International Journal of Financial Engineering
Vol. 4, Nos. 2 & 3 (2017) 1750017 (30 pages)

DOI: 10.1142/S2424786317500177

World Scientific
www.worldscientific.com

Pricing spread options by generalized bivariate edgeworth expansion

Edward P. C. Kao*

Department of Mathematics, University of Houston
Houston, Texas, 77204, USA

Weiwei Xie

Department of Mathematical Sciences, Worcester Polytechnic Institute
Worcester, Massachusetts, 01605, USA

Received: 21 August 2016; Accepted: 4 December 2016
Published: 8 June 2017

Abstract

A spread option is a contingent claim whose underlying is the price difference between two assets. For a call, the holder of the option receives the difference, if positive, between the price difference and the strike price. Otherwise, the holder receives nothing. Spread options trade in large volume in financial, fixed-income, commodity, and energy industries. It is well known that pricing of spread options does not admit closed-form solutions even under a geometric Brownian motion paradigm. When price dynamics experience stochastic volatilities and/or jumps, the valuation process becomes more challenging. Following the seminal work of Jarrow and Judd, we propose the use of Edgeworth expansion to approximate the call price. In the spirit of Pearson, we reduce the cumbersome computation inherent in Edgeworth expansion to single numerical integrations. For an arbitrary bivariate price process, we show that once its product cumulants are available, either by virtue of the structural properties of the underlying processes or by empirical estimation using market data, the approach enables analysts to approximate the call price easily. Specifically, the call prices so estimated capture the correlation, skewness, and kurtosis of the two underlying price processes. As such, the approach is useful for approximate valuations based on Lévy-based models.

Keywords: Spread options; edgeworth expansion; Lévy processes.

*Corresponding author.
Email address: *edkao@math.uh.edu

1. Introduction

A spread option is a financial contract between two parties. For the energy/power and commodity markets, spread options are in the bread-and-butter tool kit for risk management. In this paper, we consider a spread option of European type. For a call option, at maturity the holder has the right to exercise the option and collect the payoff. The payoff is the difference in price between the two underlying assets and the strike price. If the payoff is negative, the holder forgoes the right. Spread options trade extensively in organized exchanges and between counter parties. Its popularity many times stems from the need to hedge the price difference between the two underlying assets and/or speculative motives between two parties with differing beliefs about future price movements. It is well known that pricing of spread options does not have closed-form solutions even under a geometric Brownian motion (GBM) paradigm. Various approaches on pricing spread options center around two age-old thrusts — bounds and approximations. Earlier work on the subject include those of Shimko (1994), Kirk (1995), Pearson (1995), Mbanefo (1997), among others. In the recent past, Carmona and Durrleman (2003) provided a procedure requiring the solution of nonlinear equations for estimating price bounds. Dempster and Hong (2000) and Hong (2001) employed the Fast Fourier Transform for valuation of spread options. Deng *et al.* (2001), Hikspoors and Jaimungal (2007) and Benth *et al.* (2008) studied the pricing issues in energy markets. Li *et al.* (2008) and Bjerksund and Stensland (2008) gave two different closed-form approximations under a GBM paradigm. A very efficient numerical procedure using Fourier space time-stepping approach was suggested by Jackson *et al.* (2008/09). In addition, another Fourier transform-based approach for numerical solution of spread option appeared in this paper was proposed by Hurd and Zhou (2010). Kouam (2007) explored the use of Monte Carlo simulation to study scenarios where the bivariate price process contains stochastic volatility and jumps. The extension of the underlying paradigm to one that includes Levy processes has been considered by Benth and Kettler (2011) in the context of spark spread.

For a European option, the prices at maturity are all what is needed for pricing. Hence the stochastic process governing the price dynamics is relevant only to the extent it produces the price distributions of underlying assets at maturity. For an option with a single underlying asset, under the classical Black–Scholes–Merton paradigm (1973, 1990), the price at maturity follows a lognormal distribution. It is widely acknowledged that market data deviate from such a paradigm and with a consequence of frequent and serious mispricings. To circumvent this difficulty, Jarrow and Rudd (1982) pioneered the use of Edgeworth expansion to approximate arbitrary price distributions at maturity using the lognormal as the *approximating* distribution. As a result, the skewness and kurtosis of the price process are reflected in the option price so computed. The term Edgeworth expansion was coined after Edgeworth (1904) in his study of law of error. Its use for approximating probability distributions spans more than one hundred years in the statistical literature (e.g., see Cramér (1946), Kendall (1949), McCullagh (1987), and Stuart and Ord (1994)). Following the lead of Jarrow and Rudd, others have used the approach for applications in option pricing — particularly options with a single underlying asset. These include the work of Duan *et al.* (1999) under a GARCH framework and Turnbull and Wakeman (1991) for pricing average options, among others. In the past 15 years, the use of Edgeworth expansion in financial mathematics has dwindled to a trickle. Notable exceptions are the papers by Zhang *et al.* (2011), and a thesis done at Imperial College, London, by Ehrlich (2012) for pricing basket options with smile.

In this paper, we present an approach for approximating the price of a spread option by bivariate Edgeworth expansion. In Sec. 2, we introduce a few needed notations and results pertaining to product cumulants. In Sec. 3, we present the bivariate Edgeworth expansion and state its implications in computation. In Sec. 4, we address the issues concerning pricing spread options. The derivations of the results extend the idea of Pearson (1995) (also see Ravindran (1995)). Simply stated, it exploits the law of total probability in computing expectation. We will see that after elaborate manipulations, the potentially cumbersome computation boils down to single numerical integrations. Moreover, numerical differentiation embedded in the expansion is completely avoided. In Sec. 5, we demonstrate the application of the procedure for correlated price processes that include jumps of the type considered by Merton (1976). In Sec. 6, we present another example based on the double exponential jump diffusion model of Kou (2002). There, the model enables us to capture the asymmetric heavy tails of returns resulting "volatility smile". For an arbitrary bivariate price process, we emphasize that once its product cumulants are known, either by virtue of the structural properties of the underlying processes (as illustrated in Secs. 5 and 6) or by empirical estimation using market data, the approach enables analysts to approximate the call price easily. Specifically, the call prices so estimated capture the correlation, skewness, and kurtosis of the two underlying price processes. In our implementation, we match the first two moments of the actual and approximating distributions. Our approach is similar in spirit to that of the Tankov and Ménassé (2015). There, they stated that "Our closed formula represents the indifference price as a linear combination of the Black–Scholes price and correction terms which depend on the variance, skewness and kurtosis of the underlying Lévy process, and the derivatives of the Black–Scholes price."

2. Preliminaries

Let Y_1 and Y_2 be two random variables and F_{Y_1,Y_2} be the joint distribution function. We use A_{Y_1,Y_2} to denote an approximating distribution of F_{Y_1,Y_2}. We consider the cases when the respective density functions f_{Y_1,Y_2} and a_{Y_1,Y_2} exist.

Define the (r,s)th moment of (Y_1,Y_2) about the origin $(0,0)$ under F_{Y_1,Y_2} by

$$\mu'_{r,s}(F) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1^r y_2^s dF_{Y_1,Y_2}(y_1,y_2).$$

It follows that $\mu'_{r,0}$ is the rth moment of Y_1 and $\mu'_{0,s}$ is the sth moment of Y_2. Also, $\mu_{r,s}$ is known as the (r,s)th product moment of (Y_1,Y_2). The moments of (Y_1,Y_2) are defined by the moment generating function (m.g.f)

$$\mathbb{M}(F,t_1,t_2) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mathbb{E}[e^{t_1y_1+t_2y_2}] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\mu'_{r,s}(F)t_1^r t_2^s}{r!s!}. \tag{1}$$

Similarly, we define the (r,s)th product moment of (Y_1,Y_2) about the mean (μ'_{10},μ'_{02}) under F_{Y_1,Y_2} by

$$\mu_{r,s}(F) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y_1-\mu'_{10})^r(y_2-\mu'_{0,2})^s dF_{Y_1Y_2}(y_1,y_2).$$

In this paper, we consider the cases when these moments exist for all non-negative integers r and s such that $0 < r+s \le 4$.

Let $\Phi(F,t_1,t_2)$ denote the characteristic function (c.f.) of F_{Y_1,Y_2}, i.e.,

$$\Phi(F,t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it_1y_1+it_2y_2}dF_{Y_1,Y_2}(y_1,y_2),$$

where $i^2=-1$. The c.f. is the Fourier transform of the density. The joint c.f. and joint m.g.f. of the two random variables are related by

$$\Phi(F,t_1,t_2) = \mathbb{M}(F,it_1,it_2).$$

The bivariate *cumulants* $\kappa_{r,s}$, also known as the product cumulants, of the two random variables are defined by

$$\ln\mathbb{M}(F,t_1,t_2) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\kappa_{rs}(F)t_1^r t_2^s}{r!s!},$$

where $\kappa_{00}\equiv 0$ (the term cumulant was coined by Thiele (1903)). Hence

$$\exp\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\kappa_{rs}(F)t_1^r t_2^s}{r!s!}\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\mu'_{r,s}(F)t_1^r t_2^s}{r!s!}, \tag{2}$$

where the right side of (2) is due to (1) (the above equation is given as (3.72) in Stuart and Ord (1994)). Since $e^x = \sum_{n=0}^{\infty} x^n/n!$, the left side of (2) can be expanded as powers of t_1 and t_2. The product moments $\mu'_{r,s}(F)$ can be found by matching the coefficients of associated with the term $t_1^r t_2^s$ on the right side of (2). From these relations, the product cumulants can also be expressed in terms of product moments. These relations are listed in detail in Stuart and Ord (while the latest edition was published in 2005, the original version published under the authorship of Kendall and Stuart appeared in 1943). For the cases where $0 < r + s \leq 4$, we cite those results needed in subsequent development (e.g., see p. 90 and p. 107 in Stuart and Ord (1994)):

$$\begin{aligned}
&\kappa_{10}(F) = \mu'_{10}(F), && \kappa_{20}(F) = \mu_{20}(F), \quad \kappa_{30}(F) = \mu_{30}(F),\\
&\kappa_{40}(F) = \mu_{40}(F) - 3\mu_{20}^2(F), && \kappa_{11}(F) = \mu_{11}(F), \quad \kappa_{21}(F) = \mu_{21}(F),\\
&\kappa_{31}(F) = \mu_{31}(F) - 3\mu_{20}(F)\mu_{11}(F), && \kappa_{22}(F) = \mu_{22}(F) - \mu_{20}(F)\mu_{02}(F) - 2\mu_{11}^2(F).
\end{aligned}$$

The results for $\kappa_{01}(F), \kappa_{02}(F), \kappa_{03}(F), \kappa_{04}(F), \kappa_{12}(F)$, and $\kappa_{13}(F)$ can be found by symmetry. We also note that all the terms defined in this section are defined similarly when the approximating distribution $A_{Y_1 Y_2}$ is in effect.

3. Bivariate Edgeworth Expansion

In this section, we approximate the true density f_{Y_1,Y_2} in terms of the approximating density a_{Y_1,Y_2} and their product cumulants κ_{rs}. The key result is given in Theorem 1. Our derivation of the theorem is informal but it is intuitive. It is different than that of Shiganov (1986). The assumptions about the existence of all orders of product moments and partial derivatives of density functions substantially simplify the proof. A formal and rigorous proof of the theorem for the multidimensional case can be found in Shiganov (1986). There the expansion was stated as Edgeworth–Cramér expansion.

Theorem 1. *Assume that*

$$\frac{d^{(r+s)}}{dy_1^r dy_2^s} a_{Y_1,Y_2}(y_1, y_2)$$

exist for nonnegative integers r and s such that $r + s \geq 0$. The series expansion for f_{Y_1,Y_2} in terms of a_{Y_1,Y_2} and their product cumulants κ_{rs} is given by

$$f_{Y_1,Y_2}(y_1, y_2) = a_{Y_1,Y_2}(y_1, y_2) + \sum_{r+s>0} E_{rs} \frac{(-1)^{r+s}}{r!s!} \frac{d^{(r+s)}}{dy_1^r dy_2^s} a_{Y_1,Y_2}(y_1, y_2). \qquad (3)$$

Proof. See Appendix A. □

In our application of Edgeworth expansion for pricing spread options, we are only interested in including product moments of order $0 < r+s \leq 4$. This choice captures correlation, skewness, and kurtosis of the two price dynamics. The following lemma prescribes the approximation formula and expresses the correction factors E_{rs} in terms of product cumulants.

Lemma 1. *In pricing a spread option we only include product cumulants κ_{rs} such that $0 < r+s \leq 4$. In such a case, we approximate the actual price distribution f_{Y_1,Y_2} using the approximating price distribution a_{Y_1,Y_2} with*

$$f_{Y_1,Y_2}(y_1,y_2) \approx a_{Y_1,Y_2}(y_1,y_2) + \sum_{0<r+s\leq 4} E_{rs} \frac{(-1)^{r+s}}{r!s!} \frac{d^{(r+s)}}{dy_1^r dy_2^s} a_{Y_1,Y_2}(y_1,y_2). \tag{4}$$

Moreover, the correction factors E_{rs} are related to product cumulants. Their relations are specified in Appendix B.

Proof. See Appendix B. □

In the spirit of Jarrow and Rudd (1982), we use a bivariate lognormal as the approximating price distribution at maturity. As $Y_i = \log(S_i)$, $i = 1,2$, $(Y_1,Y_2) \sim BVN((\mu_1,\mu_2),(\rho,\sigma_1,\sigma_2))$. Since BVN is completely characterized by the last five parameters and the product cumulants beyond order two of BVN vanish (e.g., Holmquist (1988) simplifies the approximation formula (4) further).

Lemma 2. *If we set $\kappa_{0i}(F) = \kappa_{0i}(A)$ and $\kappa_{i0}(F) = \kappa_{i0}(A)$ for $i = 1$ and 2, and $\rho(F) = \rho(A)$, then* (4) *reduces to*

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) \approx\ & a_{Y_1,Y_2}(y_1,y_2) - \frac{\kappa_{12}(F)}{2!} \frac{d^{(3)}}{dy_1 dy_2^2} a_{Y_1,Y_2}(y_1,y_2) \\ & - \frac{\kappa_{21}(F)}{2!} \frac{d^{(3)}}{dy_1^2 dy_2} a_{Y_1,Y_2}(y_1,y_2) - \frac{k_{30}(F)}{3!} \frac{d^{(3)}}{dy_1^3} a_{Y_1,Y_2}(y_1,y_2) \\ & - \frac{\kappa_{03}(F)}{3!} \frac{d^{(3)}}{dy_2^3} a_{Y_1,Y_2}(y_1,y_2) + \frac{\kappa_{13}(F)}{3!} \frac{d^{(4)}}{dy_1 dy_2^3} a_{Y_1,Y_2}(y_1,y_2) \\ & + \frac{\kappa_{31}(F)}{3!} \frac{d^{(4)}}{dy_1^3 dy_2} a_{Y_1,Y_2}(y_1,y_2) + \frac{\kappa_{40}(F)}{4!} \frac{d^{(4)}}{dy_1^4} a_{Y_1,Y_2}(y_1,y_2) \\ & + \frac{\kappa_{04}(F)}{4!} \frac{d^{(4)}}{dy_2^4} a_{Y_1,Y_2}(y_1,y_2) + \frac{\kappa_{22}(F)}{2!2!} \frac{d^{(4)}}{dy_1^2 dy_2^2} a_{Y_1,Y_2}(y_1,y_2). \end{aligned} \tag{5}$$

Proof. Using $E_{ij} = \kappa_{ij}(F)$ for $i+j = 3$ and 4, and 0 otherwise, and the assumptions of the lemma, (5) follows. □

Lemma 2 shows that if we match the means, variances, and correlations of the true distribution F and approximating distribution A, then the difference between the two distributions can be approximated by a series expansion on the right side of (5) involving higher order cumulants of the two distributions and the derivatives of the approximating density a_{Y_1,Y_2}.

4. Pricing Spread Options

We now demonstrate the application of bivariate Edgeworth expansion to the pricing of spread option. Let S_1 and S_2 denote the price processes of the two underlying assets. Under the risk-neutral measure Q, the price of the call is given by

$$\begin{aligned} c(F) &= e^{-rT}E^{Q}[(S_2(T)-S_1(T)-K)^{+}] \\ &= e^{-rT}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(s_2-s_1-K)^{+}f_{S_1,S_2}(s_1,s_2)ds_1ds_2, \end{aligned} \tag{6}$$

where f_{S_1,S_2} denote the joint density of S_1 and S_2 under Q. In the context of Edgeworth expansion, f_{S_1,S_2} denotes the true density. As before, we let a_{S_1,S_2} denote the approximating bivariate density.

Substituting the Edgeworth expansion (4) into (6), we obtain

$$\begin{aligned} c(F) \approx c(A) + e^{-rT}\sum_{0<r+s\leq 4} E_{rs}\frac{(-1)^{r+s}}{r!s!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(s_2-s_1-K)^{+} \\ \times \frac{d^{(r+s)}}{ds_1^r ds_2^s}a_{S_1,S_2}(s_1,s_2)ds_1ds_2, \end{aligned} \tag{7}$$

where

$$c(A) = e^{-rT}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(s_2-s_1-K)^{+}a_{S_1,S_2}(s_1,s_2)ds_1ds_2. \tag{8}$$

When the conditions of Theorem 1 are met, the price of a spread option whose underlying price processes follow arbitrary distributions can be approximated by the Edgeworth expansion (7) with *any* arbitrary chosen approximating price distributions. As many option valuation results pertaining to lognormal price distributions are known, in what follows we choose the geometric Brownian motion paradigm for evaluating $c(A)$. Lemma 2 implies that the right side of (7) can be decomposed into three parts, the price of the spread option under GBM, an adjustment term involving skewness (terms with $r+s=3$), and an adjustment term involving skewness (terms with $r+s=4$). We refer them as Terms 1, 2, and 3, respectively.

4.1. Pricing spread options under GBM

Let $Y_i = \log S_i(T), i = 1, 2$. We assume $(Y_1, Y_2) \sim BVN((\mu_1, \mu_2), (\rho, \sigma_1, \sigma_2))$. We use (8) to write

$$c(A) = e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ a_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2, \tag{9}$$

where a_{Y_1, Y_2} follows the bivariate normal density. We now follow Pearson's (1995) approach to reduce the double integration to a single integration. Our derivation differs from that used by Pearson but after some algebraic manipulations it would produce the identical result.

For simplicity, we state the double integral in (9) as

$$\begin{aligned} \text{Int}A &\equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ a_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2 \\ &= \int_{-\infty}^{\infty} G_{Y_1}(y_1) a_{Y_1}(y_1) dy_1, \end{aligned}$$

where we define

$$G_{Y_1}(y_1) = \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ a_{Y_2|Y_1}(y_2|y_1) dy_2.$$

The marginal density for Y_1 is $N(\mu_1, \sigma_1)$ and the conditional density is given by

$$a_{Y_2|Y_1}(y_2|y_1) = \frac{1}{\sqrt{2\pi\hat{\sigma}_2^2}} \exp\left\{ -\frac{1}{2\hat{\sigma}_2^2} [y_2 - M_2(y_1)]^2 \right\}, \tag{10}$$

where

$$\widehat{\sigma}_2^2 \equiv \sigma_2^2(1 - \rho^2), \quad \sigma_2^*(y_1) = \frac{\sigma_2 \rho (y_1 - \mu_1)}{\sigma_1}, \quad M_2(y_1) \equiv \mu_2 + \sigma_2^*(y_1).$$

The following theorem gives a closed form expression of G_{Y_1} and hence eliminates the need to perform a numerical integration.

Theorem 2. *We have*

$$G_{Y_1}(y_1) = e^{A_2(y_1)} N(x_1(y_1)) - (e^{y_1} + K) N(x_2(y_1)), \tag{11}$$

where

$$\begin{aligned} A_2(y_1) &= M_2(y_1) + \frac{\widehat{\sigma}_2^2}{2}, \\ x_1(y_1) &= x_2(y_1) + \widehat{\sigma}_2, \\ x_2(y_1) &= \frac{M_2(y_1) - \ln(e^{y_1} + K)}{\widehat{\sigma}_2}, \end{aligned}$$

where $N(\cdot)$ is the distribution function of $N(0, 1)$.

Proof. See Appendix C. ◻

We see that Term 1 for approximating the time-0 call price of the spread option $c(F)$, which appears on the right side of (7), can be found from

$$c(A) = e^{-rT} \int_{-\infty}^{\infty} G_{Y_1}(y_1) a_{Y_1}(y_1) dy_1.$$

As a matter of fact, Term 1 can also be computed in closed form using the formulas derived in Li (2008) and Li *et al.* (2008).

4.2. Adjustment for skewness

To adjust for the discrepancy caused by the difference in skewness between the uses of actual and approximating distributions in evaluating a call price, we include the following (terms relating to E_{rs}, where $r+s=3$)

$$\begin{aligned}
&e^{-rT} \sum_{r+s=3} E_{rs} \frac{(-1)^{r+s}}{r!s!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ \frac{d^{(r+s)}}{dy_1^r dy_2^s} a_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2 \\
&\quad = -e^{-rT} \left[\frac{\kappa_{03}(F)}{3!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ \frac{d^3}{dy_2^3} a_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2 \right. \\
&\qquad + \frac{\kappa_{12}(F)}{2!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ \frac{d^3}{dy_1 dy_2^2} a_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2 \\
&\qquad + \frac{\kappa_{21}(F)}{2!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ \frac{d^3}{dy_1^2 dy_2} a_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2 \\
&\qquad \left. + \frac{\kappa_{30}(F)}{3!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ \frac{d^3}{dy_1^3} a_{Y_1,Y_2}(y_1,y_2) dy_1 dy_2 \right].
\end{aligned}$$

After multiple differentiations of the approximating bivariate normal density and performing the conditioning similar to those shown earlier, Term 2 can be found by single integration using

$$\begin{aligned}
&-e^{-rT} \left[q_1 \int_{-\infty}^{\infty} (y_1-\mu_1) f_{Y_1}(y_1) G_{Y_1}(y_1) dy_1 + q_2 \int_{-\infty}^{\infty} (y_2-\mu_2) f_{Y_2}(y_2) G_{Y_2}(y_2) dy_2 \right. \\
&\quad + q_3 \int_{-\infty}^{\infty} (y_1-\mu_1)^3 f_{Y_1}(y_1) G_{Y_1}(y_1) dy_1 + q_4 \int_{-\infty}^{\infty} (y_2-\mu_2)^3 f_{Y_2}(y_2) G_{Y_2}(y_2) dy_2 \\
&\quad + q_5 \int_{-\infty}^{\infty} (y_1-\mu_1)^2 f_{Y_1}(y_1) H_{Y_1}(y_1) dy_1 + q_6 \int_{-\infty}^{\infty} (y_2-\mu_2)^2 f_{Y_2}(y_2) H_{Y_2}(y_2) dy_2,
\end{aligned}$$

where $G_{Y_1}(y_1)$ is given by (11) and $G_{Y_2}(y_2)$ is similarly defined, and $q_i, i = 1,\ldots,6,$ and $H_{Y_i}, i = 1, 2$ are given in Appendix D.

4.3. Adjustment for kurtosis

To adjust for the discrepancy caused by the difference in kurtosis between the uses of actual and approximating distributions in evaluating a call price, we add the following (terms relating to E_{rs}, where $r+s=4$)

$$\begin{aligned}
e^{-rT}\sum_{r+s=4} E_{rs}\frac{(-1)^{r+s}}{r!s!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{y_2}-e^{y_1}-K)^{+}\frac{d^{(r+s)}}{dy_1^r dy_2^s}a_{Y_1,Y_2}(y_1,y_2)dy_1dy_2 \\
= e^{-rT}\Bigg[\frac{\kappa_{04}(F)}{4!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{y_2}-e^{y_1}-K)^{+}\frac{d^4}{dy_2^4}a_{Y_1,Y_2}(y_1,y_2)dy_1dy_2 \\
+\frac{\kappa_{13}(F)}{3!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{y_2}-e^{y_1}-K)^{+}\frac{d^4}{dy_1dy_2^3}a_{Y_1,Y_2}(y_1,y_2)dy_1dy_2 \\
+\frac{\kappa_{22}(F)}{2!2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{y_2}-e^{y_1}-K)^{+}\frac{d^4}{dy_1^2dy_2^2}a_{Y_1,Y_2}(y_1,y_2)dy_1dy_2 \\
+\frac{\kappa_{31}(F)}{3!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{y_2}-e^{y_1}-K)^{+}\frac{d^4}{dy_1^3dy_2}a_{Y_1,Y_2}(y_1,y_2)dy_1dy_2 \\
+\frac{\kappa_{40}(F)}{4!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^{y_2}-e^{y_1}-K)^{+}\frac{d^4}{dy_1^4}a_{Y_1,Y_2}(y_1,y_2)dy_1dy_2\Bigg].
\end{aligned}$$

After multiple differentiations of the approximating bivariate normal density and performing the conditioning similar to those shown earlier, Term 3 can be found by single integration using

$$\begin{aligned}
e^{-rT}\Bigg[u_1\int_{-\infty}^{\infty}G_1(y_1)a_{Y_1}(y_1)dy_1+u_2\int_{-\infty}^{\infty}(y_1-\mu_1)^2G_1(y_1)a_{Y_1}(y_1)dy_1 \\
+u_3\int_{-\infty}^{\infty}(y_1-\mu_1)H_{Y_1}(y_1)a_{Y_1}(y_1)dy_1+u_4\int_{-\infty}^{\infty}(y_2-\mu_2)^2G_2(y_2)a_{Y_2}(y_2)dy_2 \\
+u_5\int_{-\infty}^{\infty}(y_1-\mu_1)^2J_{Y_2}(y_1)a_{Y_1}(y_1)dy_1+u_6\int_{-\infty}^{\infty}(y_1-\mu_1)^3H_{Y_1}(y_1)a_{Y_1}(y_1)dy_1 \\
+u_7\int_{-\infty}^{\infty}(y_2-\mu_2)^3H_{Y_2}(y_2)a_{Y_2}(y_2)dy_2+u_8\int_{-\infty}^{\infty}(y_2-\mu_2)^4G_2(y_2)a_{Y_2}(y_2)dy_2 \\
+u_9\int_{-\infty}^{\infty}(y_1-\mu_1)^4G_1(y_1)a_{Y_1}(y_1)dy_1\Bigg],
\end{aligned}$$

where, as before, G_{Y_i} and $H_{Y_i}, i=1,2$ are defined earlier, and $u_i, i=1,\ldots,9$, and $J_{Y_2}(y_1)$ are given in Appendix E.

5. Spread Options Under Merton's Jump Diffusion

Consider a spread option involving two underlying assets whose price processes follow Merton's jump diffusion paradigm (Merton, 1990). Under risk neutral measure, the two price processes are governed by the following two stochastic differential equations

$$dS_i(t) = (r - q_i - \lambda_i E[e^{J_i} - 1])S_i(t)dt + \sigma_i S_i(t)dW_i(t) + (e^{J_i} - 1)S_i(t)dN_i(t), \tag{12}$$

where the two jump processes $N_i \sim \text{Poisson}(\lambda_i), i = 1$ and 2, the two jump size distributions $J_i \sim N(m_i, s_i^2), i = 1$ and 2, and q_1 and q_2 are the instantaneous dividend yields, and r is the riskfree rate. For each asset, the respective price process specified by (12) is exactly that considered by Merton (1976) (also, see Jarrow and Rudd (1982), p. 359). We assume (i) N_1 and N_2 are independent, (ii) J_1 and J_2 are independent, (iii) W_1 and W_2 are correlated with $dW_1 dW_2 = \rho dt$, and (iv) $\{N_j\}, \{J_j\}$, and $\{W_j\}$ are mutually independent.

Since $Y_i(t) = \log S_i(t)$, we apply Itô formula for the jump diffusion process and obtain

$$dY_i(t) = \left(r - q_i - \frac{1}{2}\sigma_i^2 - \lambda_i E[e^{J_i} - 1]\right)dt + \sigma_i W_i(T) + J_i(t)dN_i(t), \quad i = 1, 2.$$

Integrating the above, we obtain

$$\begin{aligned} Y_i(T) &= Y_i(0) + \left(r - q_i - \frac{1}{2}\sigma_i^2 - \lambda_i E[e^{J_i} - 1]\right)T + \sigma_i W_i(T) + \sum_{k=1}^{N(T)} J_i(k) \\ &= Y_i(0) + \left(r - q_i - \frac{1}{2}\sigma_i^2 - \lambda_i\left[e^{m_i + \frac{s_i^2}{2}}\right]\right)T + \sigma_i W_i(T) + \sum_{k=1}^{N(T)} J_i(k), \end{aligned} \tag{13}$$

where the last equality is due to $J_i \sim N(m_i, s_i^2)$ and an application of the m.g.f of a normal density. For notational convenience, we write the above as

$$Y_i(T) = \Lambda_i(T) + L_i,$$

where

$$\Lambda_i(T) := \log S_i(0) + \left(r - q_i - \frac{1}{2}\sigma_i^2 - \lambda_i\left[e^{m_i + \frac{s_i^2}{2}}\right]\right)T + \sigma_i W_i(T)$$

and

$$L_i \equiv \sum_{k=1}^{N_i(T)} J_i(k).$$

To simplify the above further, we write

$$Y_i(T) = \mu_i + \Omega_i Z_i + L_i,$$

where

$$\mu_i := \log S_i(0) + \left(r - q_i - \frac{1}{2}\sigma_i^2 - \lambda_i\left[e^{m_i+\frac{s_i^2}{2}}\right]\right)T,$$
$$\Omega_i := \sigma_i\sqrt{T}$$

and $Z \sim N(0,1)$.

To apply the Edgeworth expansion for pricing the spread option, we assume that the jump diffusion processes (13) generates the true distribution F_{Y_1,Y_2} at maturity. Since the valuation of the spread option is done by (5), what remain to be determined are the product cumulants $\kappa_{rs}(F)$, where $r+s=3$ and 4.

5.1. Moments of (L_1, L_2) under F

As L_i is a compound Poisson process, its m.g.f. is given by $E[e^{tL_i}] = \exp\{\lambda_i T(M_{J_i}(t)-1)\}$ (e.g., see Shreve (2000), p. 471), where $M_{J_i}(t)$ is the m.g.f. of jump size J_i. Since $J_i \sim N(m_i, s_i^2)$, we have

$$M_{J_i}(t) = \exp\left\{m_i t + \frac{s_i^2 t}{2}\right\}.$$

This gives the m.g.f. of L_i

$$E[e^{tL_i}] = \exp\left\{\lambda_i T\left(\exp\left(m_i t + \frac{s_i^2 t}{2}\right) - 1\right)\right\}.$$

Expanding the above yields the first four moments of L_i

$$\begin{aligned}
E[L_i] &= m_i\lambda_i T,\\
E[L_i^2] &= (m_i^2 + s_i^2)\lambda_i T + m_i^2\lambda_i^2 T^2,\\
E[L_i^3] &= (m_i^3 + 3m_i s_i^2)\lambda_i T + (3m_i^3 + 3m_i s_i^2)\lambda_i^2 T^2 + m_i^3\lambda_i^3 T^3,\\
E[L_i^4] &= (m_i^4 + 6m_i^2 s_i^2 + 3s_i^4)\lambda_i T + (7m_i^4 + 18m_i^2 s_i^2 + 3s_i^2)\lambda_i^2 T^2\\
&\quad + (6m_i^4 + 6m_i^2 s_i^2)\lambda_i^3 T^3 + m_i^4\lambda_i^4 T^4.
\end{aligned} \tag{14}$$

5.2. Product moments of (Y_1, Y_2) under F

The first two moments of Y_i are

$$E[Y_i] \equiv M_i = \mu_i + \lambda_i m_i T, \quad \mathrm{Var}[Y_i^2] \equiv \Pi_i^2 = \Omega_i^2 + (m_i^2 + s_i^2)\lambda_i T$$

and hence $Y_i - E[Y_i] = \Omega_i Z_i + L_i - \lambda_i m_i T$. The (r,s)th product moment of (Y_1, Y_2) is defined by

$$\mu_{rs}(F) = E[(\Omega_1 Z_1 + (L_1 - \lambda_1 m_1 T))^r(\Omega_2 Z_2 + (L_2 - \lambda_2 m_2 T))^s],$$

where r and s are nonnegative integers. Recall that for a standard bivariate normal distribution, for $0 < r+s \leq 4$, we have

$$\begin{aligned} &E[Z_1^2 Z_2^0] = 1, &&E[Z_1^4 Z_2^0] = 3,\\ &E[Z_1 Z_2] = \rho, &&E[Z_1^2 Z_2^2] = 1+2\rho^2,\\ &E[Z_1 Z_2^3] = 3\rho, \end{aligned} \tag{15}$$

$E[Z_1^0 Z_2^2], E[Z_1^0 Z_2^3]$, and $E[Z_1^3 Z_2^1]$ can be stated by symmetry, and 0 otherwise. Using (14) and (15), and by invoking the independence assumptions made at the outset of the section, we find the product moments for $3 \leq r+s \leq 4$ below:

$$\begin{aligned} \mu_{21}(F) &= 0, \quad \mu_{30}(F) = (m_1^3 + 3m_1 s_1^2)\lambda_1 T,\\ \mu_{40}(F) &= 3\Omega_1^4 + (m_1^4 + 6m_1^2 s_1^2 + 3s_1^4 + 6\Omega_1^2(m_1^2 + s_1^2))\lambda_1 T + 3(m_1^2+s_1^2)^2\lambda_1^2 T^2,\\ \mu_{22}(F) &= \Omega_1^2\Omega_2^2(1+2\rho^2) + \Omega_2^2(m_1^2+s_1^2)\lambda_1 T + \Omega_1^2(m_2^2+s_2^2)\lambda_2 T\\ &\quad + (m_1^2+s_1^2)(m_2^2+s_2^2)\lambda_1\lambda_2 T^2,\\ \mu_{31}(F) &= 3\rho\Omega_1^3\Omega_2 + 3\rho\Omega_1\Omega_2(s_1^2+m_1^2)\lambda_1 T, \end{aligned}$$

where $\mu_{12}(F), \mu_{03}(F), \mu_{04}(F)$, and $\mu_{31}(F)$ can be found by symmetry.

5.3. Product cumulants of (Y_1, Y_2) under F

Using the results shown in Sec. 2, we find the product cumulants of (Y_1, Y_2) under F below:

$$\begin{aligned} \kappa_{21}(F) &= \mu_{21}(F) = 0,\\ \kappa_{30}(F) &= \mu_{30}(F) = (m_1^3 + 3m_1 s_1^2)\lambda_1 T,\\ \kappa_{31}(F) &= \mu_{31}(F) - 3\mu_{20}(F)\mu_{11}(F) = 0,\\ \kappa_{22}(F) &= \mu_{22}(F) - \mu_{20}(F)\mu_{02}(F) - 2\mu_{11}^2(F) = 0,\\ \kappa_{40}(F) &= \mu_{40}(F) - 3\mu_{20}^2(F) = (m_1^4 + 6m_1^2 s_1^2 + 3s_1^4)\lambda_1 T, \end{aligned}$$

where $\kappa_{03}(F), \kappa_{13}(F)$, and $\kappa_{04}(F)$ can be found by symmetry.

5.4. A numerical example

Consider a numerical example with the following parameters:

	$i=1$	$i=2$
$S_i(0)$	96	100
q_i	0.05	0.05
σ_i	0.1	0.2
λ_i	0.25	0.5

(*Continued*)

	$i=1$	$i=2$
m_i	-0.13	0.11
s_i	0.37	0.41

$r = 0.1, T = 1, \rho = 0.5$, and $K = 2$.

For the above example, we use Matlab to compute the call price of the spread option. The results are summarized below:

	Values
Term 1 (the GBM solution)	15.3437
Term 2 (adjustment for skewness)	0.7450
Term 3 (adjustment for kurtosis)	-1.0463
Approximated call value	15.0424

In a recent paper, Jackson *et al.* (2008/09) proposed a Fourier transform-based algorithm for solving option pricing problems under the Lévy paradigm. Using their approach to solve the above problem, we find the call price is 15.0292. The Edgeworth-based approach presented in this paper enables us to assess the relatively contributions made by the skewness and kutosis of the underlying price processes. Finally, we remark that a Monte Carlo simulation using 200,000 samples produced a solution of 15.071.

5.5. Computations based on simulated prices at maturity

For applications, one may know the underlying stochastic differentiation equations governing the price processes and/or their parameters. If the first four moments of the price process can be assessed empirically, then the approach proposed in the paper can be useful in coming up with approximating prices. To explore this point further, we use the parameters specified in the last subsection to generate random sample of $S_1(T)$ and $S_2(T)$ of size 100,000. Based on this data set, we computed the sample estimates of the first four moments as the input to the Edgeworth-based method. With a model specification of $(K, r, T) = (2, 0.1, 1)$, we find the calculated values as follows

	Values
Term 1 (the GBM solution)	15.2760
Term 2 (adjustment for skewness)	0.6890
Term 3 (adjustment for kurtosis)	-1.0087

(*Continued*)

	Values
Approximated call value	14.9563

As expected, the results differ only slightly from those reported in the last subsection. It is important to note that in this case the model does not "know" that it is Merton's jump diffusion that generated the prices at maturity. It only uses the price data at maturity.

If traders can speculate the magnitudes of the first four moments of the price processes, we see the approach provides a nice way to the produce the prices and their components. For quantitative analysts proposing models that are only solvable by Monte Carlo simulations or intricate approximating algorithms, they can obtain empirically estimated first four moments and use the Edgeworth approach to cross-validate their results. This is a very nice feature of the Edgeworth-based approach.

6. Spread Options Under Kou's Double Jump Diffusion

Spread options are commonly used in energy and power industries to hedge fluctuations of prices of two correlated commodities, e.g., prices of electricity and coal. It is well known that price volatilities in power and energy markets are substantially higher than that observed in the equity and fixed-income markets. Moreover, there the phenomena known as asymmetric leptokurtic feature and volatility smile are more pronounced. Kou (2002) proposed the use of a double exponential jump diffusion model to mitigate these problems. Under the paradigm, we are dealing with a version of Lévy processes (e.g., see p. 112 of Cont and Tankov (2004)). In this section, we will use this paradigm to model price movements of the two underlying assets of a spread option.

6.1. The Kou's double jump diffusion

For Asset $i, i = 1, 2$, the SDEs governing the price processes are identical to that of (12). Under Kou's assumption, J_i follows an asymmetric double exponential distribution, i.e., the density of J_i is given by

$$f_{J_i}(y) = p_i \cdot \eta_{i1} e^{-\eta_{i1} y} \mathbf{1}_{\{y \leq 0\}} + q_i \cdot \eta_{i2} e^{-\eta_{i2} y} \mathbf{1}_{\{y < 0\}}, \quad i = 1, 2,$$

where $\eta_{i1} > 1 \eta_{i2} > 0, p_i \geq 0, q_i \geq 0,$ and $p_i + q_i = 1,$ for $i = 1, 2.$ In this construct, we have

$$J_i \stackrel{d}{=} \begin{cases} \xi_i^+ & \text{with probability } p_i, \\ -\xi_i^- & \text{with probability } q_i, \end{cases}$$

where $\xi_i^+ \sim \exp(\eta_{i1})$ and $\xi^- \sim \exp(\eta_{i2})$, with respective means $1/\eta_{i1}$ and $1/\eta_{i2}$. We see

$$E[J_i] = \frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}}, \quad i = 1, 2$$

and for $i = 1, 2$, we have

$$E[e^{J_i}] = \frac{p_i \eta_{i1}}{\eta_{i1} - 1} + \frac{q_i \eta_{i2}}{\eta_{i2} + 1}, \quad \eta_{i1} > 1, \quad \eta_{i2} > 0.$$

Using (22) and applying Itô formula, the risk-neutral log price (i.e., the return) of Asset i at time T can be found. It is given by

$$\begin{aligned} Y_i(T) &= Y_i(0) + \left[r - q_i - \frac{1}{2}\sigma_i^2 - \lambda_i \left(\frac{p_i \eta_{i1}}{\eta_{i1} - 1} + \frac{q_i \eta_{i2}}{\eta_{i2} + 1} - 1 \right) \right] T \\ &\quad + \sigma_i W_i(T) + \sum_{k=1}^{N_i(T)} J_i(k) \end{aligned}$$

(e.g., see Kou (2002)). For notational simplicity, for Asset i, we define

$$Y_i(T) := \Lambda_i(T) + L_i := \mu_i + \Omega_i Z_i + L_i,$$

where

$$\begin{aligned} \mu_i &:= \log S_i(0) + \left[r - q_i - \frac{1}{2}\sigma_i^2 - \lambda_i \left(\frac{p_i \eta_{i1}}{\eta_{i1} - 1} + \frac{q_i \eta_{i2}}{\eta_{i2} + 1} - 1 \right) \right] T, \\ \Omega_i &:= \sigma_i \sqrt{T}, \\ \Lambda_i(T) &:= \mu_i + \Omega_i Z_i, \\ L_i &:= \sum_{k=1}^{N_i(T)} J_i(k) \end{aligned}$$

and $Z_i \sim N(0, 1)$.

6.2. Moments of (Y_1, Y_2) under F

In order to derive product cumulants and the correction factors under Kou's model, we first need to find the m.g.f. of jump size. For Asset i, we see the m.g.f. can be derived as follows:

$$\begin{aligned}M_{J_i}(t) &= E[e^{tJ_i}] = \int_{-\infty}^{\infty} e^{tx}(p_i\eta_{i1}e^{-\eta_{i1}x}\mathbf{1}_{\{x\geq 0\}} + e^{tx}q_i\eta_{i2}e^{\eta_{i2}x}\mathbf{1}_{\{x<0\}})dx \\ &= \int_{-\infty}^{0} e^{tx}q_i\eta_{i2}e^{\eta_{i2}x}dx + \int_{0}^{\infty} e^{tx}p_i\eta_{i1}e^{-\eta_{i1}x}dx \\ &= q_i\eta_{i2}\int_{-\infty}^{0} e^{(t+\eta_{i2})x}dx + p_i\eta_{i1}\int_{0}^{\infty} e^{-(\eta_{i1}-t)x}dx \\ &= \frac{q_i\eta_{i2}}{t+\eta_{i2}} - \frac{p_i\eta_{i1}}{t-\eta_{i1}} \quad t+\eta_{i2}\geq 0 \quad \text{and} \quad t-\eta_{i2}\leq 0\end{aligned}$$

or $-\eta_{i2} \leq t \leq \eta_{i1}$. Using series expansion on t, the above m.g.f. can be stated as

$$M_{J_i}(t) = 1 + \sum_{j=1}^{\infty}\left(\frac{p_i}{\eta_{i1}^j} + (-1)^j\frac{q_i}{\eta_{i2}^j}\right)t^j.$$

By definition of the m.g.f., we have

$$E[e^{tL_i}] = 1 + tm_{i1} + \frac{t^2}{2!}m_{i2} + \frac{t^3}{3!}m_{i3} + \frac{t^4}{4!}m_{i4} + \cdots,$$

where m_{ij} is the jth moment of L_i. Since L_i is a compound Poisson process, we apply (11.3.2) of Shreve (2000, p. 471) and write the m.g.f. of L_i as

$$E[e^{tL_i}] = \exp\{\lambda_i T(M_{J_i}(t) - 1)\} \tag{16}$$

$$= \exp\left\{\lambda_i T\sum_{j=1}^{\infty}\left(\frac{p_i}{\eta_{i1}^j} + (-1)^j\frac{q_i}{\eta_{i2}^j}\right)t^j\right\}. \tag{17}$$

We expand (17) as a polynomial in t and match the respective coefficients with those in (16). This yields the first four moments of L_i. They are

$$\begin{aligned}m_{i1} &= \lambda_i T\left(\frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}}\right),\\ m_{i2} &= 2\lambda_i T\left(\frac{p_i}{\eta_{i1}^2} + \frac{q_i}{\eta_{i2}^2}\right) + (\lambda_i T)^2\left(\frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}}\right)^2,\\ m_{i3} &= 6\lambda_i T\left(\frac{p_i}{\eta_{i1}^3} - \frac{q_i}{\eta_{i2}^3}\right) + 6(\lambda_i T)^2\left(\frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}}\right)\left(\frac{p_i}{\eta_{i1}^2} + \frac{q_i}{\eta_{i2}^2}\right) + (\lambda_i T)^3\left(\frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}}\right)^3,\\ m_{i4} &= 24\lambda_i T\left(\frac{p_i}{\eta_{i1}^4} + \frac{q_i}{\eta_{i2}^4}\right) + 12(\lambda_i T)^2\left[2\left(\frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}}\right)\left(\frac{p_i}{\eta_{i1}^3} - \frac{q_i}{\eta_{i2}^3}\right) + \left(\frac{p_i}{\eta_{i1}^2} + \frac{q_i}{\eta_{i2}^2}\right)^2\right]\\ &\quad + 12(\lambda_i T)^3\left(\frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}}\right)^2\left(\frac{p_i}{\eta_{i1}^2} + \frac{q_i}{\eta_{i2}^2}\right) + (\lambda_i T)^4\left(\frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}}\right)^4.\end{aligned}$$

We are now ready to derive product moments of (Y_1, Y_2) under F. Recall

$Y_i(T) = \mu_i + \Omega_i Z_i + L_i$. We take expectation of the above and find

$$E_i[Y_i(T)] = \mu_i + \Omega_i E[Z_i] + E[L_i] = \mu_i + \lambda_i T\left(\frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}}\right).$$

Now

$$Y_i(T) - E[Y_i(T)] = (\mu_i + \Omega_i Z_i + L_i) - (\mu_i + E[L_i]) = \Omega_i Z_i + L_i - E[L_i]$$
$$= \Omega_i Z_i + L_i - m_{i1}$$

and therefore

$$\begin{aligned}\mathrm{Var}[Y_i(T)] &= E[(\Omega_i Z_i + L_i - m_{i1})^2] \\ &= \Omega_i^2 + E[L_i^2] - m_{i1}^2 \\ &= \Omega_i^2 + m_{i2} - m_{i1}^2 \\ &= \Omega_i^2 + 2\lambda_i T\left(\frac{p_i}{\eta_{i1}^2} + \frac{q_i}{\eta_{i2}^2}\right).\end{aligned}$$

Therefore the product moment $\mu_{11}(F)$ of (Y_1, Y_2) under F is given by

$$\mu_{11}(F) = E\left[\prod_{i=1}^{2}(\Omega_i Z_i + L_i - m_{i1})\right] = \rho\Omega_1\Omega_2.$$

The higher order product moments are

$$\begin{aligned}\mu_{21}(F) &= 0, \\ \mu_{30}(F) &= 6\lambda_1 T\left(\frac{p_1}{\eta_{11}^3} - \frac{q_1}{\eta_{12}^3}\right), \\ \mu_{40}(F) &= 3\Omega_1^4 + 12\lambda_1 T\left[2\left(\frac{p_1}{\eta_{11}^4} + \frac{q_1}{\eta_{12}^4}\right) + \Omega_1^2\left(\frac{p_1}{\eta_{11}^4} - \frac{q_1}{\eta_{12}^4}\right)\right] \\ &\quad + 12(\lambda_1 T)^2\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right)^2 - (\lambda_1 T)^4\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^4, \\ \mu_{22}(F) &= (1+2\rho^2)(\Omega_1^2\Omega_2^2) + \Omega_1^2 \cdot (2\lambda_2 T)\left(\frac{p_2}{\eta_{21}^2} + \frac{q_2}{\eta_{22}^2}\right) \\ &\quad + \Omega_2^2 \cdot (2\lambda_1 T)\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right) + 4\lambda_1\lambda_2 T^2\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right)\left(\frac{p_2}{\eta_{21}^2} + \frac{q_2}{\eta_{22}^2}\right), \\ \mu_{31}(F) &= 3\rho\Omega_1^3\Omega_2 + 3\rho\Omega_1\Omega_2 \cdot (2\lambda_1 T)\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right).\end{aligned}$$

In Appendix F, we give their derivations. The product cumulants $\kappa_{rs}, r+s=3$ and 4, under the true distribution F are found accordingly. They are

$$\begin{aligned}
\kappa_{11}(F) &= \mu_{11}(F) = \rho\Omega_1\Omega_2,\\
\kappa_{21}(F) &= \mu_{21}(F) = 0,\\
\kappa_{30}(F) &= \mu_{30}(F) = 6\lambda_1 T\left(\frac{p_1}{\eta_{11}^3} - \frac{q_1}{\eta_{12}^3}\right),\\
\kappa_{31}(F) &= \mu_{31}(F) - 3\mu_{20}(F)\mu_{11}(F),\\
&= 3\rho\Omega_1^3\Omega_2 + 3\rho\Omega_1\Omega_2(2\lambda_1 T)\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right)\\
&\quad - 3\rho\Omega_1\Omega_2\left[\Omega_1^2 + 2\lambda_1 T\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right)\right] = 0,\\
\kappa_{22}(F) &= \mu_{22}(F) - \mu_{20}(F)\mu_{02}(F) - 2\mu_{11}^2(F) = 0,\\
\kappa_{40}(F) &= \mu_{40}(F) - 3\mu_{20}^2(F) = 24\lambda_1 T\left(\frac{p_1}{\eta_{11}^4} + \frac{q_1}{\eta_{12}^4}\right) - (\lambda_1 T)^4\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^4
\end{aligned}$$

and finally we obtain the nonzero correction factors

$$\begin{aligned}
E_{30} &= \kappa_{30}(F) = 6\lambda_1 T\left(\frac{p_1}{\eta_{11}^3} - \frac{q_1}{\eta_{12}^3}\right),\\
E_{40} &= \kappa_{40}(F) = 24\lambda_1 T\left(\frac{p_1}{\eta_{11}^4} + \frac{q_1}{\eta_{12}^4}\right) - (\lambda_1 T)^4\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^4.
\end{aligned}$$

These results form the input for application of the Edgeworth expansion.

6.3. A numerical example

Consider a numerical example with the following model specifications

	$i=1$	$i=2$
$S_i(0)$	100	97
q_i	0.02	0.02
σ_i	0.16	0.10
λ_i	1.0	1.5
p_i	0.4	0.3
η_{i1}	10	8
η_{i2}	5	6

In addition, we assume $r = 0.05, \rho = 0.05, T = 4$, and $K = 4$.

For the above example, we use Matlab to compute the call price of the spread option. The results are summarized below:

	Values
Term 1 (the GBM solution)	7.4814
Term 2 (adjustment for skewness)	−0.9095
Term 3 (adjustment for kurtosis)	−1.3511
Approximated call value	5.2207

The above tables show that the effects of skewness and kurtosis are relatively large. If we had used the two-dimensional GBM to price the call, the call price would have been 7.4814. The two higher moments bring it down to 5.2207. Here, we demonstrated that the Edgeworth-based approach presented in this paper enables us to assess the relatively contributions made by the skewness and kurtosis of the underlying return processes. We mention in passing, the computation took 1.19 s. A Monte Carlo simulation using 300,000 samples produced a solution of 5.6453. This simulation took 222.49 s.

6.4. Computations based on empirically observed prices at maturity

In applications, one may make assumptions about the underlying stochastic differentiation equations governing the price processes and/or their parameters. If the first four moments of the price process can be assessed empirically, then the approach proposed in the paper can be useful in coming up with approximating prices. To explore this point further, we use the parameters specified in the last subsection to generate random sample of $S_1(T)$ and $S_2(T)$ of size 300,000. Based on this data set, we computed the sample estimates of the first four moments as the input to the Edgeworth-based method. With a model specification of $(K, r, T) = (4, 0.05, 0.5)$, we find the calculated values as follows

	Values
Term 1 (the GBM solution)	7.4834
Term 2 (adjustment for skewness)	−0.9402
Term 3 (adjustment for kurtosis)	−1.3509
Approximated call value	5.1923

As expected, the results differ only slightly from those reported earlier. It is important to note that in this case the model does not "know" that it is Kou's double exponential jump diffusion model that generated the prices at maturity. It only uses the price data at maturity.

If traders can speculate the magnitudes of the first four moments of the price processes, we see the approach provides a nice way to produce the prices and their components. For quantitative analysts proposing models that are only solvable by Monte Carlo simulations or intricate approximating algorithms, they can obtained empirically estimated first four moments and use the Edgeworth approach to cross-validate their results. This is a very nice feature of the Edgeworth-based approach.

7. Conclusion

This paper proposes a bivariate generalized Edgeworth expansion for pricing spread option. This paper is a generalization of Jarrow and Rudd (1992) who introduced the idea of using univariate Edgeworth expansion for pricing call options when the underling price process deviates from the lognormal distribution. As in Jarrow and Rudd, we see that when the first four product cumulants are known, the expansion yields adequate approximations for problems that do not have closed-form solutions. In theory, the higher orders of differentiation and integration can be problematic for numerical implementation. The paper demonstrates that after proper manipulations, the computation simplifies to integrations involving single integrals. Hence it significantly reduces the cost of computation and potentials for computation errors. More importantly, the option price obtained by Edgeworth expansion provides a valuable insight about its constituents — namely, the relative contributions made by the various moments of the asset prices (e.g., correlation, skewness, and kurtosis). This advantage is not shared by methods such as numerical solutions of PDE or Monte Carlo simulation. For financial analysts with real-world applications in mind, using market data, one can estimate empirical product cumulants (e.g., see Stuart and Ord (1994)) and proceed with the aforementioned approximation. In summary, the approach described in this paper would provide a viable alternative to the solution of the pricing problem.

Appendix A. Proof of Theorem 1

$$\Phi(F, t_1, t_2) = \mathbb{M}(F, it_1, it_2),$$

we use (2) to write

$$\ln \Phi(F, t_1, t_2) = \sum_{r+s \geq 0} \kappa_{r,s}(F) \frac{(it_1)^r}{r!} \frac{(it_2)^s}{s!} \tag{A.1}$$

and

$$\ln \Phi(A, t_1, t_2) = \sum_{r+s\geq 0} \kappa_{r,s}(A) \frac{(it_1)^r}{r!} \frac{(it_2)^s}{s!}. \qquad \text{(A.2)}$$

Subtracting (A.2) from (A.1), we find

$$\ln \Phi(F, t_1, t_2) = \ln \Phi(A, t_1, t_2) + \sum_{r+s\geq 0} (\kappa_{r,s}(F) - \kappa_{r,s}(A)) \frac{(it_1)^r}{r!} \frac{(it_2)^s}{s!}.$$

Hence, we have

$$\Phi(F, t_1, t_2) = \exp\left(\sum_{r+s\geq 0} (\kappa_{r,s}(F) - \kappa_{r,s}(A)) \frac{(it_1)^r}{r!} \frac{(it_2)^s}{s!} \right) \Phi(A, t_1, t_2). \qquad \text{(A.3)}$$

The bivariate inverse Fourier transforms are defined as

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(F, t_1, t_2) \exp(-it_1 y_1 - it_2 y_2) dt_1 dt_2 \qquad \text{(A.4)}$$

and

$$a_{Y_1,Y_2}(y_1, y_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(A, t_1, t_2) \exp(-it_1 y_1 - it_2 y_2) dt_1 dt_2. \qquad \text{(A.5)}$$

We take derivatives with respect to y_1 and y_2 and obtain

$$\begin{aligned}(-1)^{r+s} & \frac{d^{(r+s)} a(y_1, y_2)}{dy_1^r dy_2^s} \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(A, t_1, t_2) \exp(-it_1 y_1 - it_2 y_2)(it_1)^r (it_2)^s dt_1 dt_2. \qquad \text{(A.6)}\end{aligned}$$

Multiplying (A.3) by $\exp(-it_1 y_1 - it_2 y_2)$, integrating the resulting expression, and applying (A.4)–(A.6) yield

$$f_{Y_1,Y_2}(y_1, y_2) = a_{Y_1,Y_2}(y_1, y_2) + \sum_{r+s>0} E_{rs} \frac{(-1)^{r+s}}{r!s!} \frac{d^{(r+s)} a(y_1, y_2)}{dy_1^r dy_2^s}.$$

The above amounts to the inversions of the Fourier transforms involved. □

Appendix B

The relations between the correction factors and the product cumulants are:

$$\begin{aligned} E_{10} &= \kappa_{10}(F) - \kappa_{10}(A), \\ E_{20} &= [\kappa_{20}(F) - \kappa_{20}(A)] + E_{10}^2, \end{aligned}$$

$$
\begin{aligned}
E_{30} &= [\kappa_{30}(F) - \kappa_{30}(A)] + 3E_{10}[\kappa_{20}(F) - \kappa_{20}(A)] + E_{10}^3, \\
E_{40} &= [\kappa_{40}(F) - \kappa_{40}(A)] + 4E_{10}[\kappa_{30}(F) - \kappa_{30}(A)] + 3[\kappa_{20}(F) - \kappa_{20}(A)]^2 \\
&\quad + 6E_{10}^2[\kappa_{20}(F) - \kappa_{20}(A)] + E_{10}^4, \\
E_{11} &= \kappa_{11}(F) - \kappa_{11}(A) + E_{10}E_{01}, \\
E_{21} &= [\kappa_{21}(F) - \kappa_{21}(A)] + E_{01}[\kappa_{20}(F) - \kappa_{20}(A)] + 2E_{10}[\kappa_{11}(F) - \kappa_{11}(A)] \\
&\quad + E_{01}E_{10}^2,
\end{aligned}
$$

$$
\begin{aligned}
E_{31} &= [\kappa_{31}(F) - \kappa_{31}(A)] + E_{01}[\kappa_{30}(F) - \kappa_{30}(A)] + 3[\kappa_{11}(F) - \kappa_{11}(A)]E_{10}^2 \\
&\quad + 3[\kappa_{11}(F) - \kappa_{11}(A)][\kappa_{20}(F) - \kappa_{20}(A)] + 3E_{10}E_{01}[\kappa_{20}(F) - \kappa_{20}(A)] \\
&\quad + 3E_{10}[\kappa_{21}(F) - \kappa_{21}(A)], \\
E_{22} &= [\kappa_{22}(F) - \kappa_{22}(A)] + 2E_{10}[\kappa_{12}(F) - \kappa_{12}(A)] + 2[\kappa_{11}(F) - \kappa_{11}(A)]^2 \\
&\quad + [\kappa_{20}(F) - \kappa_{20}(A)][\kappa_{02}(F) - \kappa_{02}(A)] + 4E_{10}E_{01}[\kappa_{11}(F) - \kappa_{11}(A)] \\
&\quad + 2[\kappa_{21}(F) - \kappa_{21}(A)]E_{01} + E_{10}^2[\kappa_{02}(F) - \kappa_{02}(A)] \\
&\quad + E_{01}^2[\kappa_{20}(F) - \kappa_{20}(A)] + E_{10}^2E_{01}^2.
\end{aligned}
$$

The relations for $E_{01}, E_{02}, E_{03}, E_{04}, E_{12}$, and E_{13} can be found by symmetry and hence omitted.

Proof of Lemma 1. Truncating the summation term of (3) at $r + s = 4$ yields (4). To derive the rest of the results, we note that the exponential function in (A.3) is analytic and hence expandable. Thus we have

$$
\begin{aligned}
&\exp\left(\prod_{r+s\geq 0}(\kappa_{r,s}(F) - \kappa_{r,s}(A))\frac{(it_1)^r}{r!}\frac{(it_2)^s}{s!}\right) \\
&= \sum_{n=0}^{\infty}\left(\prod_{r+s\geq 0}(\kappa_{r,s}(F) - \kappa_{r,s}(A))\frac{(it_1)^r}{r!}\frac{(it_2)^s}{s!}\right)^n \frac{1}{n!} \\
&= \sum_{r+s\geq 0} E_{rs}\frac{(it_1)^r}{r!}\frac{(it_2)^s}{s!}.
\end{aligned}
$$

By matching the coefficients for the term $t_1^r t_2^s$, we obtain the relations for the case when $0 < r + s \leq 4$. □

Appendix C

Proof of Theorem 2. As $e^{y_2} - e^{y_1} - K \geq 0$ implies $y_2 \geq \ln(e^{y_1} - K)$, it follows that

$$\begin{aligned} G_{Y_1}(y_1) &= \int_{\ln(e^{y_1}-K)}^{\infty} (e^{y_2} - e^{y_1} - K)a(y_2|y_1)dy_2 \\ &= \int_{\ln(e^{y_1}-K)}^{\infty} e^{y_2}a(y_2|y_1)dy_2 - (e^{y_1+K}) \int_{\ln(e^{y_1}-K)}^{\infty} a(y_2|y_1)dy_2. \end{aligned}$$

Using (10), we do a complete-the-square, and a change of variable. This will produce the identity

$$\int_{\ln(e^{y_1}-K)}^{\infty} e^{y_2}a(y_2|y_1)dy_2 = e^{A_2(y_1)}N(x_1(y_{1)}).$$

The identity

$$\int_{\ln(e^{y_1}-K)}^{\infty} a(y_2|y_1)dy_2 = N(x_2(y_1))$$

follows from a similar change-of-variable. $\square$

Appendix D

The following are the results needed for computation of adjustment for skewness shown in Sec. 4.2. The $q_i's$ are given in closed form

$$q_1 = \frac{1}{3!}\left(\frac{3E_{30}}{(1-\rho^2)^2\sigma_1^4} - \frac{3\rho E_{03}}{(1-\rho^2)^2\sigma_2^3\sigma_1}\right) + \frac{1}{2!}\left(\frac{(1+2\rho^2)E_{12}}{(1-\rho^2)^2\sigma_1^2\sigma_2^2} - \frac{3\rho E_{21}}{(1-\rho^2)^2\sigma_1^3\sigma_2}\right),$$

$$q_2 = \frac{1}{3!}\left(\frac{3E_{03}}{(1-\rho^2)^2\sigma_2^4} - \frac{3\rho E_{30}}{(1-\rho^2)^2\sigma_1^3\sigma_2}\right) + \frac{1}{2!}\left(\frac{(1+2\rho^2)E_{21}}{(1-\rho^2)^2\sigma_1^2\sigma_2^2} - \frac{3\rho E_{12}}{(1-\rho^2)^2\sigma_1\sigma_2^3}\right),$$

$$q_3 = \frac{1}{3!}\left(\frac{\rho^3E_{03}}{(1-\rho^2)^3\sigma_1^3\sigma_2^3} - \frac{E_{30}}{(1-\rho^2)^3\sigma_1^6}\right) + \frac{1}{2!}\left(\frac{\rho E_{21}}{(1-\rho^2)^3\sigma_1^5\sigma_2} - \frac{\rho^2E_{12}}{(1-\rho^2)^3\sigma_1^4\sigma_2^2}\right),$$

$$q_4 = \frac{1}{3!}\left(\frac{\rho^3E_{30}}{(1-\rho^2)^3\sigma_1^3\sigma_2^3} - \frac{E_{03}}{(1-\rho^2)^3\sigma_2^6}\right) + \frac{1}{2!}\left(\frac{\rho E_{12}}{(1-\rho^2)^3\sigma_1\sigma_2^5} - \frac{\rho^2E_{21}}{(1-\rho^2)^3\sigma_1^2\sigma_2^4}\right),$$

$$q_5 = \frac{1}{3!}\left(\frac{3\rho E_{30}}{(1-\rho^2)^3\sigma_1^5\sigma_2} - \frac{3\rho^2E_{03}}{(1-\rho^2)^3\sigma_2^4\sigma_1^2}\right) + \frac{1}{2!}\left(\frac{\rho(2+\rho^2)E_{12}}{(1-\rho^2)^3\sigma_1^3\sigma_2^3} - \frac{(1+2\rho^2)E_{21}}{(1-\rho^2)^3\sigma_1^4\sigma_2^2}\right),$$

$$q_6 = \frac{1}{3!}\left(\frac{3\rho E_{03}}{(1-\rho^2)^3\sigma_2^5\sigma_1} - \frac{3\rho^2E_{30}}{(1-\rho^2)^3\sigma_1^4\sigma_2^2}\right) + \frac{1}{2!}\left(\frac{\rho(2+\rho^2)E_{21}}{(1-\rho^2)^3\sigma_1^3\sigma_2^3} - \frac{(1+2\rho^2)E_{12}}{(1-\rho^2)^3\sigma_1^2\sigma_2^4}\right)$$

and $H_{Y_i}(y_i), i = 1, 2$, are defined by

$$H_{Y_1}(y_1) = \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+(y_2 - \mu_2) a_{Y_2|Y_1}(y_2|y_1) dy_2,$$
$$H_{Y_2}(y_2) = \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+(y_1 - \mu_1) a_{Y_1|Y_2}(y_1|y_2) dy_1.$$

The derivations of the above can be found in Xie (2008).

Appendix E

The following are the results needed for computation of adjustment for kurtosis shown in Sec. 4.3. The $u_i's$ are given in closed form

$$u_1 = \frac{1}{(1-\rho^2)^2}\left[\frac{E_{40}}{8\sigma_1^4} + \frac{E_{04}}{8\sigma_2^4} + \frac{E_{22}(1+2\rho^2)}{4\sigma_1^2\sigma_2^2} - \frac{E_{31}\rho}{2\sigma_1^4\sigma_2} - \frac{E_{13}\rho}{2\sigma_1\sigma_2^3}\right],$$
$$u_2 = \frac{1}{(1-\rho^2)^3\sigma_1^2}\left[-\frac{E_{40}}{4\sigma_1^4} - \frac{E_{04}\rho^2}{4\sigma_2^4} - \frac{E_{22}(1+5\rho^2)}{4\sigma_1^2\sigma_2^2} + \frac{E_{31}\rho}{\sigma_1^3\sigma_2} + \frac{E_{13}\rho(1+\rho^2)}{2\sigma_1\sigma_2^3}\right],$$
$$u_3 = \frac{\rho}{(1-\rho^2)^3\sigma_1\sigma_2}\left[\frac{E_{40}}{2\sigma_1^4} + \frac{E_{04}}{2\sigma_2^4} + \frac{E_{22}(2+\rho^2)}{\sigma_1^2\sigma_2^2} - \frac{E_{31}(1+3\rho^2)}{2\sigma_1^3\sigma_2} - \frac{E_{13}(1+3\rho^2)}{2\sigma_1\sigma_2^3}\right],$$
$$u_4 = \frac{1}{(1-\rho^2)^3\sigma_2^2}\left[-\frac{E_{40}\rho^2}{4\sigma_1^4} - \frac{E_{04}}{4\sigma_2^4} - \frac{E_{22}(1+5\rho^2)}{4\sigma_1^2\sigma_2^2} + \frac{E_{31}(1+\rho^2)}{2\sigma_1^3\sigma_2} + \frac{E_{13}\rho}{\sigma_1\sigma_2^3}\right],$$
$$u_5 = \frac{1}{(1-\rho^2)^4\sigma_1^2\sigma_2^2}\left[\frac{E_{40}\rho^2}{4\sigma_1^4} + \frac{E_{04}\rho^2}{4\sigma_2^4} + \frac{E_{22}(1+4\rho^2+\rho^4)}{4\sigma_1^2\sigma_2^2} - \frac{E_{31}\rho(1+\rho^2)}{2\sigma_1^3\sigma_2}\right.$$
$$\left. - \frac{E_{13}\rho(1+\rho^2)}{2\sigma_1\sigma_2^3}\right],$$
$$u_6 = \frac{\rho}{2(1-\rho^2)^4\sigma_1\sigma_2}\left[-\frac{E_{40}}{3\sigma_1^6} - \frac{E_{04}\rho^2}{3\sigma_1^2\sigma_2^4} - \frac{E_{22}(1+\rho^2)}{\sigma_1^4\sigma_2^2} + \frac{E_{31}(1+3\rho^2)}{\sigma_1^5\sigma_2}\right.$$
$$\left. + \frac{E_{13}\rho^2(3+\rho^2)}{3\sigma_1^3\sigma_2^3}\right],$$
$$u_7 = \frac{\rho}{2(1-\rho^2)^4\sigma_1\sigma_2}\left[-\frac{E_{40}\rho^2}{3\sigma_1^4\sigma_2^2} - \frac{E_{04}}{3\sigma_2^6} - \frac{E_{22}(1+\rho^2)}{\sigma_1^2\sigma_2^4} + \frac{E_{31}\rho(3+\rho^2)}{\sigma_1^2\sigma_2^4}\right.$$
$$\left. + \frac{E_{13}(3+\rho^2)}{3\sigma_1\sigma_2^5}\right],$$
$$u_8 = \frac{1}{(1-\rho^2)^4\sigma_2^4}\left[\frac{E_{40}\rho^4}{24\sigma_1^4} + \frac{E_{04}}{24\sigma_2^4} + \frac{E_{22}\rho^2}{4\sigma_1^2\sigma_2^2} - \frac{E_{31}\rho^3}{6\sigma_1^3\sigma_2^5} - \frac{E_{13}\rho}{6\sigma_1^2\sigma_2^7}\right],$$
$$u_9 = \frac{1}{(1-\rho^2)^4\sigma_1^4}\left[\frac{E_{40}}{24\sigma_1^4} + \frac{E_{04}\rho^4}{24\sigma_2^4} + \frac{E_{22}\rho^2}{4\sigma_1^2\sigma_2^2} - \frac{E_{31}\rho^2}{6\sigma_1^7\sigma_2} - \frac{E_{13}\rho^3}{6\sigma_1^5\sigma_2^3}\right]$$

and $J_{Y_2}(y_1)$ is defined by

$$J_{Y_2}(y_1) = \int_{-\infty}^{\infty} (e^{y_2} - e^{y_1} - K)^+ (y_2 - \mu_2)^2 a_{Y_2|Y_1}(y_2|y_1) dy_2.$$

The derivations of the above can be found in Xie (2008).

Appendix F. Derivations for results shown in Sec. 6.2

The following are the derivations of higher moments of (Y_1, Y_2) under F:

$$\begin{aligned}
\mu_{21}(F) &= E[(\Omega_1 Z_1 + L_1 - m_{11})^2(\Omega_2 Z_2 + L_2 - m_{21})] \\
&= E[\Omega_1^2 \Omega_2 Z_1^2 Z_2 + \Omega_1^2 Z_1^2 (L_2 - m_{21}) + 2\Omega_1\Omega_2 Z_1 Z_2 (L_1 - m_{11}) \\
&\quad + 2\Omega_1 Z_1 (L_1 - m_{11})(L_2 - m_{21}) + (L_1 - m_{11})^2 \Omega_2 Z_2 \\
&\quad + (L_1 - m_{11})^2 (L_2 - m_{21})] \\
&= 0, \\
\mu_{30}(F) &= E[(\Omega_1 Z_1 + L_1 - m_{11})^3] \\
&= E[\Omega_1^3 Z_1^3] + E[(L_1 - m_{11})^3] + E[3(\Omega_1 Z_1)^2 (L_1 - m_{11})] \\
&\quad + E[3\Omega_1 Z_1 (L_1 - m_{11})^2] \\
&= E[(L_1 - m_{11})^3] \\
&= E[L_1^3] + 3m_{11}^2 E[L_1] - 3m_{11} E[L_1^2] - m_{11}^3 \\
&= m_{13} + 3m_{11}^3 - 3m_{11}m_{12} - m_{11}^3 \\
&= m_{13} + 2m_{11} - 3m_{11}m_{12} \\
&= 6\lambda_1 T\left(\frac{p_1}{\eta_{11}^3} - \frac{q_1}{\eta_{12}^3}\right) + 6(\lambda_1 T)^2 \left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right) \\
&\quad + (\lambda_1 T)^3 \left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^3 + 2(\lambda_1 T)^3 \left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^3 \\
&\quad - 3(\lambda_1 T)\left(\frac{p_i}{\eta_{i1}} - \frac{q_i}{\eta_{i2}}\right)\left[2\lambda_1 T\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right) + (\lambda_1 T)^2 \left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^2\right] \\
&= 6\lambda_1 T\left(\frac{p_1}{\eta_{11}^3} - \frac{q_1}{\eta_{12}^3}\right),
\end{aligned}$$

$$\begin{aligned}
\mu_{40}(F) &= E[(\Omega_1 Z_1 + L_1 - m_{11})^4] \\
&= E[\Omega_1^4 Z_1^4] + E[(L_1 - m_{11})^4] + 4E[(\Omega_1 Z_1)^3 (L_1 - m_{11})] \\
&\quad + 4E[\Omega_1 Z_1 (L_1 - m_{11})^3] + 6E[\Omega_1^2 Z_1^2 (L_1 - m_{11})^2]
\end{aligned}$$

$$
\begin{aligned}
&= 3\Omega_1^4 + E[(L_1 - m_{11})^4] + 6\Omega_1^2 E[(L_1 - m_{11})^2] \\
&= 3\Omega_1^4 + E[L_1^4] + 6m_{11}^2 E[L_1^2] - 4m_{11}E[L_1^3] - 4m_{11}^3 E[L_1] + 6\Omega_1^2(E[L_1^2] - m_{11}^2) \\
&= 3\Omega_1^4 + m_{14} + 6m_{11}^2 m_{12} - 4m_{11}m_{13} - 4m_{11}^4 + 6\Omega_1^2(m_{12} - m_{11}^2) \\
&= 3\Omega_1^4 + 24\lambda_1 T\left(\frac{p_1}{\eta_{11}^4} + \frac{q_1}{\eta_{12}^4}\right) + 12(\lambda_1 T)^2\left[2\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)\left(\frac{p_1}{\eta_{11}^3} - \frac{q_1}{\eta_{12}^3}\right)\right. \\
&\quad \left. + \left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right)^2\right] + 12(\lambda_1 T)^3\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^2\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right) \\
&\quad + (\lambda_1 T)^4\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^4 + 6(\lambda_1 T)^2\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^2 \\
&\quad \times \left[2\lambda_1 T\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right) + (\lambda_1 T)^2\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^2\right] - 4\lambda_1 T\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right) \\
&\quad \times \left[6\lambda_1 T\left(\frac{p_1}{\eta_{11}^3} - \frac{q_1}{\eta_{12}^3}\right) + 6(\lambda_1 T)^2\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right)\right. \\
&\quad \left. + (\lambda_1 T)^3\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^3\right] \\
&\quad - 4(\lambda_1 T)^4\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^4 + 6\Omega_1^2(2\lambda_1 T)\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right).
\end{aligned}
$$

After cancellations, we find

$$
\begin{aligned}
\mu_{40}(F) = 3\Omega_1^4 &+ 12\lambda_1 T\left[2\left(\frac{p_1}{\eta_{11}^4} + \frac{q_1}{\eta_{12}^4}\right) + \Omega_1^2\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right)\right] \\
&+ 12(\lambda_1 T)^2\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right)^2 - (\lambda_1 T)^2\left(\frac{p_1}{\eta_{11}} - \frac{q_1}{\eta_{12}}\right)^4,
\end{aligned}
$$

$$
\begin{aligned}
\mu_{22}(F) &= E[(\Omega_1 Z_1 + L_1 - m_{11})^2(\Omega_2 Z_2 + L_2 - m_{21})^2] \\
&= \Omega_1^2\Omega_2^2 E[Z_1^2 Z_2^2] + \Omega_1^2 E[Z_1^2(L_2 - m_{21})^2] + \Omega_2^2 E[Z_2^2(L_1 - m_{11})^2] \\
&\quad + E[(L_1 - m_{11})^2(L_2 - m_{21})^2] \\
&= (1 + 2\rho^2)\Omega_1^2\Omega_2^2 + \Omega_1^2 E[(L_2 - m_{21})^2] + \Omega_2^2 E[(L_1 - m_{11})^2] \\
&\quad + E[(L_1 - m_{11})^2(L_2 - m_{21})^2] \\
&= (1 + 2\rho^2)(\Omega_1\Omega_2)^2 + \Omega_1^2(2\lambda_2 T)\left(\frac{p_2}{\eta_{21}^2} + \frac{q_2}{\eta_{22}^2}\right) + \Omega_2^2(2\lambda_1 T)\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right) \\
&\quad + 4\lambda_1\lambda_2 T^2\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right)\left(\frac{p_2}{\eta_{21}^2} + \frac{q_2}{\eta_{22}^2}\right)
\end{aligned}
$$

and finally, we have

$$\begin{aligned}\mu_{31}(F) &= E[(\Omega_1 Z_1 + L_1 - m_{11})^3(\Omega_2 Z_2 + L_2 - m_{21})] \\ &= \Omega_1^3\Omega_2 E[Z_1^3 Z_2] + 3\Omega_1\Omega_2 E[Z_1(L_1 - m_{11})^2 Z_2] \\ &= 3\rho\Omega_1^3\Omega_2 + 3\rho\Omega_1\Omega_2(2\lambda_1 T)\left(\frac{p_1}{\eta_{11}^2} + \frac{q_1}{\eta_{12}^2}\right).\end{aligned}$$

References

Benth, FE and PC Kettler (2011). Dynamic copula models for the spark spread, *Quantitative Finance*, 11, 407–421.

Benth, FE, JS Benth and S Koelebakker (2008). *Stochastic Modelling of Electricity and Related Markets*, Singapore: World Scientific.

Bjerksund, P and G Stensland (2008). Closed form spread option valuation. Technical Report, Department of Finance, NHH, Norway.

Black, F and M Scholes (1973). The pricing of options and corporate liabilities, *Journal of Political Economy*, 81, 637–655.

Carmona, R and V Durrleman (2003). Pricing and hedging spread options, *SIAM Review*, 45, 627–685.

Cont, R and P Tankov (2004). *Financial Modelling with Jump Processes*, Boca Raton: Chapman & Hall/CRC.

Cramér, H (1946). *Mathematical Methods of Statistics*, Princeton, NJ: Princeton University Press.

Dempster, MAH and SSG Hong (2000). Spread option valuation and the fast Fourier transform, *Mathematical Finance: Bachelier Congress*, eds. Geman *et al.* NY: Springer.

Deng, SJ, B Johnson and A Sogomonian (2001). Exotic electricity options and the valuation of electricity generation and transmission assets, *Decision Support Systems*, 30, 383–392.

Duan, JC, G Gauthier and J Simonato (1999). An analytical approximation for the GARCH option pricing model, *Journal of Computational Finance*, 2, 75–116.

Edgeworth, FY (1904). The law of error, *Transactions of the Cambridge Philosophical Society*, 20, 36.

Ehrlich, I (2012). Pricing basket options with smile, Master Thesis, Department of Mathematics, Imperial College, London.

Hikspoors, S and S Jaimungal (2007). Energy spot price models and spread options pricing. *International Journal of Theoretical and Applied Finance*, 10, 1111–1135.

Holmquist, B (1988). Moments and cumulants of the multivariate normal distribution, *Stochastic Analysis and Applications*, 6, 273–278.

Hong, SSG (2001). Pricing and hedging of spread options with stochastic component correlation, Doctoral Dissertation, University of Cambridge, England.

Hurd, TR and Z Zhou (2010). A Fourier transform method for spread option pricing, *SIAM Journal on Financial Mathematics*, 1, 141–157.

Jarrow, R and A Rudd (1982). Approximate option valuation for arbitrary stochastic processes, *Journal of Financial Economics*, 10, 347–369.

Jackson, KR, S Jaimungal and V Surkov (2008/09). Fourier space time-stepping for option pricing with Lévy models, *The Journal of Computational Finance*, 12, 1–29.

Kendall, MG (1949). Rank and product-moment correlation, *Biometrika*, 36, 177–193.

Kirk, E (1995). Correlation in the energy markets, *Managing Energy Price Risk*, pp. 71–78. London: Risk Publications.

Kou, SG (2002). A jump-diffusion model for option pricing, *Management Science*, 48, 1086–1101.

Kouam, CT (2007). Pricing spread options under stochastic volatility, Doctoral Dissertation, Department of Mathematics, University of Houston, TX.

Li, M (2008). The impact of return nonnormality on exchange options, *The Journal of Futures Markets*, 28, 845–870.

Li, M, SJ Deng and JY Zhou (2008). Closed-form approximation for spread option prices and greeks, *The Journal of Derivatives*, 63, 58–80.

Tankov, P and C Ménassé (2015). Asymptotic indifference pricing in exponential Lévy models, In the abstracts of the Aarhus Conference on Probability, Statistics and Their Applications, 15-19 June, 2015, Celebrating the Scientific Achievements of Ole E. Barndorff-Nielsen, p. 26.

Mbanefo, A (1997). Co-movement term structure and the valuation of energy spread options, *Mathematics of Derivative Securities*, eds. Dempster and Pliska, editors, pp. 103–111. London: Cambridge University Press.

McCullagh, P (1987). *Tensor Methods in Statistics*, London: Chapman and Hall.

Merton, R (1990). *Continuous-Time Finance*, Cambridge: Blackwell.

Merton, R (1976). Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics*, 3, 125–144.

Pearson, N (1995). An efficient approach for pricing spread options, *The Journal of Derivatives*, 3, 76–91.

Ravindran, K (1995). Low-fat spreads, *Over the Rainbow: Developments in Exotic Options and Complex Swaps*, ed. R. Jarrow, pp. 141–142. London: Risk Publications.

Shimko, DC (1994). Options on futures spreads: Hedging, speculation, and valuation, *The Journal of Futures Markets*, 14, 183–213.

Shiganov, IS (1986). A multidimensional counterpart of the Gram–Charlier and Edgeworth–Cramér expansions, *Journal of Mathematical Sciences*, 32, 97–101 (Translated from *Problemy Ustoichivosti Stokhasticheskikh Modelei - Trudy Seminara*, 116–121, 1980).

Shreve, SE (2000). *Stochastic Calculus for Finance II: Continuous-Time Models*, New York: Springer.

Stuart, A and K Ord (1994). *Kendall's Advanced Theory of Statistics*, London: Hodder Arnold.

Thiele, TN (1903). *Theory of Observations*, London: Charles & Edwin Layton.

Turnbull, S and L Wakeman (1991). A quick algorithm for pricing European average options, *Journal of Financial and Quantitative Analysis*, 26, 377–389.

Xie, W (2008). Pricing spread options by Edgeworth expansion, Doctoral Dissertation. Department of Mathematics, University of Houston, TX.

Zhang, L, PA Mykland and Y Ait-Sahalia (2011). Edgeworth expansions for realized volatility and related estimators, *Journal of Econometrics*, 160, 190–203.