions

The *type* of a differential equation is determined by the type of unknown function appearing in the equation. When the unknown function is a function of a single variable, and thus all derivatives are with respect to this single input variable, we have an *ordinary differential equation* (ODE). When the equation features an unknown function which is a function of multiple variables, the derivatives will be partial derivatives with respect to one or more of the input variables and we have a *partial differential equation* (PDE).

Definition

The *order* of a differential equation is the order of the highest derivative of the unknown function appearing in the equation.

7. Identify the unknown function, independent variable, type, and order of each differential equation.

   a. $y' = \dfrac{xy + y}{x^2 - y^2}$

   b. $x^2 \dfrac{d^3y}{dx^3} + 4x\dfrac{dy}{dx} + 6y = \sin(x)$

   c. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial y} = \cos(xy)$

7. Identify the unknown function, independent variable, type, and order of each differential equation.

   a. $y' = \dfrac{xy + y}{x^2 - y^2}$
      First order ODE

   b. $x^2 \dfrac{d^3y}{dx^3} + 4x\dfrac{dy}{dx} + 6y = \sin(x)$
      Third order ODE

   c. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial y} = \cos(xy)$
      Second order PDE

Solution of a Differential Equation

Terminology

The focus of this class is an introduction to the study of differential equations. Our focus will be on ordinary differential equations. Thus, for the remainder of the course *differential equation* will be understood to mean *ordinary differential equation*.

Definition

A *solution of a differential equation* is a function with the property that the equation reduces to an identity when the solution function is substituted into the equation wherever the unknown function appears.

Solution of a Differential Equation

Example: Given the second-order ODE

$$x^2y'' - 2xy' + 2y = 4x^3$$

show that $y(x) = x^2 + 2x^3$ is a solution.

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show that $y(x) = x^2 + 2x^3$ is a solution.

1. From th expression of $y(x)$, we compute:
$y'(x) = 2x + 6x^2$, $y''(x) = 2 + 12x$
2. Substitute into left hand side of ODE:
$x^2(2 + 12x) - 2x(2x + 6x^2) + 2(2x + 6x^2) = (...) = 4x^3$
3. Since the result matches the right hand side of ODE, then y is a solution.

Solution of a Differential Equation

Example: Given the second-order ODE

$$x^2y'' - 2xy' + 2y = 4x^3$$

show that $z(x) = 2x^2 + 3x$ is not a solution.

Solution of a Differential Equation

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$$x^2y'' - 2xy' + 2y = 4x^3$$

show that $z(x) = 2x^2 + 3x$ is not a solution.

1. From th expression of $z(x)$, we compute:
$z'(x) = 4x + 3$, $z''(x) = 4$
2. Substitute into left hand side of ODE:
$x^2(4) - 2x(4x + 3) + 2(2x^2 + 3x) = 0$
3. Since the result is DIFFERENT from the right hand side of ODE, then z is NOT a solution.

Finding Solutions for Differential Equations

Examples:

1. Find the solutions of the differential equation:

$$y' = 3x^2 - 6\sin(3x) + 4e^{2x}$$

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$$y' = 3x^2 - 6\sin(3x) + 4e^{2x}$$

To find y, we integrate the right hand side

$$y(x) = \int (3x^3 - 6\sin(3x) + 4e^{2x})\, dx$$

$$y(x) = \frac{3}{4}x^4 + 2\cos(3x) + 2e^{2x} + C$$

Finding Solutions for Differential Equations

2. Find a value of r, if possible, such that $y = e^{rx}$ is a solution of

$$y'' - 2y' - 8y = 0$$

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$$y'' - 2y' - 8y = 0$$

1. From $y(x)$, we compute
$y'(x) = re^{rx}, \quad y''(x) = r^2 e^{rx}$
2. Substitute into left hand side of ODE:
$r^2 e^{rx} - 2re^{rx} - 8e^{rx} = (r^2 - 2r - 8)e^{rx} = 0$
3. For the equation to be 0 for all x, it must be $(r^2 - 2r - 8) = 0$
Hence it must be $r = -2$ or $r = 4$.

Finding Solutions for Differential Equations

3. Find a value of r, if possible, such that $y = x^r$ is a solution of

$$x^2y'' + 2xy' - 6y = 0$$

Finding Solutions for Differential Equations

3. Find a value of r, if possible, such that $y = x^r$ is a solution of

$$x^2 y'' + 2xy' - 6y = 0$$

1. From $y(x)$, we compute
$y'(x) = rx^{r-1}, \quad y''(x) = r(r-1)x^{r-2}$
2. Substitute into left hand side of ODE:
$r(r-1)x^2 x^{r-2} + 2rxx^{r-1} - 6x^r = (r^2 + r - 6)x^r = 0$
3. For the equation to be 0 for all x, it must be $(r^2 + r - 6) = 0$
Hence it must be $r = 2$ or $r = -3$.

4. Find a value of r, if possible, such that $y = x^r$ is a solution of

$$y'' - \frac{1}{x}y' - \frac{3}{x^2}y = 0$$

Finding Solutions for Differential Equations

4. Find a value of r, if possible, such that $y = x^r$ is a solution of

$$y'' - \frac{1}{x}y' - \frac{3}{x^2}y = 0$$

1. From $y(x)$, we compute
$y'(x) = rx^{r-1}, \quad y''(x) = r(r-1)x^{r-2}$
2. Substitute into left hand side of ODE:
$r(r-1)x^{r-2} - rx^{-1}x^{r-1} - 3x^{-2}x^r = (r^2 - 2r - 3)x^{r-2} = 0$
3. For the equation to be 0 for all x, it must be $(r^2 - 2r - 3) = 0$
Hence it must be $r = -1$ or $r = 3$.

n-Parameter Family of Solutions and General Solution

Example:

1. Find the solutions of the differential equation

$$y'' - 12x + 6\cos(2x) = 0$$

n-Parameter Family of Solutions and General Solution

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$$y'' - 12x + 6\cos(2x) = 0$$

Note that $y''(x) = 12x - 6\cos(2x)$.
To find y*, we integrate the right hand side twice*

$$y'(x) = \int (12x - 6\cos(2x))\, dx = 6x^2 - 3\sin(2x) + C_1$$

$$y(x) = \int y'(x)\, dx = \int (6x^2 - 3\sin(2x) + C_1) dx$$

$$y(x) = 2x^3 + \frac{3}{2}\cos(2x) + C_1 x + C_2$$

n-Parameter Family of Solutions and General Solution

The solution to the last example features two arbitrary constants, C_1 and C_2. We would call this a two-parameter family of solutions for the given differential equation.

Fact

To solve a DE having the form

$$y^{(n)}(x) = f(x),$$

we integrate the function f n times, producing n arbitrary constants as part of our solution.

n-Parameter Family of Solutions and General Solution

Definition

In general, given an n-th order DE of the form

$$F[x, y, y', y'', \ldots, y^{(n)}] = 0,$$

we expect to "integrate" n times, producing a solution featuring n arbitrary constants of integration. We call such a family of solutions an *n-parameter family of solutions*.

Definition

To *solve* a differential equation of the form

$$F[x, y, y', y'', \ldots, y^{(n)}] = 0$$

means to find an n-parameter family of solutions.

n-Parameter Family of Solutions and General Solution

Terminology

For most differential equations encountered in this course, the n-parameter family of solutions we find will represent all solutions of the DE. We will mostly use the term *general solution* in place of the more cumbersome n-parameter family of solutions.

n-Parameter Family of Solutions and General Solution

Example:

2. Given the differential equation

$$y' = 4x\sqrt{y-2},$$

verify that $y = (x^2 + C)^2 + 2$ is the general solution.

n-Parameter Family of Solutions and General Solution

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2. Given the differential equation

$$y' = 4x\sqrt{y-2},$$

verify that $y = (x^2 + C)^2 + 2$ is the general solution.

From the expression of $y(x)$, we compute:

$$y'(x) = 2(x^2 + C)(2x) = 4x(x^2 + C) = 4x\sqrt{y(x) - 2}$$

n-Parameter Family of Solutions and General Solution

Definition

Solutions of an n-th order DE which do not fit the form of the general solution are called *singular solutions*.

Example:

3. Given the differential equation

$$y' = 4x\sqrt{y-2},$$

verify that the constant function $y \equiv 2$ is a singular solution.

n-Parameter Family of Solutions and General Solution

Definition

Solutions of an n-th order DE which do not fit the form of the general solution are called *singular solutions*.

Example:

3. Given the differential equation

$$y' = 4x\sqrt{y-2},$$

verify that the constant function $y \equiv 2$ is a singular solution.

$y(x) = 2$ is a solution as you can verify by substitution into the ODE. Note that there is no value of C for which the expression

$$y(x) = (x^2 + C)^2 + 2 = 2 \quad \textit{for all } x$$

Definition

Given the general solution of an n-th order DE, when particular values are assigned to the arbitrary constants we have a *particular solution* of the equation.

Example:

4. The function $y = x^2 + 5$ is a particular solution of the DE $y' = 2x$.

n-Parameter Family of Solutions and General Solution

Definition

Given the general solution of an n-th order DE, when particular values are assigned to the arbitrary constants we have a *particular solution* of the equation.

Example:

4. The function $y = x^2 + 5$ is a particular solution of the DE $y' = 2x$.

The general solution is

$$y(x) = x^2 + C$$

By choosing $C = 5$, we have the particular solution

$$y(x) = x^2 + 5$$

n-Parameter Family of Solutions and General Solution

5. The function $y = C_1x + C_2x^2 + 2x^3$ is the general solution of the DE $x^2y'' - 2xy' + 2y = 4x^3$. List some particular solutions.

n-Parameter Family of Solutions and General Solution

5. The function $y = C_1 x + C_2 x^2 + 2x^3$ is the general solution of the DE $x^2 y'' - 2xy' + 2y = 4x^3$. List some particular solutions.

For $C_1 = C_2 = 0$ we have the particular solution

$$y_1(x) = 2x^3$$

For $C_1 = C_2 = 1$ we have the particular solution

$$y_2(x) = x + x^2 + 2x^3$$

For $C_1 = 1$, $C_2 = -2$ we have the particular solution

$$y_3(x) = x - 2x^2 + 2x^3$$