Math 3321

Extensions to Other First Order Equations

University of Houston

Lecture 05

Outline

Foundations

We have covered the two basic types of first order differential equations. These are linear equations

$$y' + p(x)y = q(x)$$

and separable equations

$$y' = p(x)h(y).$$

There are other types of equations which are neither linear nor separable which can be transformed into one of these types of equations by a change of variable.

In this lecture we will look at two of these cases.

Bernoulli Equations

Definition

A first order differential equation $y' = f(x, y)$ is a *Bernoulli equation* when it can be expressed in the following form

$$y' + p(x)y = q(x)y^r \tag{B}$$

where p and q are continuous functions on some interval I and r is a real number such that $r \neq 0, 1$.

Notice that equation (B) is very close to a linear equation. Further, the restriction that $r \neq 0, 1$, is due to the fact that when $r = 0$ we have a linear equation and when $r = 1$ we have an equation which can be treated as either linear or separable. We will use the substitution $v = y^{1-r}$ to transform (B) into a linear equation.

Bernoulli Equations

Remark. To show that

$$y' + p(x)y = q(x)y^r \quad r \neq 0, 1$$

is nonlinear, we can write the differential operator

$$L(y) = y' + p(x)y - q(x)y^r$$

For $c \neq 0$, we have that

$$L(cy) = cy' + cp(x)y - c^r q(x)y^r = c(y' + p(x)y - c^{r-1}q(x)y^r)$$

That is,

$$L(cy) \neq cL(y)$$

This shows linearity is not satisfied for $r \neq 0, 1$.

Solution Method for Bernoulli Equations

Solving $y' + p(x)y = q(x)y^r$ where $r \neq 0, 1$:

Our steps are as follows:

1. Identify: Can we write the given equation in the form (B): $y' + p(x)y = q(x)y^r$? If yes, do so.
2. Multiply both sides of equation (B) by y^{-r} to form

$$y^{-r}y' + p(x)y^{1-r} = q(x). \tag{1}$$

Solving $y' + p(x)y = q(x)y^r$ where $r \neq 0, 1$:

3. We now let $v = y^{1-r}$ and observe the following:

$$v' = \frac{d}{dx}\left(y^{1-r}\right) = (1-r)y^{-r}y'$$

so that $y^{-r}y' = \frac{1}{1-r}v'$.

We can now substitute for $y^{-r}y'$ and y^{1-r} in (1) to get

$$\frac{1}{1-r}v' + p(x)v = q(x)$$

or

$$v' + (1-r)p(x)v = (1-r)q(x).$$

Solution Method for Bernoulli Equations

Solving $y' + p(x)y = q(x)y^r$ where $r \neq 0, 1$:

4. Notice that $v' + (1-r)p(x)v = (1-r)q(x)$ is a linear equation. Now solve this equation using the solution method we have previously established.
5. Find the general solution of (B) by substituting $v = y^{1-r}$ into the solution from the previous step.

Examples of Solving Bernoulli Equations

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Hence

$$\begin{aligned} v(x) &= e^{-2x}\int e^{2x}\,e^x\,dx + Ce^{-2x} \\ &= e^{-2x}\frac{1}{3}e^{3x} + Ce^{-2x} \\ &= \frac{1}{3}e^x + Ce^{-2x} \end{aligned}$$

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Since $v = y^{1/2}$, *then*

$$y(x) = \left(\frac{1}{3}e^{x} + Ce^{-2x}\right)^{2}$$

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We solve it using the method of the integrating factor where

$$u(x) = e^{h(x)} = e^{\int(-1/x)dx} = \frac{1}{x}$$

Examples of Solving Bernoulli Equations

Hence, multiplying by the integrating factor

$$\frac{1}{x}v' - \frac{1}{x^2}v = \left(\frac{v}{x}\right)' = -3x$$

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$$\frac{1}{x}v' - \frac{1}{x^2}v = \left(\frac{v}{x}\right)' = -3x$$

Hence

$$\begin{aligned} \frac{v}{x} &= -\frac{3}{2}x^2 + C \\ v &= -\frac{3}{2}x^3 + Cx \end{aligned}$$

It follows from $v = y^{-1}$ that

$$y = (-\frac{3}{2}x^3 + Cx)^{-1}.$$

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$$v' + \frac{2}{x}v = -4x$$

We solve it using the method of the integrating factor where

$$u(x) = e^{h(x)} = e^{\int (2/x)dx} = e^{2\ln x} = x^2$$

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Hence, multiplying by the integrating factor

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Hence

$$\begin{aligned} x^2 v &= -x^4 + C \\ v &= -x^2 + Cx^{-2} = \frac{C - x^4}{x^2} \end{aligned}$$

It follows from $v = y^{-2}$ *that*

$$y^2 = \frac{x^2}{C - x^4}$$

Homogeneous Equations

Definition

A first order differential equation $y' = f(x, y)$ is a *homogeneous equation* if the function f has the following property:

$$f(\lambda x, \lambda y) = f(x, y) \text{ for every } \lambda > 0.$$

The strategy for solving these equations will be based on using the substitution $v = \dfrac{y}{x}$, which means $y = vx$. Using the property, we will can now say

$$f(x, y) = f(x, xv) = f(1, v)$$

which will be a key fact in our solution method.

Solution Method for Homeogeneous Equations

Solving $y' = f(x, y)$ where $f(\lambda x, \lambda y) = f(x, y)$ for every $\lambda > 0$:

Our steps are as follows:

1. Introduce a new dependent variable v by using the substition $y = vx$. Take the derivative of both sides of this substitution equation with respect to x using the product rule. This gives

$$y' = v + xv'.$$

Solution Method for Homeogeneous Equations

Solving $y' = f(x,y)$ where $f(\lambda x, \lambda y) = f(x,y)$ for every $\lambda > 0$:

2. Now we can substitute into the differential equation so that

$$y' = f(x,y)$$

becomes

$$v + xv' = f(x,y) = f(1,v).$$

Now solve for v' to find

$$v' = \frac{f(1,v) - v}{x}.$$

This is a separable equation, allowing us to write

$$\frac{1}{f(1,v) - v} dv = \frac{1}{x} dx.$$

Solution Method for Homeogeneous Equations

Solving $y' = f(x, y)$ where $f(\lambda x, \lambda y) = f(x, y)$ for every $\lambda > 0$:

3. Now integrate both sides to find the general solution of the separable equation from Step (2) in terms of v.
4. Find the general solution to the original equation by replacing v with $\frac{y}{x}$.

Examples of Solving Homeogeneous Equations

1. Show that the given equation is homogeneous and find the general solution: $y' = \dfrac{x^2 + y^2}{2xy}$

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We first show homogeneity:

$$
\begin{aligned}
f(\lambda x, \lambda y) &= \frac{\lambda^2 x^2 + \lambda^2 y^2}{2\lambda x \lambda y} \\
&= \frac{\lambda^2 (x^2 + y^2)}{\lambda^2 2xy} \\
&= \frac{(x^2 + y^2)}{2xy} \\
&= f(x, y)
\end{aligned}
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$$\begin{aligned} v' &= \tfrac{f(1,v)-v}{x} \\ &= \frac{\frac{(1+v^2)}{2v} - v}{x} \\ &= \frac{1 + v^2 - 2v^2}{2vx} \\ &= \frac{1 - v^2}{2vx} \end{aligned}$$

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Under the assumption $v \neq \pm 1$, this gives the separable ODE:

$$\frac{2\,v\,v'}{1-v^2} = \frac{1}{x}$$

Examples of Solving Homeogeneous Equations

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We solve the separable ODE

$$\begin{aligned}
\int \tfrac{2\,v}{1-v^2}\,dv &= \int \tfrac{1}{x}\,dx \\
-\ln|1-v^2| &= \ln|x| + C \\
\ln|1-v^2|^{-1} &= \ln|x| + C \\
(1-v^2)^{-1} &= k\,x \\
(1-v^2) &= \tilde{k}\,x^{-1} \\
v^2 &= 1-\tilde{k}\,x^{-1}
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\end{aligned}$$

Since $y = vx$*, then*

$$y^2 x^{-2} = 1 - \tilde{k}\,x^{-1}$$

hence

$$y^2 = x^2 - \tilde{k}\,x$$

By direct substitution, $y = \pm x$ *is a solution.*

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Direct calculation show that $f(\lambda x, \lambda y) = f(x,y)$
For the solution, we set $y = vx$. *Hence*

$$\begin{aligned} v' &= \frac{f(1,v)-v}{x} \\ &= \frac{\frac{e^v+v}{1} - v}{x} \\ &= \frac{e^v}{x} \end{aligned}$$

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This gives the separable ODE:

$$e^{-v}\, v' = \frac{1}{x}$$

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-e^{-v} &= \ln|x| + C \\
e^{-v} &= -\ln|x| + C' \\
-v &= \ln(C' - \ln|x|) \\
v &= \ln(\tfrac{1}{C' - \ln|x|})
\end{aligned}$$

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\end{aligned}$$

Since $y = vx$, then

$$y = x\ln(\tfrac{1}{C' - \ln|x|})$$