Math 3321

Systems of Linear Equations. Part II

University of Houston

Lecture 18

Outline

1. Matrix, Augmented Matrix, Matrix of Coefficients
2. Solution of systems of linear equations by Gaussian elimination
3. Reduced Row Echelon Form
4. Homogeneous Systems

Matrix of Coefficients

We have seen that the solution of a system of linear equations is efficiently handled through the manipulation of the coefficients of the of the linear equations.

Definition

A **matrix** is a rectangular array of numbers. A matrix with m rows and n columns is an $m \times n$ matrix.

We are interested to matrices associated with system of linear equations

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n = b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n = b_m \end{array}$$

Matrix of Coefficients

We define the **matrix of coefficients**

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{32} & \cdots & a_{mn} \end{pmatrix}$$

Denoting the vectors of unknowns x and the constant vector b as

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

the system of linear equations can be written as

$$Ax = b$$

Augmented matrix

We define the **augmented matrix** of the system of linear equations as

$$(A|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

To find a solution of the system of linear equations, we will apply a set of **elementary row operations** to $(A|b)$ to convert the matrix to **row echelon form.**

Augmented matrix

Elementary row operations:

1. Interchange row i and row j

$$R_i \leftrightarrow R_j.$$

2. Multiply row i by a nonzero number k

$$kR_i \to R_i.$$

3. Multiply row i by a number k and add the result to row j

$$kR_i + R_j \to R_j.$$

Row Echelon form

Row echelon form matrix

A matrix is in row echelon form if the following conditions are met:

1. Rows consisting entirely of zeros are at the bottom of the matrix.
2. The first nonzero entry in a nonzero row is a 1. It is called the *leading* 1.
3. If row i and row $i+1$ are nonzero rows, then the leading 1 in row $i+1$ is to the right of the leading 1 in row i.

Row Echelon form

Examples of matrices in row echelon form:

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccr|c} 1 & 1 & -3 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{crcr|r} 1 & -2 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

Row Echelon form

NOTE. If a matrix is in row echelon form, the following properties hold:

- All the entries below a leading 1 are zero.
- The number of leading 1's is less than or equal to the number of rows.
- The number of leading 1's is less than or equal to the number of columns.

NOTE. If a matrix is in row echelon form, the following properties hold:

- All the entries below a leading 1 are zero.
- The number of leading 1's is less than or equal to the number of rows.
- The number of leading 1's is less than or equal to the number of columns.

As we observed in the last lecture, after converting the augmented matrix to row-echelon form it is very simple to compute the solution of the corresponding linear system.

Row Echelon form

Example

Solve the system

$$\begin{aligned} 3x - 4y - z &= 3 \\ 2x - 3y + z &= 1 \\ x - 2y + 3z &= 2 \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & -4 & -1 & 3 \\ 2 & -3 & 1 & 1 \\ 1 & -2 & 3 & 2 \end{array}\right)$$

Row Echelon form

We apply elementary row operations

$$\left(\begin{array}{ccc|c} 3 & -4 & -1 & 3 \\ 2 & -3 & 1 & 1 \\ 1 & -2 & 3 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 2 & -3 & 1 & 1 \\ 3 & -4 & -1 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 2 & -10 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & -5 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

We finally have the matrix in row echelon form.

Row Echelon form

Corresponding system of equations:

$$\begin{aligned} x - 2y + 3z &= 2 \\ 0x + y - 5z &= -3 \\ 0x + 0y + 0z &= 1 \end{aligned}$$

That is

$$\begin{aligned} x - 2y + 3z &= 2 \\ y - 5z &= -3 \\ 0z &= 1 \end{aligned}$$

Solution set: no solution.

Row Echelon form

Example

Solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= -2 \\ -2x_1 + 5x_2 - x_3 + 4x_4 &= 1 \\ 3x_1 - 7x_2 + 2x_3 + x_4 &= 9 \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{rrrr|r} 1 & -2 & 1 & -1 & -2 \\ -2 & 5 & -1 & 4 & 1 \\ 3 & -7 & 2 & 1 & 9 \end{array}\right)$$

Row Echelon form

We apply elementary row operations

$$\left(\begin{array}{rrrr|r} 1 & -2 & 1 & -1 & -2 \\ -2 & 5 & -1 & 4 & 1 \\ 3 & -7 & 2 & 1 & 9 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & -2 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 & -3 \\ 0 & -1 & -1 & 4 & 15 \end{array}\right)$$

$$\to \left(\begin{array}{rrrr|r} 1 & -2 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 & -3 \\ 0 & 0 & 0 & 6 & 12 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & -2 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

We finally have the matrix in row echelon form.

Row Echelon form

Corresponding system of equations:

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= -2 \\ x_2 + x_3 + 2x_4 &= -3 \\ x_4 &= 2 \end{aligned}$$

Solution set:

$$\begin{aligned} x_1 &= -14 - 3a, \\ x_2 &= -7 - a, \\ x_3 &= a, \\ x_4 &= 2, \qquad a \text{ any real number.} \end{aligned}$$

Solution method for systems of linear equations

The method for the solution of systems of linear equations we have presented is called **Gaussian elimination with back substitution.**

It consists of the following steps.

1. Write the augmented matrix $(A|b)$ for the system.
2. Use elementary row operations to transform the augmented matrix to row echelon form.
3. Write the system of equations corresponding to the row echelon form.
4. Back substitute to find the solution set.

Systems of linear equations

Consistent/Inconsistent systems

A system of linear equations is **consistent** if it has at least one solution.

That is, a system is consistent if it has either a unique solution or infinitely many solutions.

A system that has no solutions is **inconsistent**.

Dependent/Independent systems

A consistent system is said to be **independent** if it has a unique solution.
A system with infinitely many solutions is called **dependent**.

Example

Solve the system of equations

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 &= 2 \\ 3x_1 - 9x_2 + 7x_3 - x_4 + 3x_5 &= 7 \\ 2x_1 - 6x_2 + 7x_3 + 4x_4 - 5x_5 &= 7 \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{rrrrr|r} 1 & -3 & 2 & -1 & 2 & 2 \\ 3 & -9 & 7 & -1 & 3 & 7 \\ 2 & -6 & 7 & 4 & -5 & 7 \end{array}\right)$$

Systems of linear equations

Transform to row echelon form:

$$\left(\begin{array}{rrrrr|r} 1 & -3 & 2 & -1 & 2 & 2 \\ 3 & -9 & 7 & -1 & 3 & 7 \\ 2 & -6 & 7 & 4 & -5 & 7 \end{array}\right)$$

Equivalent system:

$$\left(\begin{array}{rrrrr|r} 1 & -3 & 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Corresponding system of equations:

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 &= 2 \\ 0x_1 + 0x_2 + x_3 + 2x_4 - 3x_5 &= 1 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= 0 \end{aligned}$$

Systems of linear equations

The reduced equations simplify to

$$\begin{aligned} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 &= 2 \\ x_3 + 2x_4 - 3x_5 &= 1 \end{aligned}$$

This shows that the system is **consistent** (that is, it admits at least one solution).

Solution set:

$$\begin{aligned} x_1 &= 3a + 5b - 8c, \\ x_2 &= a, \\ x_3 &= 1 - 2b + 3c, \\ x_4 &= b, \\ x_5 &= c, \end{aligned}$$

where a, b, c are arbitrary real numbers. Hence the system is **dependent.**

Problem

For what value(s) of k, if any, does the system

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + kz &= 3 \\ x + ky + 3z &= 2 \end{aligned}$$

have:

(a) a unique solution?

(b) infinitely many solutions?

(c) no solution?

Systems of linear equations

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{array}\right)$$

Transform to row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & (k+3)(k-2) & k-2 \end{array}\right)$$

Conclusion:

- (a) Unique solution: $k \neq 2, -3$. In this case, we have 3 equations and 3 unknowns.
- (b) Infinitely many solns: $k = 2$. In this case, we have 2 equations and 3 unknowns.
- (c) No solution: $k = -3$. In this case, the last equation takes the form $0 = k - 2$

Rank of a matrix

If an $m \times n$ matrix A is expressed in row echelon form, then the number of non-zero rows in its row echelon form is called the **rank** of A.

Equivalently, the rank of a matrix is the number of leading 1's in its row echelon form.

Fact: the rank of a matrix is less than or equal to the number of rows.

Using the rank, we can establish necessary and sufficient conditions for a system of linear equations to be consistent.

Theorem

A system of linear equations is consistent if and only if the rank of the coefficient matrix **equals** the rank of the augmented matrix.

If the rank of the augmented matrix is **greater than** the rank of the coefficient matrix, then the system has no solutions.

Reduced Row Echelon Form

Reduced Row Echelon Form

A matrix is in reduced row echelon form if the following conditions are met:

1. Rows consisting entirely of zeros are at the bottom of the matrix.
2. The first nonzero entry in a nonzero row is a 1.
3. The leading 1 in row $i+1$ is to the right of the leading 1 in row i.
4. The leading 1 is the only nonzero entry in its column.

Examples

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad \left(\begin{array}{cccr|c} 1 & 0 & 0 & 4 & 9 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right) \quad \left(\begin{array}{ccrc|r} 1 & 0 & -4 & 0 & 30 \\ 0 & 1 & 1 & 0 & -18 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

Reduced Row Echelon Form

Example

Solve the system

$$\begin{aligned} x - 2y + 4z &= 12 \\ 2x - y + 5z &= 18 \\ -x + 3y - 3z &= -8 \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{array}\right)$$

Row reduce to:

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Reduced Row Echelon Form

We can continue applying elementary row operations to the converted augmented matrix until we obtain the reduced row-echelon form.

$$\left(\begin{array}{rrr|r} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Corresponding system of equations

$$\begin{aligned} x &= 2 \\ y &= 1 \\ z &= 3 \end{aligned}$$

Reduced Row Echelon Form

Example

Solve the system

$$\begin{aligned} 2x_1 + 5x_2 - 5x_3 - 7x_4 &= 8 \\ x_1 + 2x_2 - 3x_3 - 4x_4 &= 2 \\ -3x_1 - 6x_2 + 11x_3 + 16x_4 &= 0 \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{rrrr|r} 2 & 5 & -5 & -7 & 8 \\ 1 & 2 & -3 & -4 & 2 \\ -3 & -6 & 11 & 16 & 0 \end{array}\right)$$

Row echelon form:

$$\left(\begin{array}{rrrr|r} 1 & 2 & -3 & -4 & 2 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right)$$

Reduced Row Echelon Form

We can continue applying elementary row operations to the converted augmented matrix until we obtain the reduced row-echelon form.

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & -4 & 2 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 11 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 9 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right)$$

Corresponding system of equations:

$$x_1 + 4x_4 = 9$$
$$x_2 - x_4 = 1$$
$$x_3 + 2x_4 = 3$$

Solution:

$$x_3 = 3 - 2x_4,\ x_2 = 1 + x_4,\ x_1 = 9 - 4x_4, \quad x_4 \text{ any real number.}$$

Homogeneous Systems

Homogeneous systems of linear equations

The system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is **homogeneous** if

$$b_1 = b_2 = \cdots = b_m = 0,$$

otherwise, the system is **nonhomogeneous**.

Homogeneous Systems

A homogeneous system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = 0 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = 0 \end{array}$$

always has at least one solution, namely

$$x_1 = x_2 = \cdots = x_n = 0,$$

called the **trivial solution**.

That is, homogeneous systems are always consistent.

Homogeneous Systems

Example

Solve the homogeneous system

$$\begin{aligned} x - 2y + 2z &= 0 \\ 4x - 7y + 3z &= 0 \\ 2x - y + 2z &= 0 \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 4 & -7 & 3 & 0 \\ 2 & -1 & 2 & 0 \end{array}\right)$$

Row echelon form:

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Homogeneous Systems

Corresponding system of equations:

$$\begin{aligned} x - 2y + 2z &= 0 \\ y - 5z &= 0 \\ z &= 0 \end{aligned}$$

This system has the unique solution

$$\begin{aligned} x &= 0, \\ y &= 0, \\ z &= 0 \end{aligned}$$

The trivial solution is the only solution.

Homogeneous Systems

Example

Solve the homogeneous system

$$\begin{aligned} 3x - 2y + z &= 0 \\ x + 4y + 2z &= 0 \\ 7x + 4z &= 0 \end{aligned}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & -2 & 1 & 0 \\ 1 & 4 & 2 & 0 \\ 7 & 0 & 4 & 0 \end{array}\right)$$

Row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 1 & \frac{5}{14} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Homogeneous Systems

Corresponding system of equations:

$$\begin{aligned} x + 4y + 2z &= 0 \\ y + \frac{5}{14}z &= 0 \end{aligned}$$

This system has infinitely many solutions:

$$x = -\frac{2}{7}\,a, \quad y = -\frac{5}{14}\,a, \quad z = a,$$

where a is any real number.