Math 3321

Introduction to Second Order Linear ODEs

University of Houston

Lecture 07

Outline

1. Basic Terminology
2. Second Order Linear Homogeneous Equations
3. Examples

Basic Terminology

Second order differential equations can be written as

$$F(x, y', y'') = 0.$$

This chapter is concerned with a specific type of second order equations. These are the second order linear equations.

Definition

A *second order linear differential equation* is an equation which can be written in the form

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

where p, q, and f are continuous functions on some interval I.

Basic Terminology

Definitions

Given a second order linear differential equation (1), we have the following terminology. The functions p and q are called the *coefficients* of the equation. The function f is called the *forcing function* or the *nonhomogeneous term*.

Examples:

1. $y'' + 6y' + 8y = e^{3x} + \cos(2x)$

2. $x^2y'' + 7xy' + 8y = x^2$

3. $x^2y'' + 3xy^3y' - y = e^x$

Basic Terminology

Existence and Uniqueness

Given a second order linear equation (1). Let a be any point in the interval I, and let α and β be any two real numbers. The IVP

$$y'' + p(x)y' + q(x)y = f(x),\ y(a) = \alpha,\ y'(a) = \beta$$

has a unique solution.

Remark

Unlike the case of first order linear equations where we can always find a solution, *there is no general method for solving second (or higher) order linear differential equations*. There will be methods for solving certain types of second order linear equations and these will be the focus of this chapter.

Basic Terminology

Definitions

The linear ODE (1) is *homogeneous* if the function f on the right side of the equations is 0 for all $x \in I$. In this case, equation (1) becomes

$$y'' + p(x)y' + q(x)y = 0. \tag{H}$$

The equation (1) is *nonhomogeneous* if f is not the zero function on I.

As we will see moving forward, most of our attention will be devoted to solving homogeneous equations.

Terminology

The first thing we note about (H) is that the zero function $y \equiv 0$ is a solution:

$$y \equiv 0 \text{ gives } y' \equiv 0 \text{ and } y'' \equiv 0,$$

therefore we have

$$0 + p(x)0 + q(x)0 = 0.$$

We call the zero function the *trivial solution*. Our interest is in finding nontrivial solutions. Unless otherwise stated, the term “solution” will mean “nontrivial solution.”

Second Order Linear Homogeneous Equations

Theorem 1

Given any two solutions $y = y_1(x)$ and $y = y_2(x)$ for (H), then $u(x) = y_1(x) + y_2(x)$ is also a solution for (H).

Second Order Linear Homogeneous Equations

Theorem 1

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Denote the ODE as a differential operator

$$L(y) = y'' + p(x)y' + q(x)y$$

As we have seen before, L is linear.
Hence

$$L(y_1 + y_2) = L(y_1) + L(y_2)$$

So,

$$L(y_1) = 0 \textit{ and } L(y_2) = 0 \textit{ implies } L(y_1 + y_2) = 0$$

Second Order Linear Homogeneous Equations

Theorem 2

Given any solution $y = y(x)$ for (H) and C a real number, then $u(x) = Cy(x)$ is also a solution for (H).

Second Order Linear Homogeneous Equations

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As above, denote the ODE as a differential operator

$$L(y) = y'' + p(x)y' + q(x)y$$

By the linearity of L,

$$L(Cy) = C\,L(y)$$

So,

$$L(y) = 0 \text{ implies } L(Cy) = 0$$

Second Order Linear Homogeneous Equations

Definition

Let $f = f(x)$ and $g = g(x)$ be functions defined on some interval I, and let C_1 and C_2 be real numbers. We call the expression

$$C_1 f(x) + C_2 g(x)$$

a *linear combination* of f and g.

Theorem 3

Given any two solutions $y = y_1(x)$ and $y = y_2(x)$ for (H) as well as real numbers C_1 and C_2, then

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

is also a solution for (H).

Second Order Linear Homogeneous Equations

Note

Theorem 3 tells us that any linear combination of solutions of (H) is also a solution of (H).

Fact

The function $y = C_1 y_1(x) + C_2 y_2(x)$ is the general solution to (H) if and only if

$$y_1(x)y_2'(x) - y_2(x)y_1'(x) \neq 0.$$

Second Order Linear Homogeneous Equations

Definition

Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of (H). The function W defined by

$$W[y_1, y_2](x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

is called the *Wronskian* of y_1 and y_2.

We use the notation $W[y_1, y_2](x)$ to emphasize that the Wronskian is a function of x that is determined by two solutions y_1, y_2 of equation (H). When there is no danger of confusion, we will shorten the notation to $W(x)$.

There is a short-hand way to represent the Wronskian of two solutions of equation (H) using the determinant of a 2×2 matrix. We will write

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x).$$

Second Order Linear Homogeneous Equations

Examples: Find the Wronskian for the following functions.

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$$
\begin{aligned}
W(x) &= y_1(x)y_2'(x) - y_2(x)y_1'(x) \\
&= -e^{3x}e^{-x} - e^{-x}3e^{3x} = -4e^{2x}
\end{aligned}
$$

Second Order Linear Homogeneous Equations

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2. $y_1 = x^3 \; and \; y_2 = 5e^{3x}$

$$\begin{aligned} W(x) &= y_1(x)y_2'(x) - y_2(x)y_1'(x) \\ &= x^3(15e^{3x}) - 5e^{3x}(3x^2) = 15e^{3x}x^2(x-1) \end{aligned}$$

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$$\begin{aligned} W(x) &= y_1(x)y_2'(x) - y_2(x)y_1'(x) \\ &= x^3(15x^2) - 5x^3(3x^2) = 15x^5 - 15x^5 = 0 \end{aligned}$$

Second Order Linear Homogeneous Equations

Theorem 4

Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of (H), and let $W(x)$ be their Wronskian. Exactly one of the following holds:

(i) $W(x) = 0$ for all $x \in I$ and y_1 is a constant multiple of y_2 (or vice versa).

(ii) $W(x) \neq 0$ for all $x \in I$ and $y = C_1 y_1(x) + C_2 y_2(x)$ is the general solution of (H).

Second Order Linear Homogeneous Equations

Second Order Linear Homogeneous Equations

Definitions

A pair of solutions $y = y_1(x)$ and $y = y_2(x)$ of equation (H) is a *fundamental set* of solutions if

$$W[y_1, y_2](x) \neq 0 \text{ for all } x \in I.$$

A fundamental set of solutions is also called a *solution basis*.

Second Order Linear Homogeneous Equations

Definitions

Given two functions $f = f(x)$ and $g = g(x)$ defined on an interval I, we say that f and g are *linearly dependent on* I if there exists a number λ such that $g(x) = \lambda f(x)$ for all $x \in I$. When the functions are not linearly dependent, we say f and g are *linearly independent*.

Second Order Linear Homogeneous Equations

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Example 1. Consider the functions $f(x) = 2x^2$, $g(x) = -x^2$
Clearly, we have that

$$g(x) = -\frac{1}{2} f(x), \quad \text{for all } x$$

This shows that f and g are *linearly dependent.*

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Example 2. Consider the functions $f(x) = 2x^2$, $g(x) = x$
Suppose that there is a $\lambda \neq 0$ such that

$$g(x) = \lambda f(x) \Rightarrow x = \lambda 2x^2 \Rightarrow x(1 - 2\lambda x) = 0$$

on an interval. This is false for any λ.
This shows that f and g are *linearly independent*.

Theorem 5

Let $f = f(x)$ and $g = g(x)$ be differentiable functions on an interval I. If f and g are linearly dependent on I, then $W(x) = 0$ for all $x \in I$.

Equivalently, we can say:

Let $f = f(x)$ and $g = g(x)$ be differentiable functions on an interval I. If $W(x) \neq 0$ for at least one $x \in I$, then f and g are linearly independent on I.

Second Order Linear Homogeneous Equations

Theorem 4 Restated

Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of (H), and let $W(x)$ be their Wronskian. Exactly one of the following holds:

(i) $W(x) = 0$ for all $x \in I$; y_1 and y_2 are linearly dependent.

(ii) $W(x) \neq 0$ for all $x \in I$; y_1 and y_2 are linearly independent and $y = C_1 y_1(x) + C_2 y_2(x)$ is the general solution of (H).

Examples

1. Verify that the given functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions of the equation?

$$y'' - y' - 6y = 0;\ y_1 = e^{3x},\ y_2 = e^{-2x}$$

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$$y'' - y' - 6y = 0;\ y_1 = e^{3x},\ y_2 = e^{-2x}$$

We have

$$y_1'(x) = 3e^{3x}, y_1''(x) = 9e^{3x}$$

Hence

$$y_1'' - y_1' - 6y_1 = 9e^{3x} - 3e^{3x} - 6e^{3x} = 0$$

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We have

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Hence

$$y_1'' - y_1' - 6y_1 = 9e^{3x} - 3e^{3x} - 6e^{3x} = 0$$

Similarly,

$$y_2'(x) = -2e^{-2x}, y_2''(x) = 4e^{-2x}$$

Hence

$$y_2'' - y_2' - 6y_2 = 4e^{-2x} + 2e^{-2x} - 6e^{-2x} = 0$$

Examples

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$$\begin{aligned} W(x) &= y_1(x)y_2'(x) - y_2(x)y_1'(x) \\ &= e^{3x}(-2)e^{-2x} - e^{-2x}3e^{3x} = -5e^x \end{aligned}$$

Yes, they do form a fundamental set of solutions.

Examples

2. Verify that the given functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions of the equation?

$$y'' + 4y = 0;\ y_1 = \cos(2x),\ y_2 = \sin(2x)$$

Examples

2. Verify that the given functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions of the equation?

$$y'' + 4y = 0;\ y_1 = \cos(2x),\ y_2 = \sin(2x)$$

We have

$$y_1'(x) = -2\sin(2x), y_1''(x) = -4\cos(2x)$$

Hence

$$y_1'' + 4y_1 = -4\cos(2x) + 4\cos(2x) = 0$$

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Hence

$$y_1'' + 4y_1 = -4\cos(2x) + 4\cos(2x) = 0$$

Similarly,

$$y_2'(x) = 2\cos(2x), y_2''(x) = -4\sin(2x)$$

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$$\begin{aligned} W(x) &= y_1(x)y_2'(x) - y_2(x)y_1'(x) \\ &= \cos(2x)2\cos(2x) - \sin(2x)(-2)\sin(2x) \\ &= 2(\cos^2(2x) + \sin^2(2x)) = 2 \end{aligned}$$

Yes, they do form a fundamental set of solutions.

Examples

3. Show that the given functions are linearly independent on an interval I and find a second-order linear homogeneous equation having the pair as a fundamental set of solutions.

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Proof 1. *y_1 and y_2 are linearly independent on an interval I if there is no $\lambda \neq 0$ such that $y_1(x) = \lambda y_2(x)$.*
However, $e^{3x} = \lambda e^{-x}$ on an interval implies $e^{4x} = \lambda$ on an interval, which is false.

Examples

3. Show that the given functions are linearly independent on an interval I and find a second-order linear homogeneous equation having the pair as a fundamental set of solutions.

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However, $e^{3x} = \lambda e^{-x}$ on an interval implies $e^{4x} = \lambda$ on an interval, which is false.
Proof 2. *We compote the Wronskian: $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$*
We find:

$$W(x) = -e^{3x}e^{-x} - e^{-x}3e^{3x} = -4e^{2x}$$

Since $W(x) \neq 0$ then the given functions are linearly independent on an interval I.

Examples

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If the second-order linear homogeneous equation

$$y'' + ay' + by = 0$$

has 2 solutions of the form e^{rx}*, then*

$$r^2 e^{rx} + are^{rx} + be^{rx} = 0$$

Hence it must be

$$r^2 + ar + b = 0$$

with solutions $r = 3, -1$. *Thus it must be*

$$r^2 + ar + b = (r-3)(r+1) = r^2 - 2r - 3$$

Thus is must be $y'' - 2y' - 3y = 0$