Math 3321

IVPs and Finding ODEs

University of Houston

Lecture 02

Outline

The Differential Equation of an n-Parameter Family

Definition

Given an n-parameter family of curves, the *differential equation of the family* is an n-th order differential equation that has the given family as its general solution.

General strategy for finding the DE for a given n-parameter family

To find the differential equation for a given n-parameter family we have the following general strategy:

(1) Differentiate the family n times. This produces a system of $n+1$ equations.

(2) Choose any n of the equations and solve for the parameters.

(3) Substitute the "values" for the parameters in the remaining equation.

The Differential Equation of an n-Parameter Family

Examples:

1. $y = Cx^3 - 2x$

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We compute

$$y'(x) = 3Cx^2 - 2$$

We have

$$y'(1) = 3C - 2 \quad \Rightarrow \quad C = \frac{1}{3}(y'(1) + 2)$$

The Differential Equation of an n-Parameter Family

2. $y = C_1 x + C_2 x^3$

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We compute

$$y'(x) = C_1 + 3C_2 x^2 \quad \Rightarrow \quad y'(0) = C_1$$

$$y''(x) = 6C_2 x \quad \Rightarrow \quad y''(1) = 6C_2$$

3. $y = C_1 \cos(2x) + C_2 \sin(2x)$

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We compute

$$y'(x) = -2C_1 \sin(2x) + 2C_2 \cos(2x) \quad \Rightarrow \quad y'(0) = 2C_2$$

$$y''(x) = -4C_1 \cos(2x) - 4C_2 \sin(2x) \quad \Rightarrow \quad y''(0) = -4C_1$$

Note that we can also choose

$$y(0) = C_1, \quad y(\pi/4) = C_2$$

The Differential Equation of an n-Parameter Family

4. $y = C_1 e^{2x} + C_2 e^{3x} + C_3$

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We compute

$$y(0) = C_1 + C_2 + C_3$$

$$y'(x) = 2C_1e^{2x} + 3C_2e^{3x} \quad \Rightarrow \quad y'(0) = 2C_1 + 3C_2$$

$$y''(x) = 4C_1e^{2x} + 9C_2e^{3x} \quad \Rightarrow \quad y''(0) = 4C_1 + 9C_2$$

To express C_1, C_2, C_3 *in terms of* $y(0), y'(0), y''(0)$, *we need to solve a linear system of 3 equations.*

Initial-Value Problems

Examples:

1. $y = C_1 \cos(2x) + C_2 \sin(2x)$ is the general solution of the ODE $y'' + 4y = 0$.

   a. Find a solution which satisfies $y(0) = 2$.

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   a. Find a solution which satisfies $y(0) = 2$.

We have

$$y(0) = C_1 = 2,$$

Hence

$$y = 2\cos(2x) + C_2 \sin(2x)$$

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1. $y = C_1 \cos(2x) + C_2 \sin(2x)$ is the general solution of the ODE $y'' + 4y = 0$.

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    b. Find a solution which satisfies $y(0) = 2$, $y'(0) = 2$.

We have

$$y(0) = C_1 = 2, y'(0) = 2C_2 = 2 \Rightarrow C_2 = 1$$

Hence

$$y = 2\cos(2x) + \sin(2x)$$

Initial-Value Problems

A problem such as the last example is called an initial-value problem (IVP).

Definition

An *n-th order initial-value problem* consists of an n-th order ODE

$$F[x, y, y', y'', \ldots, y^{(n)}] = 0,$$

together with n initial conditions of the form

$$y(c) = k_0,\ y'(c) = k_1,\ y''(c) = k_2, \ldots,\ y^{(n-1)}(c) = k_{n-1}$$

where c and k_0, k_1, $k_2, \ldots$, k_{n-1} are given numbers.

Initial-Value Problems

IVP Facts

1. An n-th order ODE can always be written in the form

$$F[x, y, y', y'', \ldots, y^{(n)}] = 0.$$

   This can be accomplished by bringing all terms to the left-hand side of the equation.

2. The initial conditions provided in the IVP allow us to determine a particular solution of the ODE by using the general solution and its derivatives to create n equations for our n unkown coefficients from the general solution.

Initial-Value Problems

Examples:

2. Find a solution to the initial-value problem:

$$y' = 6x^2 + 4e^{2x},\ y(0) = -2$$

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We find the general solution by integration

$$y(x) = 2x^3 + 2e^{2x} + C$$

Hence

$$y(0) = 2 + C = -2 \quad \Rightarrow \quad C = -4$$

Hence the IVP solution is

$$y(x) = 2x^3 + 2e^{2x} - 4$$

Initial-Value Problems

3. Given that $y = C_1 e^{2x} + C_2 e^{-3x}$ solves the differential equation, find a solution to the IVP:

$$y'' - y' - 6y = 0, \; y(0) = -2, \; y'(0) = 4$$

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$$y'' - y' - 6y = 0,\ y(0) = -2,\ y'(0) = 4$$

We compute

$$y(0) = C_1 + C_2 = -2$$

$$y'(x) = 2C_1e^{2x} + 3C_2e^{3x} \quad \Rightarrow \quad y'(0) = 2C_1 + 3C_2 = 4$$

We need to solve a system in 2 unknowns and 2 equations. We find

$$C_1 = -10, \quad C_2 = 8$$

Existence and Uniqueness

Fundamental Questions

When studying initial-value problems, there are two fundamental questions.

1. Does a given initial-value problem *have* a solution? That is, do solutions to the problem *exist*?
2. If a solution does exist, is it *unique*? That is, have we found the only solution to the problem or is it the case that more than one solution can be found?

A general treatment on the subjects of existence and uniqueness of initial-value problems is beyond the scope of this class and will be addressed only in the specific cases of interest to us. However, the next two examples will demonstrate that not every IVP has a solution and when a solution to an IVP can be found, the solution may not be unique.

1. The general solution of the differential equation $y' = \dfrac{2y}{x}$ is the one-parameter family of curves given by $y = Cx^2$. Does the IVP below have a solution?

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The ODE is not defined at $x = 0$, the IVP has no solution.

Existence and Uniqueness

2. Letting $a > 0$, show that every member of the family of curves

$$y_a(x) = \begin{cases} 0 & \text{if } x \leq a \\ (x-a)^2 & \text{if } x > a \end{cases}$$

is a solution for the IVP

$$y' = 2\sqrt{y},\ y(0) = 0.$$

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$$y' = 2\sqrt{y},\ y(0) = 0.$$

By direct computation

$$y_a' = 2\sqrt{y_a}$$

Since $a > 0$, then $y_a(0) = 0$ for any choice of $a > 0$.
This shows that the the IVP has no unique solution. For any $a > 0$, $y_a(x)$ is a solution.