# Math 3321

## Second Order Linear Nonhomogeneous Equations (Variation of Parameters)

University of Houston

Lecture 09

# Outline

# Introduction

In this lecture, we will be focused on the general second order linear nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{N}$$

where $p$, $q$, and $f$ are continuous functions on some interval $I$.

We would like to determine the structure of solutions for (N) and we hope to develop a method for finding a solution for (N) using two linearly independent solutions of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0. \tag{H}$$

**Definition**

There is a connection between (N) and (H). Given an equation of type (N), the corresponding equation of form (H) is called the **reduced equation** for equation (N) of **homogeneous part** of equation (N).

# General Results

**Theorem 1**

Given any two solutions $z = z_1(x)$ and $z = z_2(x)$ for (N), then

$$y(x) = z_1(x) - z_2(x)$$

is a solution of the reduced equation (H).

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*If $z = z_1(x)$ and $z = z_2(x)$ are solutions for (N), then*

$$z_1'' + p(x)z_1' + q(x)z_1 = f(x)$$
$$z_2'' + p(x)z_2' + q(x)z_2 = f(x)$$

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$$z_1'' + p(x)z_1' + q(x)z_1 = f(x)$$
$$z_2'' + p(x)z_2' + q(x)z_2 = f(x)$$

*Hence, by linearity, with* $y(x) = z_1(x) - z_2(x)$ *we have*

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (z_1 - z_2)'' + p(x)(z_1 - z_2)' + q(x)(z_1 - z_2) \\ &= z_1'' + p(x)z_1' + q(x)z_1 - z_2'' - p(x)z_2' - q(x)z_2 \end{aligned}$$

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# General Results

**Theorem 2**

Let $y = y_1(x)$ and $y = y_2(x)$ be linearly independent solutions of the reduced equation (H) and let $z = z(x)$ be a particular solution of (N). If $u = u(x)$ is *any* solution of (N), then there exist constants $C_1$ and $C_2$ such that

$$u(x) = C_1 y_1(x) + C_2 y_2(x) + z(x).$$

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$$u(x) = C_1 y_1(x) + C_2 y_2(x) + z(x).$$

*Denote*

$$L(y) = y'' + p(x)y' + q(x)y$$

*By hypothesis,*

$$L(C_1 y_1 + C_2 y_2) = C_1 L(y_1) + C_2 L(y_)) = 0$$

$$L(z) = f$$

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$$L(z) = f$$

*Thus*

$$L(u) = L(C_1y_1 + C_2y_2 + z) = L(C_1y_1 + C_2y_2) + L(z) = f$$

# General Results

Theorem 2 tells us that when we have $y = y_1(x)$ and $y = y_2(x)$ are two linearly independent solutions for (H) and $z = z(x)$ is a particular solution of (N), then all solutions of (N) can be expressed as

$$y = C_1 y_1(x) + C_2 y_2(x) + z(x). \quad (1)$$

That is, (1) is the **general solution** of equation (N):

$$y'' + p(x)y' + q(x)y = f(x) \quad (N)$$

# General Results

The next result is known as the *superposition principle*. It can be useful for finding particular solutions of nonhomogeneous equations.

**Theorem 3**

Given the second order linear nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x) + g(x), \tag{2}$$

if $z = z_f(x)$ and $z = z_g(x)$ are particular solutions of

$$y'' + p(x)y' + q(x)y = f(x) \text{ and } y'' + p(x)y' + q(x)y = g(x),$$

respectively, then $z(x) = z_f(x) + z_g(x)$ is a particular solution of (2).

# General Results

Justification of superposition principle:

*As above, denote*

$$L(y) = y'' + p(x)y' + q(x)y$$

*By linearity, if*

$$L(z_f) = f \quad \text{and} \quad L(z_g) = g$$

*then*

$$L(z_f + z_g) = f + g$$

# General Results

In general, if $z = z_1(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x),$$

$z = z_2(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x),$$

$$\vdots$$

$z = z_n(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_n(x),$$

then $z = z_1(x) + z_2(x) + \cdots + z_n(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x) + \cdots + f_n(x).$$

# Variation of Parameters

Based on our work so far, to find a solution of (N) we need to find:

(i) a linearly independent pair of solutions $y_1$, $y_2$ of the reduced equation (H), and

(ii) a particular solution $z$ of (N).

Our first method for finding such a particular solution is the *method of variation of parameters.* We will use two linearly independent solutions of (H) to construct a particular solution of (N).

# Variation of Parameters

We start with $y_1(x)$ and $y_2(x)$, two linearly independent solutions of the reduced equation

$$y'' + p(x)y' + q(x)y = 0.$$

This means $C_1y_1(x) + C_2y_2(x)$ will be the general solution of the reduced equation. We replace $C_1$ and $C_2$ with functions of $x$, to form

$$z = u(x)y_1(x) + v(x)y_2(x).$$

We will impose two conditions on $u$ and $v$.

C1: $z = uy_1 + vy_2$ solves (N). That is,

$$z'' + p(x)z' + q(x)z = f(x).$$

C2: The second condition will help with our calculations. We require

$$y_1u' + y_2v' = 0.$$

# Variation of Parameters

## Method of Variation of Parameters

Starting with the differential equation (N), we find two linearly independent solutions $y_1(x)$ and $y_2(x)$ for the reduced equation (H). Letting $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$ denote the Wronskian of the pair of solutions, we will find a solution for (N) of the form

$$z(x) = u(x)y_1(x) + v(x)y_2(x)$$

where

$$u' = \frac{-y_2 f}{W} \text{ and } v' = \frac{y_1 f}{W}.$$

Then the general solution to (N) will be

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + z(x).$$

# Examples

1. Given that $\{y_1(x) = x^2,\, y_2(x) = x^4\}$ is a fundamental set of solutions of the reduced equation, find the general solution of

$$y'' - \frac{5}{x}y' + \frac{8}{x^2}y = 4x^3.$$

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*We compute the Wronskian:*

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = x^2(4x^3) - x^4(2x) = 2x^5$$

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*By the method of variation of parameters, a particular solution is*

$$z(x) = u(x)x^2 + v(x)x^4$$

*where*

$$u' = \frac{-y_2 f}{W} \text{ and } v' = \frac{y_1 f}{W}$$

# Examples

*We compute*

$$\begin{aligned} u(x) = \int \frac{-y_2(x)f(x)}{W(x)}dx &= \int \frac{-x^4(4x^3)}{2x^5}dx \\ &= -\int 2x^2 dx = -\frac{2}{3}x^3 \\ v(x) = \int \frac{y_1(x)f(x)}{W(x)}dx &= \int \frac{x^2(4x^3)}{2x^5}dx \\ &= \int 2dx = 2x \end{aligned}$$

# Examples

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$$u(x) = \int \frac{-y_2(x)f(x)}{W(x)}dx = \int \frac{-x^4(4x^3)}{2x^5}dx$$
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$$v(x) = \int \frac{y_1(x)f(x)}{W(x)}dx = \int \frac{x^2(4x^3)}{2x^5}dx$$
$$= \int 2dx = 2x$$

*Hence*

$$z(x) = u(x)x^2 + v(x)x^4 = -\tfrac{2}{3}x^3(x^2) + 2x(x^4) = \tfrac{4}{3}x^5$$

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$$v(x) = \int \frac{y_1(x)f(x)}{W(x)}dx = \int \frac{x^2(4x^3)}{2x^5}dx = \int 2dx = 2x$$

*Hence*

$$z(x) = u(x)x^2 + v(x)x^4 = -\tfrac{2}{3}x^3(x^2) + 2x(x^4) = \tfrac{4}{3}x^5$$

*The general solution is*

$$y(x) = c_1 x^2 + c_2 x^4 + \tfrac{4}{3}x^5$$

# Examples

2. Find the general solution of

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*We first find the solution of the homogeneous system by solving the characteristic equation*

$$r^2 + r - 6 = 0 \Rightarrow r = -3, r = 2$$

*Hence the function*

$$y_1(x) = e^{-3x}, \quad y_2(x) = e^{2x}$$

*are independent solutions of the homogeneous system*

# Examples

*To find a particular solution, we compute the Wronskian:*

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = e^{-3x}2e^{2x} - e^{2x}(-3e^{-3x})) = 5e^{-x}$$

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*By the method of variation of parameters, a particular solution is*

$$z(x) = u(x)e^{-3x} + v(x)e^{2x}$$

*where*

$$u' = \frac{-y_2 f}{W} \text{ and } v' = \frac{y_1 f}{W}$$

# Examples

*We compute*

$$u(x) = \int \frac{-y_2(x)f(x)}{W(x)}dx = \int \frac{-e^{2x}(3e^{2x})}{5e^{-x}}dx$$
$$= -\frac{3}{5}\int e^{5x}dx = -\frac{3}{25}e^{5x}$$
$$v(x) = \int \frac{y_1(x)f(x)}{W(x)}dx = \int \frac{e^{-3x}(3e^{2x})}{5e^{-x}}dx$$
$$= \int \frac{3}{5}dx = \frac{3}{5}x$$

# Examples

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$$u(x) = \int \frac{-y_2(x)f(x)}{W(x)}dx = \int \frac{-e^{2x}(3e^{2x})}{5e^{-x}}dx$$

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$$v(x) = \int \frac{y_1(x)f(x)}{W(x)}dx = \int \frac{e^{-3x}(3e^{2x})}{5e^{-x}}dx$$

$$= \int \tfrac{3}{5}dx = \tfrac{3}{5}x$$

*Hence*

$$z(x) = u(x)e^{-3x} + v(x)e^{2x} = -\tfrac{3}{25}e^{5x}e^{-3x} + \tfrac{3}{5}xe^{2x} = \tfrac{3}{5}(x - \tfrac{1}{5})e^{2x}$$

# Examples

*We compute*

$$u(x) = \int \frac{-y_2(x)f(x)}{W(x)}dx = \int \frac{-e^{2x}(3e^{2x})}{5e^{-x}}dx$$

$$= -\frac{3}{5}\int e^{5x}dx = -\frac{3}{25}e^{5x}$$

$$v(x) = \int \frac{y_1(x)f(x)}{W(x)}dx = \int \frac{e^{-3x}(3e^{2x})}{5e^{-x}}dx$$

$$= \int \frac{3}{5}dx = \frac{3}{5}x$$

*Hence*

$$z(x) = u(x)e^{-3x} + v(x)e^{2x} = -\frac{3}{25}e^{5x}e^{-3x} + \frac{3}{5}xe^{2x} = \frac{3}{5}(x - \frac{1}{5})e^{2x}$$

*The general solution is*

$$y(x) = c_1e^{-3x} + c_2e^{2x} + \frac{3}{5}(x - \frac{1}{5})e^{2x}$$

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*We first find the solution of the homogeneous system by solving the characteristic equation*

$$r^2 + 9 = 0 \Rightarrow r = \pm 3i$$

*Hence the function*

$$y_1(x) = \cos(3x), \quad y_2(x) = \sin(3x)$$

*are independent solutions of the homogeneous system*

# Examples

*To find a particular solution, we compute the Wronskian:*

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = 3\cos(3x)\cos(3x) + 3\sin(3x)\sin(3x) = 3$$

# Examples

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$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = 3\cos(3x)\cos(3x) + 3\sin(3x)\sin(3x) = 3$

*By the method of variation of parameters, a particular solution is*

$$z(x) = u(x)\cos(3x) + v(x)\sin(3x)$$

*where*

$$u' = \frac{-y_2 f}{W} \text{ and } v' = \frac{y_1 f}{W}$$

# Examples

*We compute*

$$
\begin{aligned}
u(x) = \int \frac{-y_2(x) f(x)}{W(x)} dx &= \int \frac{-\sin(3x)\tan(3x)}{3} dx \\
&= \int \frac{-\sin^2(3x)}{3\cos(3x)} dx \\
&= \int \frac{\cos^2(3x) - 1}{3\cos(3x)} dx \\
&= \frac{1}{3} \int (\cos(3x) - \sec(3x)) dx \\
&= \frac{1}{9}\sin(3x) - \frac{1}{9}\ln|\sec(3x) + \tan(3x)| \\
v(x) = \int \frac{y_1(x) f(x)}{W(x)} dx &= \int \frac{\cos(3x)\tan(3x)}{3} dx \\
&= \frac{1}{3}\int \sin(3x) dx = -\frac{1}{9}\cos(3x)
\end{aligned}
$$

# Examples

*Hence*

$$\begin{aligned} & z(x) \\ = \; & u(x)\cos(3x) + v(x)\sin(3x) \\ = \; & \tfrac{1}{9}\sin(3x)\cos(3x) - \tfrac{1}{9}\ln|\sec(3x)+\tan(3x)|\cos(3x) - \tfrac{1}{9}\cos(3x)\sin(3x) \\ = \; & -\tfrac{1}{9}\cos(3x)\ln|\sec(3x)+\tan(3x)| \end{aligned}$$

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*The general solution is*

$$y(x) = c_1\cos(3x) + c_2\sin(3x) - \tfrac{1}{9}\cos(3x)\ln|\sec(3x)+\tan(3x)|$$