Math 3321

Initial-Value Problems with Piecewise Continuous Nonhomogeneous Terms

University of Houston

Lecture 16

Outline

1. Introduction

2. Examples

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Recall the following:

$$\mathcal{L}[y'] = sY(s) - y(0)$$

and

$$\mathcal{L}[y''] = s^2Y(s) - sy(0) - y'(0),$$

where $\mathcal{L}[y(x)] = Y(s)$.

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where $\mathcal{L}[y(x)] = Y(s)$.

We will also need

$$\mathcal{L}[f(x-c)u(x-c)] = e^{-cs}F(s)$$

and

$$\mathcal{L}^{-1}[e^{-cs}F(s)] = f(x-c)u(x-c).$$

Table of Laplace Transforms

$f(x)$	$F(s) = \mathcal{L}[f(x)]$
k (constant)	$\frac{k}{s}, \quad s > 0$
$e^{\alpha x}$	$\frac{1}{s-\alpha}, \quad s > \alpha$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}, \quad s > 0$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}, \quad s > 0$
$e^{\alpha x} \cos \beta x$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$e^{\alpha x} \sin \beta x$	$\frac{\beta}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$x^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$x^n e^{rx}, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-r)^{n+1}}, \quad s > r$
$x \cos \beta x$	$\frac{s^2-\beta^2}{(s^2+\beta^2)^2}, \quad s > 0$
$x \sin \beta x$	$\frac{2\beta s}{(s^2+\beta^2)^2}, \quad s > 0$

Examples

1. Solve $y'' + 2y' + y = f(x);\ y(0) = y'(0) = 0$, where

$$f(x) = \begin{cases} 1, & 0 \le x < 2 \\ x - 1, & 2 \le x. \end{cases}$$

Examples

1. Solve $y'' + 2y' + y = f(x);\ y(0) = y'(0) = 0$, where

$$f(x) = \begin{cases} 1, & 0 \leq x < 2 \\ x - 1, & 2 \leq x. \end{cases}$$

Using the idea presented in Lecture 15, we can write f as follows

$$f(x) = f_1(x) + f_2(x), \qquad x \geq 0$$

where each term is a continuous function on an interval, namely

$$\begin{aligned} f_1(x) &= 1 - 1\,u(x-2) \\ f_2(x) &= (x-1)u(x-2) \end{aligned}$$

Hence

$$f(x) = 1 + (x-2)u(x-2) \qquad x \geq 0$$

Examples

Taking the Laplace transform of the LHS of the equation, we get

$$\begin{aligned}
\mathcal{L}[y'' + 2y' + y] &= \mathcal{L}[y''] + 2\mathcal{L}[y'] + \mathcal{L}[y] \\
&= s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + \mathcal{L}[y] \\
&= s^2\mathcal{L}[y] + 2s\mathcal{L}[y] + \mathcal{L}[y] \\
&= (s+1)^2\mathcal{L}[y]
\end{aligned}$$

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We also have that

$$\mathcal{L}[f(x)] = \mathcal{L}[1] + \mathcal{L}[(x-2)u(x-2)] = \tfrac{1}{s} + e^{-2s}\tfrac{1}{s^2}$$

Examples

Taking the Laplace transform of the LHS of the equation, we get

$$\begin{aligned}\mathcal{L}[y''+2y'+y] &= \mathcal{L}[y'']+2\mathcal{L}[y']+\mathcal{L}[y]\\ &= s^2\mathcal{L}[y]-sy(0)-y'(0)+2s\mathcal{L}[y]-2y(0)+\mathcal{L}[y]\\ &= s^2\mathcal{L}[y]+2s\mathcal{L}[y]+\mathcal{L}[y]\\ &= (s+1)^2\mathcal{L}[y]\end{aligned}$$

We also have that

$$\mathcal{L}[f(x)]=\mathcal{L}[1]+\mathcal{L}[(x-2)u(x-2)]=\tfrac{1}{s}+e^{-2s}\tfrac{1}{s^2}$$

Hence the Laplace transform of the differential equation gives

$$(s+1)^2\mathcal{L}[y]=\tfrac{1}{s}+e^{-2s}\tfrac{1}{s^2}$$

so that

$$Y(s)=\mathcal{L}[y]=\tfrac{1}{s(s+1)^2}+e^{-2s}\tfrac{1}{s^2(s+1)^2}$$

Examples

We can write

$$Y(s) = \frac{1}{s(s+1)^2} + e^{-2s}\frac{1}{s^2(s+1)^2} = Y_1(s) + e^{-2s}Y_2(s),$$

where

$$Y_1(s) = \frac{1}{s(s+1)^2}, \quad Y_2(s) = \frac{1}{s^2(s+1)^2},$$

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so that the solution will be of the form

$$y(x) = y_1(x) + y_2(x-2)u(x-2)$$

where

$$y_1(x) = \mathcal{L}^{-1}[Y_1(s)], \quad y_2(x) = \mathcal{L}^{-1}[Y_2(s)].$$

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To find the inverse Laplace transform of Y_1 *and* Y_2*, we will apply partial fractions.*

Examples

Using partial fractions

$$Y_1(s) = \frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

Hence

$$y_1(x) = \mathcal{L}^{-1}[\tfrac{1}{s}] - \mathcal{L}^{-1}[\tfrac{1}{s+1}] - \mathcal{L}^{-1}[\tfrac{1}{(s+1)^2}] = 1 - e^{-s} - xe^{-s}$$

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Similarly, using partial fractions, we get

$$Y_2(s) = \frac{1}{s^2(s+1)^2} = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{1}{(s+1)^2}$$

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Hence

$$\begin{aligned} y_2(x) &= -2\mathcal{L}^{-1}[\tfrac{1}{s}] + \mathcal{L}^{-1}[\tfrac{1}{s^2}] + 2\mathcal{L}^{-1}[\tfrac{1}{s+1}] + \mathcal{L}^{-1}[\tfrac{1}{(s+1)^2}] \\ &= -2 + x + 2e^{-x} + xe^{-x} \end{aligned}$$

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Finally

$$\begin{aligned} y &= 1 - e^{-x} - xe^{-x} + (-2 + (x-2) + 2e^{-(x-2)} + (x-2)e^{-(x-2)})u(x-2) \\ &= 1 - (x+1)e^{-x} + ((x-4) + xe^{-(x-2)})u(x-2) \end{aligned}$$

Examples

2. Solve $y'' + 2y' + y = f(x);\ y(0) = 3,\ y'(0) = -1$, where

$$f(x) = \begin{cases} e^x, & 0 \le x < 1 \\ e^x - 1, & 1 \le x. \end{cases}$$

2. Solve $y'' + 2y' + y = f(x);\ y(0) = 3,\ y'(0) = -1$, where

$$f(x) = \begin{cases} e^x, & 0 \le x < 1 \\ e^x - 1, & 1 \le x. \end{cases}$$

Using the same idea as above, we can write f as

$$f(x) = f_1(x) + f_2(x) \qquad x \ge 0$$

where

$$\begin{aligned} f_1(x) &= e^x - e^x\, u(x-1) \\ f_2(x) &= (e^x - 1)u(x-1) \end{aligned}$$

Hence

$$f(x) = e^x - u(x-1) \qquad x \ge 0$$

We have

$$\mathcal{L}[f(x)] = \tfrac{1}{s-1} - e^{-s}\tfrac{1}{s}$$

Examples

Taking the Laplace transform of the LHS of the equation, we get

$$\begin{aligned}\mathcal{L}[y'' + 2y' + y] &= s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + \mathcal{L}[y] \\ &= s^2\mathcal{L}[y] + 2s\mathcal{L}[y] + \mathcal{L}[y] - 3(s+2) + 1 \\ &= (s+1)^2\mathcal{L}[y] - 3s - 5\end{aligned}$$

Examples

Taking the Laplace transform of the LHS of the equation, we get

$$\begin{aligned}\mathcal{L}[y''+2y'+y] &= s^2\mathcal{L}[y]-sy(0)-y'(0)+2s\mathcal{L}[y]-2y(0)+\mathcal{L}[y]\\ &= s^2\mathcal{L}[y]+2s\mathcal{L}[y]+\mathcal{L}[y]-3(s+2)+1\\ &= (s+1)^2\mathcal{L}[y]-3s-5\end{aligned}$$

Hence the Laplace transform of the differential equation gives

$$(s+1)^2\mathcal{L}[y]-3s-5=\tfrac{1}{s-1}-e^{-s}\tfrac{1}{s}$$

so that

$$\begin{aligned}Y(s) &= \mathcal{L}[y]=\tfrac{1}{(s-1)(s+1)^2}+\tfrac{3s+5}{(s+1)^2}-e^{-s}\tfrac{1}{s(s+1)^2}\\ &= \tfrac{3s^2+4s-4}{(s-1)(s+1)^2}-e^{-s}\tfrac{1}{s(s+1)^2}\\ &= \tfrac{9}{4}\tfrac{1}{s+1}+\tfrac{5}{2}\tfrac{1}{(s+1)^2}+\tfrac{3}{4}\tfrac{1}{s-1}-e^{-s}\left(\tfrac{1}{s}-\tfrac{1}{s+1}-\tfrac{1}{(s+1)^2}\right)\end{aligned}$$

Examples

To find the solution of the IVP we need to compute

$$y(x) = \mathcal{L}^{-1}[\tfrac{9}{4}\tfrac{1}{s+1} + \tfrac{5}{2}\tfrac{1}{(s+1)^2} + \tfrac{3}{4}\tfrac{1}{s-1} - e^{-s}(\tfrac{1}{s} - \tfrac{1}{s+1} - \tfrac{1}{(s+1)^2})]$$

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Hence

$$y(x) = \tfrac{9}{4}e^{-x} + \tfrac{5}{2}xe^{-x} + \tfrac{3}{4}e^{x} - (1 - e^{-(x-1)} - (x-1)e^{-(x-1)})u(x-1)$$

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which simplifies to

$$y(x) = (\tfrac{9}{4} + \tfrac{5}{2}x)e^{-x} + \tfrac{3}{4}e^{x} - (1 - xe^{-(x-1)})u(x-1)$$