Finite Element Method for Pricing Swing Options under Stochastic Volatility

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October 7, 2017

Abstract

This paper studies the pricing of a swing option under the stochastic volatility. A swing option is an American-style contract with multiple exercise rights. As such, it is an optimal multiple-stopping time problem. In this paper, we reduce the problem to a sequence of optimal single stopping time problems. We propose an algorithm based on the finite element method to value the contract in a Black-Scholes-Merton framework. In many real-world applications, the volatility is typically not a constant. Stochastic volatility models are commonly used for modeling dynamic changes of volatility. Here we introduce an approach to handle this added complication and present numerical results to demonstrate that the approach is accurate and efficient.

Key words: swing options, the stochastic volatility, the finite element method, and optimal multiple stopping times

1 Introduction

An option is a financial contract between two counterparties. For a call option, it gives the buyer the right but not the obligation to acquire the underlying asset for a certain price within a specified period. In return, the buyer has to pay a premium to the seller to obtain the right. A put option is defined analogously. There are two commonly used options - European options and American options. For European options, holders are allowed to exercise their rights only on the option maturity date. For American options, holders can exercise their

*This paper was presented on June 15, 2017 in the Commodity and Energy Markets Annual Meeting held at the Mathematical Institute, University of Oxford, UK. The authors would like to thank seminar participants for their thoughtful comments and suggestions. The authors are responsible for any remaining errors.

rights at any time prior to the maturity date.

In this paper, we consider a generalization of the American style option commonly known as the swing option. An American style option gives the holder only one exercise right at any time until the maturity date, whereas a swing option gives the holder a prespecified number of opportunities to exercise the option before maturity. Between any two consecutive exercises, we assume that there is a minimum waiting time requirement. After each exercise, the option holder may receive a gain based on the specification of the payoff function. Swing options are many times used in energy markets, particularly in the power sector and natural gas industries. Since energy markets frequently experience high volatility, a swing option gives the holder some added optionality as the price of the underlying fluctuates. Hence it is a useful tool for risk management.

Option pricing plays a prominent role in the financial market. In the early seventies, Fischer Black, Robert Merton, and Myron Scholes introduced the idea of option valuation based on the construction of a riskfree hedging portfolio. Under their paradigm, they developed the well-known Black-Sholes-Merton partial differential equation (BSM PDE) for the European call option, and gave a closed-form solution (e.g., see Merton [27], Duffie[14], or Bjork[5]). For American options, there are no closed-form solutions. It is an optimal stopping time problem as the option holder can exercise the right at any time prior to maturity. As a consequence the holder does not know when to exercise the right *a priori* as a function of time. To find the exercise boundary, it is a free boundary problem for the associated BSM PDE. There are several numerical methods to solve free boundary problem, e.g., see [2],[15],[16],[18], and [24]. Numerical solutions for American options can be found once exercise boundaries are identified.

Since for a single stopping time problem, closed-form solutions do not exist. For the more complicated multiple stopping time problem, we expect that at best we may find approximate solutions for swing options by numerical methods or Monte Carlo simulations. In [7], Carmona and Touzi gave a thorough analysis of optimal multiple stopping problems. They proved the existence of multiple exercise policies. Under the risk neutral paradigm, they also sketch a general solution strategy for swing options. Furthermore, in [6] Carmona and Dayanik studied the optimal multiple stopping problem for a standard diffusion process. Recently, Wilhelm and Winter [32] developed an algorithm using the finite element method (the FEM) to value a swing option with up to seven exercise rights. They compared their results with those obtained by Monte Carlo simulations and a lattice methodand found that FEM performed well.

In financial, commodity, and energy markets, it is well known that the volatility is not a constant. This phenomenon is substantially more pronounced in the power sector. The constant volatility assumption is undoubtedly for modeling convenience. Almost always it yields crude approximations. In this paper, we consider the volatility as a stochastic process. Several researchers have studied American options under stochastic volatility (SV). Winkler, Apel and Wystup [34] used FEM to valuate European Options under Heston's stochastic volatility paradigm. Chockalingam and Muthuraman [8] studied the American options under stochastic volatility. They transformed the free boundary problem associated with American options under SV model to a converging sequence of fixed-boundary problems which were easy to valuate. Ikonen and Toivanen [18] provided five numerical methods for solving time-dependent Linear Complementarity Problems (LCP) which arose in evaluating the American options under stochastic volatility.

In this paper, we propose an approach based on FEM to compute the price of a swing put option under stochastic volatility. Our approach uses a key idea given in Carmona and Touzi [7], namely, transforming the optimal multiple stopping time problem to a single optimal stopping time problem. Here, we develop an algorithm based on LCP to solve the swing put under stochastic volatility using FEM. To validdate the accuracy of our alogithm, We consider two special cases. In the first case, we reduce the exercise right to one and solve the resulting American option problem under SV. In the second case, we consider a swing options with constant volatility (CV). We use an algorithm using the Fourier space time-stepping approach proposed in [19] for finding the put prices. We then compare our results with those reported in [32]. Finally, we present a numerical approach for a general swing put option under SV. For all three cases, we also compute the option prices using Monte Carlo simulations and compare the results against those obtained by the FEM-based approach proposed in this paper. Our comparisons indicate that the FEM given hereis accurate and noticeably reduce the computing time.

2 Pricing the Swing Options

In this section, we briefly review the pricing the standard swing options based on the work of [7, 32]. In next section, we will introduce the swing options under stochastic volatility.

Let $(\Omega,\mathcal{F},P)$ be a complete probability space. and $\mathbb{F}=\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration generated by a standard Brownian motion $(W_t)_{t\geq 0}$. $\mathbb{F}$ is an increasing continuous family of σ-algebras of $\mathcal{F}_t$. Let $S = \{S_t\}_{t\geq 0}$ be the risky asset price which is adapted to the $\mathbb{F}$ filtration. It is the solution of the following stochastic differential equation:

$$dS_t = \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t \tag{1}$$

with initial value $S_0 = s$

Let the bank account process B_t be the price of risk free asset such that

$$dB_t = r_t B_t dt, \qquad B_0 = 1$$

where r_t is an adapted process.

Applying Girsanov's theorem, there exists a risk-neutral probability measure Q, such that Q is equivalent to P. Under the risk-neutral measure Q, the discounted price process $\tilde{S}_t = S_t/B_t$ is a martingale following the stochastic differential equation (SDE)

$$dS_t = r_t S_t dt + \sigma(t, S_t)S_t dW_t \tag{2}$$

A Swing option is a contract that gives the option holder the right to exercise up to p times until maturity, where $p \in \mathbb{N}$ is a prespecified number. Between any two consecutive exercises, we impose a delivery waiting time, known as the refraction tim, for the swing option. In commodity and energy markets, this requirement is sometimes necessary. It prevents the holder to exercise all its rights at the same time. Since a swing option is a multiple stopping time problem, the holder may choose to exercise up to p times, but is not obligated to exercise them at all - contingent on the price movement of the underlying asset.

Assume that the contract originates from time t, the swing option expires at time T. Let $\mathcal{T}_t^{(p)}$ be the sequence of admissible stopping time for the swing option with up to $p \in \mathbb{N}$ exercise rights. Let the refraction time be $\delta > 0$. Using the definition in [32], the admissible stopping time set is defined as follows:

$$\begin{aligned}\mathcal{T}_t^{(p)} := \{\tau^{(p)} = (\tau_1, \tau_2, \cdots, \tau_p)\,|\tau_i \geq t \ for\ i = 1, \cdots, p \\ \tau_1 \leq T \ \ a.s. \ \ and \ \ \tau_{i+1} - \tau_i \geq \delta \ for\ i = 1, \cdots, p-1\}. \quad (3)\end{aligned}$$

Assuming the payoff process of the swing option $\phi(S) : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the integrability condition:

$$\mathbb{E}\{\phi(\bar{S})^\alpha\} < \infty \qquad \text{for some } \alpha \geq 1 \tag{4}$$

where $\phi(\bar{S}) = \sup_{t\geq 0} \phi(S_t)$ and $\phi(S_t) = 0$ for $t > T$.

Let $F^{(p)}(t, s)$ be the value of a swing option with up to p exercise rights, which starts at time t, with starting asset value s, and maturity date T. Under the risk-neutral measure Q, $F^{(p)}(t, s)$ is the supremum of the expected discounted payoff at each stopping time, i.e.

$$F^{(p)}(t, s) = \sup_{\tau^{(p)} \in \mathcal{T}_t^{(p)}} \mathbb{E}^Q \left[\sum_{i=1}^{p} e^{-r(\tau_i - t)} \phi\left(S_{\tau_i}\right) |S_t = s \right] \tag{5}$$

for all $t \in [0, T]$, and S_t has the same dynamics as (2).

Carmona and Touzi [7] proved the following existence theorem about the swing option pricing process.

Theorem 2.1 *Assuming the filtration $\mathbb{F}$ is left continuous and every $\mathbb{F}$-adapted martingale has continuous sample paths. If the payoff process of the swing option $\phi(S_t)$ is continuous almost surely, and (4) holds, then for any $p \in \mathbb{N}$, there exists $\tau^* = (\tau_1^*, \cdots, \tau_p^*) \in \mathcal{T}_t^{(p)}$ such that*

$$F^{(p)}(t, s) = \mathbb{E}^Q \left[\sum_{i=1}^{p} e^{-r(\tau_i^* - t)} \phi(S_{\tau_i^*}) |S_t = s \right]. \tag{6}$$

Applying the result of Theorem 2.1, Carmona and Touzi reduced the optimal multiple stopping time problem to a sequence of optimal single stopping time problems. Following Wilhelm and Winter [32], we state the following dynamic programming recursion

$$F^{(p)}(t,s) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^Q \left[e^{-r(\tau-t)} \Phi^{(p)}(\tau, S_\tau) | S_t = s \right], \tag{7}$$

with

$$\Phi^{(p)}(t,s) := \begin{cases} \phi(s) + e^{-r\delta} \mathbb{E}\left[F^{(p-1)}(t+\delta, S_{t+\delta}) | S_t = s \right] & \text{if } t \leq T - \delta \\ \phi(s) & \text{if } t \in (T-\delta, T]. \end{cases} \tag{8}$$

When $p = 0$, there is no exercise right remaining, it follows $F^{(0)}(t,s) := 0$.

In [32] Wilhelm and Winter also proved that the only price of a swing option with p exercise rights which is arbitrage free is given by (5). Thus the arbitrage free price of a swing option can be determined by a sequence of single optimal stopping time problems. We now elaborate on the solution procedure. To begin with, in (7) we see that the value of the swing option with p exercise rights is the value of an American option with payoff process $\Phi^{(p)}(\tau, S_\tau)$. Then (8) shows that the payoff process $\Phi^{(p)}(\tau, S_\tau)$ is the sum of swing option payoff process and the value of a European option (in the parlance of dynamic programming, the two terms correspond to the immediate payoff and the value of the optimal return function in the subsequent stage). With regard to this European option, the payoff function is none other than the value of the swing option with $p-1$ exercise rights following the refraction time δ.

Based on the above analysis, we are able to compute the value of a swing option with p exercise rights recursively. The algorithm is summarized below:

Assuming that the price of a swing option under the stochastic volatility model with m exercise rights has been calculated.

Step1: calculate the value of the corresponding European option with the payoff process defined by the price of the swing option with m exercise rights;

Step2: calculate the payoff process for $\Phi^{(m+1)}(\tau, S_\tau)$ using (8);

Step3: calculate the swing option with $m+1$ exercise rights using (7), and let $m = m+1$, stop if $m = p$; else go to Step 1.

3 Swing Options under SV model

Of all the parameters in a Black-Scholes model for option pricing, volatility is the only parameter that cannot be directly observed from the market. In the Black-Scholes formula, volatility is assumed to ba a constant. Historical volatility or implied volatility is typically used as an approximation. Historical volatility gives an average volatility for the given time interval. It does not reflect future volatility movement. It is well known that implied volatility exhibits smile effects, i.e., the at-the-money options tend to have a lower implied volatility than in-the-money or out-of-the-money options. In assessing the volatility of underlying assets for option pricing, traders almost always adjust its value according to their own experiences and expectations about the market. This process is nevertheless ad-hoc. Taking the time varying nature of volatility change in a formal framework invariably renders the model more realistic.

There are several ways to model the change of volatility value over time. The GARCH model and its variants are used by many practitioners. Another choice is the stochastic volatility model (SV). In an SV model, it is commonly assumed that volatility follows a mean-reverting Brownian Process. In [11], Danielsson compared SV models with GARCH models and found SV models provide a better estimation and observed that SV models could capture the market behavior more accurately. In this paper, we assume the swing option under the stochastic volatility paradigm.

Under the risk neutral measure Q, the price process S_t of the underlying asset and the volatility process σ_t follow the SDEs

$$dS_t = rS_t dt + \sigma_t S_t dW_{1t} \tag{9}$$

$$\sigma_t = f(Y_t) \tag{10}$$

$$dY_t = \mu(t, Y_t)dt + \hat{\sigma}(t, Y_t)d\hat{W}_t \tag{11}$$

where $(\hat{W}_t)$ is a Brownian motion which may be correlated with W_{1t} with a correlation coefficient ρ. Thus $\hat{W}_t$ can be written as a linear combination of W_{1t} and another independent Brownian motion W_{2t}

$$\hat{W}_t = \rho W_{1t} + \sqrt{1-\rho^2}W_{2t} \tag{12}$$

Stochastic volatility models have appeared in the literature for more than twenty years. In Table 1, we summarize the parameter specifications for (10) and (11) used in several commonly cited models.

Table 1: Stochastic Volatility Models

	$f(y)$	$\mu(t,y)$	$\hat{\sigma}(t,y)$	ρ
Ball and Roma (1994)	$\sqrt{y}$	$\alpha(m-y)$	$\beta\sqrt{y}$	$\rho = 0$
Heston(1993)	$\sqrt{y}$	$\alpha(m-y)$	$\beta\sqrt{y}$	$\rho \neq 0$
Stein and Stein(1991)	$\lvert y\rvert$	$\alpha(m-y)$	β	$\rho = 0$
Scott(1987)	e^y	$\alpha(m-y)$	β	$\rho = 0$
Hull and White(1987)	$\sqrt{y}$	μy	βy	$\rho = 0$

Following the approach sketched in Section 2, we can similarly determine the price of swing options under stochastic volatility. We first calculate the prices of the corresponding European and American options under SV, then use (7) and (8) to compute the price of the corresponding swing option accordingly.

Consider a European option under SV with dynamics (9),(10) and (11). Suppose the expiration date is T and the payoff function is $g(S_T)$. At time t, let $F(S_t, Y_t, t)$ denote the price of the swing option when the price of the underlying asset is S_t and the volatility process is at a level Y_t. The corresponding partial differential equation for the European option under the stochastic volatility model is (see L. Jiang[21])

$$
\begin{aligned}
&\frac{\partial F}{\partial t}+\frac{1}{2}f^2(Y)S^2\tfrac{\partial^2 F}{\partial S^2}+\rho f(Y)\hat{\sigma}S\frac{\partial^2 F}{\partial S\partial Y}\\
+&\frac{1}{2}\hat{\sigma}^2\frac{\partial^2 F}{\partial Y^2}+rS\frac{\partial F}{\partial S}+(\mu-\Lambda\hat{\sigma})\frac{\partial F}{\partial Y}-rF=0 \quad (0\le t<T, S>0, Y\in\mathbb{R})\\
&F(S,Y,T)=g(S_T) \quad (t=T, S>0, Y\in\mathbb{R})
\end{aligned}
\tag{13}
$$

where the function $g(S_T)$ is the boundary condition, and $\Lambda(S,Y,t)$ represents the market price of volatility risk. Sometimes it is also called the volatility risk premium.

Comparing with European options under SV, the American options under SV share the same partial differential equation and the maturity date payoff process. The only difference is that under the latter, the exercise is permitted at any time during the life of the option. The early exercise possibility results in a free boundary problem for American-style options (e.g., see Peskir and Shiryaev [28]). The free boundary splits the whole region into two parts - the exercise region and the continuation region. When S_t is in the continuation region, the price $F(S,Y,t)$ satisfies the partial differential equation (13). When S_t is in the exercise region, the option should be exercised since it is worth more. Based on these relations, the pricing of American option under the stochastic volatility model can be transformed to a time dependent linear complementarity problem (e.g., see Wilmott, Dewynne, and Howison [33]).

Define the generalized Black-Scholes operator $\mathcal{A}$ as

$$
\begin{aligned}
\mathcal{A}F=\frac{1}{2}f^2(Y)S^2\frac{\partial^2 F}{\partial S^2}+\rho f(Y)\hat{\sigma}S\frac{\partial^2 F}{\partial S\partial Y}+\frac{1}{2}\hat{\sigma}^2\frac{\partial^2 F}{\partial Y^2}\\
+rS\frac{\partial F}{\partial S}+(\mu-\Lambda\hat{\sigma})\frac{\partial F}{\partial Y}-rF.
\end{aligned}
\tag{14}
$$

Then the linear complementarity problem (LCP) for the American option under the stochastic volatility model can be characterized as

$$
\begin{aligned}
\frac{\partial F}{\partial t}+\mathcal{A}F\le 0 &\quad (0\le t<T, S>0, Y\in\mathbb{R})\\
F\ge g &\quad (0\le t<T, S>0, Y\in\mathbb{R})\\
(\frac{\partial F}{\partial t}+\mathcal{A}F)(F-g)=0 &\quad (0\le t<T, S>0, Y\in\mathbb{R})
\end{aligned}
\tag{15}
$$

with initial data

$$F|_{t=T}=g(S_T).$$

The asymptotic behavior of $F(S,Y,t)$ depends on the payoff process $g(S)$. For example, for a put option, i.e., $g(S)=(K-S)^+$, where K is the strike price, $F(S,Y,t)$ should satisfy the following conditions:

$$\lim_{S\to\infty}\frac{\partial F(S,Y,t)}{\partial S}=0 \tag{16}$$

and

$$\lim_{Y \to \infty} \frac{\partial F(S, Y, t)}{\partial Y} = 0. \tag{17}$$

If we denote the free boundary by the critical curve $S^* = S^*(t)$ for $t \in [0, T]$, then we can identify the behavior of $F(S, Y, t)$ for a put option when the underlying asset price approaches $S^*(t)$

$$\lim_{S \to S^*(t)} F(S, Y, t) = K - S^*(t) \tag{18}$$

and the so-called smooth-pasting condition

$$\lim_{S \to S^*(t)} \frac{\partial F(S, Y, t)}{\partial S} = -1. \tag{19}$$

The pricing of swing option under SV can be described as a sequence of solving European options under SV and American option under SV. Once we solve the European/American option under SV, based on (7) and (8), we can find the price the swing option under SV. In next Section, we will describe the algorithm in detail.

4 An Algorithm for Swing Options under Stochastic Volatility

we now propose an algorithm for pricing a swing option under SV. There are several alternative approaches (e.g., the finite-difference method, a Fourier transform-based method, or Monte Carlo simulations). In this paper, we choose the finite element method. Our choice is based on the degree of the precision and the computation time needed for solving the problem. Before applying the FEM, we first specify the specific SV model chosen for illustration. We emphasize that our approach is applicable to other models (e.g., those shown in Table 1).

To illustrate the application of our proposed procedure, we consider a swing put option under the SV model proposed by Stein and Stein [30]. There the volatility is a function of a mean reverting Orstein-Uhlenbeck process,

$$\begin{aligned}
&dS_t = rS_t dt + \sigma_t S_t dW_{1t} \\
&\sigma_t = |Y_t| \\
&dY_t = \alpha(m - Y_t)dt + \beta d\hat{W}_t
\end{aligned} \tag{20}$$

where α, m, and β are positive numbers. The parameter α is the rate of the mean reversion, m is the long-term mean variance level, and the ratio $\frac{\beta^2}{\alpha}$ is the long-term behavior of the variance of Y_t. In the Stein-Stein stochastic volatility model, the correlation coefficient ρ between the two Brownian motions is assumed to be 0. The various properties of the Stein and Stein SV are discussed in [30]

Let t denote the time to maturity, i.e. $t = T - \tau$, where τ is the current time. Based on the variable t, we transform the backward PDE to a forward PDE. For simplicity, we assume the market price of volatility risk is zero, i.e., we set $\Lambda(S, Y, t) = 0$. Let $F(S, Y, t)$ be the price of a swing option under SV. Define the generalized Black-Scholes operator $\mathcal{A}$ as

$$\mathcal{A}F = -\frac{1}{2}Y^2S^2\frac{\partial^2 F}{\partial S^2} - \rho\beta S|Y|\frac{\partial^2 F}{\partial S\partial Y} - \frac{1}{2}\beta^2\frac{\partial^2 F}{\partial Y^2} - rS\frac{\partial F}{\partial S} - \alpha(m - Y)\frac{\partial F}{\partial Y} + rF. \quad (21)$$

The payoff process $g(S, t)$ is now defined by

$$g(S, t) = (K - S_t)^+ = \max(K - S_t, 0). \quad (22)$$

Before developing the algorithm for a swing put option under the stochastic volatility model, we use the finite element method to solve the pricing problems for European and American put options under SV.

4.1 European put option under SV

Following the development in the last section, the European put option under SV can be written as

$$\begin{aligned} &\frac{\partial F}{\partial t} + \mathcal{A}F = 0 &&\text{in } \Omega \times (0, T] \\ &F(S, Y, 0) = g(S, 0) &&\text{in } \Omega \end{aligned} \quad (23)$$

where $g(S, t) = (K - S_t)^+$, and $\Omega = \{S > 0, Y \in \mathbb{R}\}$.

There is no need to impose a boundary condition on $S = 0$ because of the degeneracy of the equation and for $S \to \infty$, or $Y \to \infty$

$$\lim_{S\to\infty} \frac{\partial F(S, Y, t)}{\partial S} = 0$$

and

$$\lim_{Y\to\infty} \frac{\partial F(S, Y, t)}{\partial Y} = 0.$$

Achdou, Franchi and Tchou [1] proved the existence of a unique solution to (23). Using this observation, we propose an algorithm based on FEM and apply the Galerkin scheme to obtain the numerical solution. We rewrite (23) in a variational form, $\forall v \in W$

$$\begin{aligned} &\left(\frac{\partial F}{\partial t}, v\right) + (\mathcal{A}F, v) = 0 &&\text{in } \Omega \times (0, T] \\ &F(S, Y, 0) = g(S, 0) &&in\ \Omega \end{aligned} \quad (24)$$

where W is the weighted Sobolev space:

$$W = \left\{ v : \left(\sqrt{1+Y^2}v, \frac{\partial v}{\partial Y}, S|Y|\frac{\partial v}{\partial S} \right) \in (L^2(\Omega))^3 \right\} \tag{25}$$

with the norm

$$||v||_W = \left(\int_\Omega (1+Y^2)v^2 + \left(\frac{\partial v}{\partial Y}\right)^2 + S^2Y^2\left(\frac{\partial v}{\partial S}\right)^2 \right)^{\frac{1}{2}}. \tag{26}$$

Define the space V as the closed subspace of W which vanishes on the Dirichlet boundary, i.e.,

$$V = \{v \in W : v|_{\Gamma d} = 0\}. \tag{27}$$

4.1.1 Time Discretization

Since (23) is a time-dependent problem, for the time domain, we use the time difference method. We partition the time interval [0,T] into subintervals $[t_{m-1}, t_m]$, $1 \leq m \leq M$, such that $0 = t_0 < t_1 < \cdots < t_M = T$. Define $\Delta t_i = t_i - t_{i-1}$. Denote the numerical solution at time t_m as F^m.

A variety of techniques for the numerical solution to (28) can be employed. Here we write (28) in a generalized weighted implicit form with parameter θ.

$$\left(\frac{F^m - F^{m-1}}{\Delta t_m}, v \right) + \theta(\mathcal{A}F^m, v) + (1-\theta)(\mathcal{A}F^{m-1}, v) = 0. \tag{28}$$

When $\theta = 0$, this is an explicit scheme, whereas when $\theta = 1$, it becomes an implicit scheme. In particular, when $\theta = \frac{1}{2}$, it is the well-known Crank-Nicolson (CN) scheme. In this paper, we choose the CN scheme.

4.1.2 Discretization on the S-Y domain

Assuming the number of the vertices is N_V, and the number of vertices lying in the open domain Ω is N_{Vo}. We introduce two spaces of finite dimensions, W_h and V_h. We use piecewise linear functions for the FE method implementation, then

$$W_h = \{v \in \mathcal{C}^0(\bar{\Omega}) : v \text{ is linear on any triangular } E_i\} \tag{29}$$

and

$$V_h = \{v \in W_h : v|_{\Gamma d} = 0\}. \tag{30}$$

The solution $F(t_m)$ to the swing put option under SV can be approximated by a function $F_h^m \in W_h$

$$F(t_m, \cdot) \approx F_h^m(\cdot) = \sum_{i=1}^{N_V} F_i^m \phi_i(\cdot) \qquad m = 0, 1, \cdots, M. \tag{31}$$

where F_h^m is the numerical solution at time t_m, the F_i^ms are undetermined values and $\phi_i,\ i = 1, \cdots, N_V$ are the pyramid-shaped linear functions.

Substituting F_h^m into the variational form (28), we obtain the discretization form: $\forall v \in V_h$

$$\left(\frac{F_h^m - F_h^{m-1}}{\Delta t_m}, v\right) + \frac{1}{2}(\mathcal{A}F_h^m, v) + \frac{1}{2}(\mathcal{A}F_h^{m-1}, v) = 0. \tag{32}$$

Applying $v = \phi_i$ for $i = 1, \cdots, N_{Vo}$, into (32), after some calculations, we will obtain a linear system like $AF_h^m = b$ for $m = 0, \cdots, M$. The linear system has to be solved for each time step to obtain the price of a European option under SV at $t = M$

4.2 American put Stochastic Volatility

In contrast to a European option, an American-type option can be exercised at any time prior to maturity. This is an optimal stopping time problem and the arbitrage free price of an American type option with the payoff process $g(t, S_t)$ is given by:

$$F(t, s, y) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^Q[e^{-r(\tau - t)} g(\tau, S_\tau) | S_t = s, Y_t = y]. \tag{33}$$

Since it is a free boundary problem as mentioned in Section 3, we transform the free boundary problem to a linear complementarity problem. Consequently, the dependence of the solution on the optimal exercise boundary is removed. Therefore the American put option under SV can be stated as a time dependent linear LCP form:

$$\begin{aligned} \frac{\partial F}{\partial t} + \mathcal{A}F \geq 0 \quad & \text{in } \Omega \times (0, T] \\ F(S, Y, t) \geq g(S, t) \quad & \text{in } \Omega \times (0, T] \\ (\frac{\partial F}{\partial t} + \mathcal{A}F)(F(S, Y, t) - g(S, t)) = 0 \quad & \text{in } \Omega \times (0, T] \\ F(S, Y, 0) = g(S, 0) \quad & \text{in } \Omega. \end{aligned} \tag{34}$$

There are several approaches for handling time dependent LCPs. In [18], Ikonen, and Toivanen discussed five of them for dealing with LCPs for American options under SV. These approaches include the projected SOR method, the operator splitting method, the penalty method, among others. The basic idea stems from noting the fact that the value of American option is always no less than the payoff process. At each time step t_m, after solving the variational problem, the condition $F^m(S, T, t) \geq g(S, t)$ is to be enforced.

The procedure to discretize an American option under SV is similar to that used in the evaluation of its European counterpart under SV. We use the same time scheme for the American option under SV and the same S-Y domain discretization. By solving the LCP problem and enforcing the payoff condition, the price of the American option under SV at each discrete point (S_i, Y_j, t_m) is obtained accordingly. In other words, we find

$$F^m(S_i, Y_j, t_m) = \max(F_e^m(S_i, Y_j, t_m), g(S_i, t_m)). \tag{35}$$

At each time step t_m , after F^m is calculated, we can also capture the information about the optimal exercise boundary. Thus the latter is obtained as a byproduct.

4.3 Algorithm for Swing Option under SV model

We are now ready to develop an algorithm for the evaluation of a swing put option under SV. Let $F^{(n)}(S,Y,t)$ be the value of a swing put option under SV with the payoff process $g(S,t)$, where $n \in \mathbb{N}$ is the number of exercise rights remaining, $t \in [0,T]$ is the time to maturity, and $g(S,t) = \max(K - S_t, 0)$. Following (7), the swing option price can be determined as a price of an American option whose pricing function $\Psi(S,Y,t)$ is characterized by

$$\begin{aligned} \frac{\partial F^{(n)}}{\partial t} + \mathcal{A}F^{(n)} \geq 0 &\quad \text{in } \Omega \times (0,T] \\ F^{(n)} \geq \Psi^{(n)} &\quad \text{in } \Omega \times (0,T] \\ \left(F^{(n)} - \Psi^{(n)}\right)\left(\frac{\partial F^{(n)}}{\partial t} + \mathcal{A}F^{(n)}\right) = 0 &\quad \text{in } \Omega \times (0,T] \\ F^{(n)}(S,Y,0) = \Psi^{(n)}(S_0,0) &\quad \text{in } \Omega. \end{aligned} \tag{36}$$

According to (8), the n^{th} payoff process can be obtained by

$$\begin{aligned} &\Psi^{(n)}(S,Y,t) := \begin{cases} g(S,t) + F_e^{(n)}(S,Y,t,\delta) & \text{for } t \in [\delta,T) \\ g(S,t) & \text{for } t \in [0,\delta) \end{cases} \\ &\Psi^{(0)}(S,Y,t) := 0 \end{aligned} \tag{37}$$

where $F_e^{(n)}$ is the price of a European put option under SV satisfying the following PDE

$$\begin{aligned} \frac{\partial F_e^{(n)}}{\partial t} + \mathcal{A}F_e^{(n)} = 0 &\quad \text{in } \Omega \times (0,\delta) \\ F_e^{(n)}(S,Y,0,\delta) = F^{(n-1)}(S,Y,t-\delta) &\quad \text{in } \Omega. \end{aligned} \tag{38}$$

The discretizations of the time and the S-Y domain are almost the same as we have done for the European/American put option with SV. There is only one more requirement for the refraction time δ such that $\delta/\triangle t \in \mathbb{N}$.

For each iteration when the exercise number is $i, i = 1, 2, \cdots, n$, the American option with SV is calculated for the complete time domain, i.e., t from 0 to T, whereas for the European option with SV, it is calculated only for the time domain where $t \in (0, \delta)$.

Using (36),(37) and (38), we present an algorithm for pricing the swing put option under SV. We summarize the solution procedure as following:

for $l = 1 : n$
  for $t = 0 : \triangle t : \delta - 1$
    $\Psi^{(l)}(S,Y,t) = g(S,t)$

end

for $t = \delta : \triangle t : T$

if $l > 1$, calculate $F_e^{(l)}(S, Y, \tau)$ using

$$\begin{aligned} \frac{\partial F_e^{(l)}}{\partial \tau} + \mathcal{A} F_e^{(l)} &= 0 \quad \tau \in (0, \delta) \\ F_e^{(l)}(S, Y, 0) = F^{(l-1)}(S, Y, t-\delta) & \quad in \Omega \end{aligned}$$

else

$F_e^{(l)}(S, Y, \delta) = 0$

end if

$$\Psi^{(l)}(S, Y, t) = g(S, t) + F_e^{(l)}(S, Y, \delta) \qquad \forall t \in (\delta, T]$$

end

Calculate $F^{(l)}(S, Y, t)$ with boundary condition $\Psi^{(l)}(S, Y, t)$

$$\begin{aligned} \frac{\partial F^{(l)}}{\partial t} + \mathcal{A} F^{(l)} \geq 0 & \quad \text{in } \Omega \times (0, T] \\ F^{(l)} \geq \Psi^{(l)} & \quad \text{in } \Omega \times (0, T] \\ \left(F^{(l)} - \Psi^{(l)}\right)\left(\frac{\partial F(l)}{\partial t} + \mathcal{A} F^{(l)}\right) = 0 & \quad \text{in } \Omega \times (0, T] \\ F^{(l)}(S, Y, 0) = \Psi^{(l)}(S, 0) & \quad \text{in } \Omega \end{aligned}$$

end

5 Numerical Results

To validate our FEM-based algorithm for pricing a swing option under SV, we first consider the two special cases where alternative approaches for producing comparative results are known. The first is when the number of exercise opportunity is one. Then the problem reduces to an American option under SV. The second case is a swing put option with a constant volatility. In both cases, we will see that our algorithm performs satisfactorily. When the swing option has more than one swing exercise right, at the absence of other viable means to cross check the approach, we use Monte Carlo simulations to produce results for comparison. We will see that prices obtained from the proposed approach stay within the confidence intervals that can be established from simulations.

5.1 American Option under Stochastic Volatility

When $n = 1$, the swing option under SV reduces to an American option under SV. We set the parameters for BSM PDE as following: the risk free rate of interest $r = 0.05$, the strike price $K = 100$, the time to maturity $T = 1$. We consider the Stein-Stein Stochastic volatility model with $\alpha = 1$, $m = 0.16$, the correlation coefficient $\rho = 0$, and $\beta = \frac{\sqrt{2}}{2}$. We set the market price of volatility risk $\Lambda = 0$. The S-plane and the Y-plane are partitioned into 100 mesh points respectively and the number of time steps is 70. Figure 1 plots

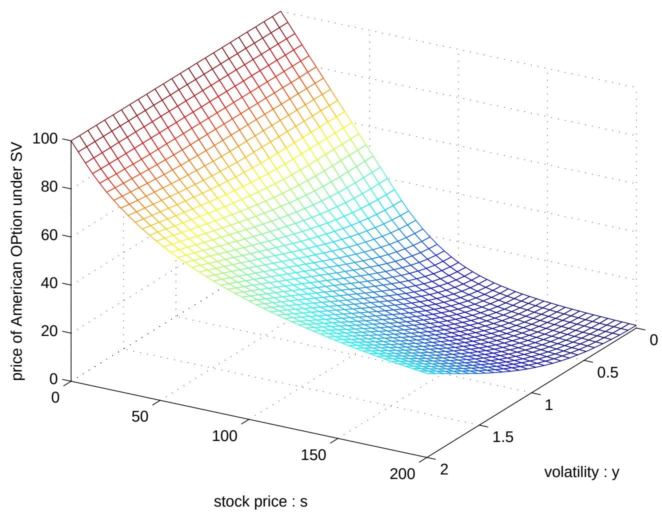

Figure 1:

Table 2: Prices of American option under SV

Stock Price	Volatility	the FE method	Monte Carlo [stand.dev]
80	0.16	22.9124	22.9249 [0.24]
80	0.40	25.4355	25.2324 [0.26]
90	0.16	16.8695	17.2265 [0.25]
90	0.40	19.8516	19.8874 [0.26]
100	0.16	12.4061	12.9463 [0.36]
100	0.40	15.5671	15.7207 [0.31]
110	0.16	9.26419	9.9865 [0.27]
110	0.40	12.3741	12.3188 [0.19]

the price of a American Put option with one year to maturity. For comparison, we employ a least-square based Monte Carlo simulation (e.g., see Longstaff and Schwartz [25]) with 10 time steps, 2,000 simulations, and 10 different seeds. The basis functions chosen are $1, x, x^2$. Table 2 summuarizes the numerical results obtained under both methods.

As mentioned in Section 4, once we find the price of American put option under SV, we can also capture the information of optimal exercise boundary. Figure 2 plots the optimal exercise boundary. In Figure 3, we compare the American option under SV and American option with constant volatility. We explore the price difference at two specific σ values when $T = 1$. In the figure, we can see when in the optimal exercise region, the prices of these two models (SV versus Constant volatility) are the identical. Outside the region, the prices are different. The prices of constant volatility model could be underpriced, or overpriced.

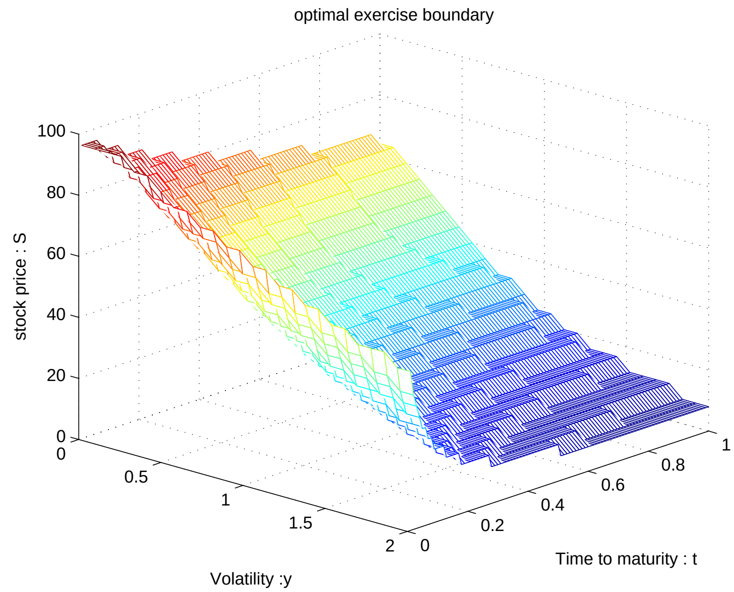

Figure 2:

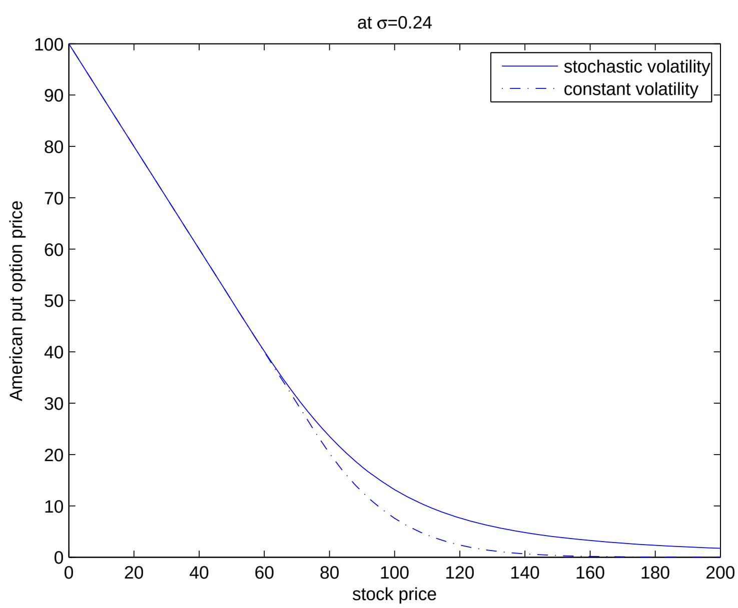

Figure 3:

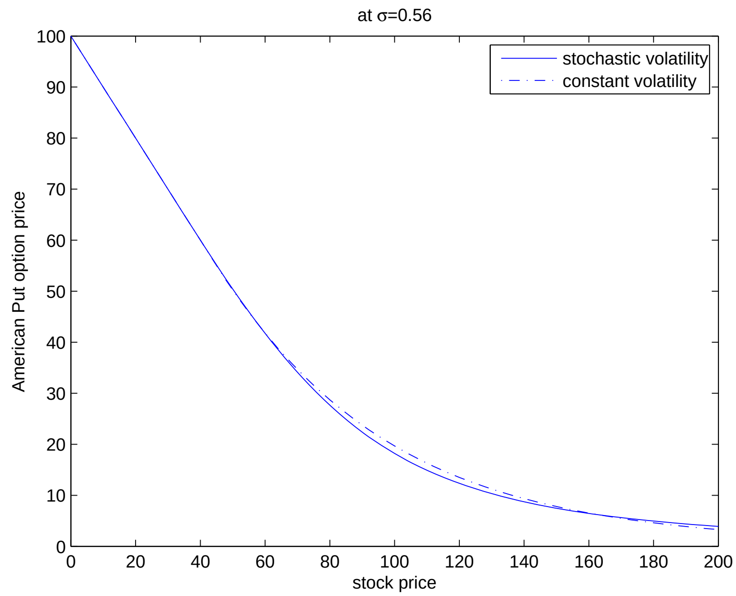

Figure 4:

5.2 Swing Put Option with Constant Volatility

When $\alpha = 0$, $m = 0$, and $\beta = 0$. this model is reduced to a swing put option with the constant volatility. Suppose the number of exercise rights is $p = 3$. We first use this reduced model to obtain the numerical solution for the price of the swing option. We then develop an algorithm using the Fourier Space Time-stepping method (FST) described in [19] to compute the price under the same setting. In this experiment, we choose $K = 100,\ r = 0.05,\ \sigma = 0.3,\ \delta = 0.1,\ T = 1$. For the FE method, we choose 400 mesh points and 200 time steps, while for the FST method, we use 1000 time steps and 800 frequency points. Figure 4 plots the numerical prices of the swing option obtained from these two approaches.

In Figure 4, we observe that the results obtained from the FE method and the FST method match well for the case of a swing option with up to 3 exercise rights under the constant volatility. The price behavior is similar to that of an American option.

We also study the convergence behaviors of this reduced model, the FST method, and the Monte Carlo simulation, when the spot price is at the money. We use the numerical result in [32] as a benchmark, which uses 4000 mesh points and 1000 time steps. These swing option prices are $F^{(1)}(100, 0, 0)) = 9.8700$, $F^{(2)}(100, 0, 0)) = 19.2550$, and $F^{(3)}(100, 0, 0)) = 28.1265$. Let N_t be the number of time steps, N be the number of frequency points, and M be the number of simulation paths. The unit of computing time is the second.

We show the absolute errors and the computing time for the FE method.

Table 3: Absolute errors and the computing time using the FEM-based method for a swing put under the CV with 400 mesh points.

	$N_t = 100$		$N_t = 200$		$N_t = 400$		$N_t = 800$	
Rights	Error	Time	Error	Time	Error	Time	Error	Time
$p = 1$	0.0216	0.134	0.0111	0.279	5.64e-03	0.422	2.86e-03	0.858
$p = 2$	0.0193	0.166	9.9e-03	0.369	4.8e-03	0.658	2.2e-03	1.725
$p = 3$	0.0122	0.288	5.7e-03	0.442	1.7e-03	0.915	2.0e-04	3.849

Table 4: Absolute errors and the computing time using the FST method for a swing option under the constant volatility with 400 time steps.

	$N = 100$		$N = 200$		$N = 400$		$N = 800$	
Rights	Error	Time	Error	Time	Error	Time	Error	Time
$p = 1$	0.0852	0.05	0.0132	0.06	0.0057	0.15	0.0102	0.22
$p = 2$	0.1835	0.26	0.0427	0.35	0.0084	0.57	0.0004	0.95
$p = 3$	0.3261	0.46	0.1003	0.61	0.0451	0.92	0.0308	1.36

Notice that the computing time in Table 3 is for calculating the swing option prices at all 400 mesh points. In the table 3, we only show the price behavior when the spot price is at the money.

In Table 4, we show the behavior of the FST method. The computing time in this table is the time needed to calculate the price only at a single spot price. From this view point, the FE method is much fast than the FST method.

Table 5: Absolute errors and the computing time using the Monte Carlo simulation for a swing option under the constant volatility

	$M = 2000$		$M = 4000$		$M = 8000$	
Rights	Error(std)	Time	Error(std)	Time	Error(std)	Time
$p = 1$	0.0605(0.1276)	1.48	0.0452(0.0959)	2.25	0.0283(0.0838)	2.91
$p = 2$	0.1132(0.2843)	1.77	0.0906(0.2190)	3.16	0.0490(0.1032)	5.38
$p = 3$	0.1362(0.3888)	3.24	0.0967(0.2621)	4.88	0.0647(0.1554)	7.32

Table 5 produces similar results for pricing a swing put under CV using Monte Carlo simulation. The Monte Carlo method is an extension of the Least Square Method for American options. In simulation, we choose $1, x, x^2$ as the basis functions. Similar to the FST method, the computing times the table are the times needed to calculate the price of a single spot price. Based on the figure shown in Tables 3-5, we demonstrate that the accuracies of the FEM are noticeably higher and the computing times are substantially shorter than the other two approaches. Although the FST method is relatively easy to implement, its applicability is constrained by the requirements that the coefficients of the partial differential equation are constants. While Monte Carlo simulation is easy to construct, it demand a larger amount of computing time to achieve a desired degree of accuracy. To illustrate the effect of the number of exercise rights on swing put prices as a function of the spot price, in Figure 5, we plot the prices for swing prices under CV using the FEM based method when exercise

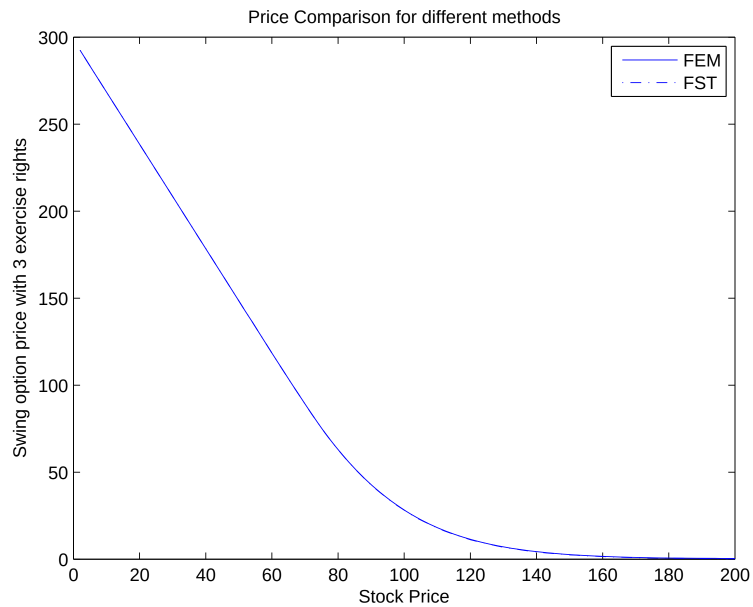

Figure 5:

rights up to 3.

5.3 Swing Put Option under Stochastic Volatility

We now consider the 'fully-fledged' (by this, we mean the case when the number of swing rights can be greater than one) swing put option under the stochastic volatility model. We set the parameters as follows: $\alpha = 1$, $m = 0.16$, $\beta = \frac{\sqrt{2}}{2}$, and $r = 0.05,\ T = 1,\ K = 100$.

Let N be the number of partition of S-plane, M be the number of partition of Y-plane, and N_t be the number of time steps. In our experiment, $N_t = 70, N = M = 101$. Again, we use the standard Stein-Stein stochastic volatility model where the correlation coefficient $\rho = 0$. Thus, the two Brownian Motions are uncorrelated. Figure 6 plots the prices for the swing put option under SV with exercise rights $n = 3$.

For comparison, we developed a Monte Carlo simulation for pricing the swing option under SV. We use the same parameters for the SV model as in the FE method. In the simulation, we use 10 time steps, 2000 simulation paths, and 10 different seeds. We choose $1, x, x^2$ as the basis functions for the Least Square method. Table 6 displays the results obtained from the simulation.

From the above computational results, we remark that our algorithm for the swing option under SV works well. In addition, it took around 140 seconds to obtain the numerical results for all (S, Y) points using the FE method, whereas using the Monte Carlo simulation, it took around 1.2 seconds to calculate the

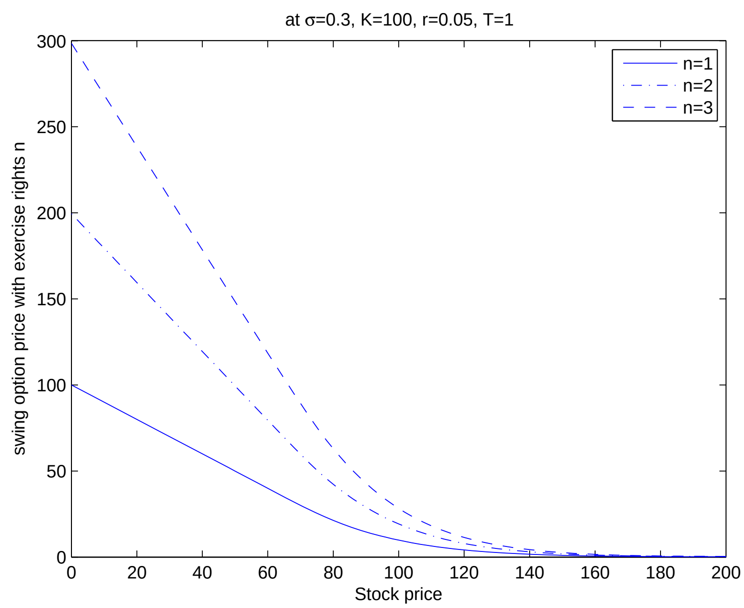

Figure 6:

Table 6: Prices of swing option under SV

Stock Price	Volatility	the FE method	Monte Carlo [stand.dev]
80	0.16	67.2005	67.6835 [0.44]
80	0.40	74.5725	73.9330 [0.62]
90	0.16	48.4735	49.8099 [0.89]
90	0.40	57.3988	57.2476 [0.51]
100	0.16	34.799	36.4638 [0.87]
100	0.40	44.306	43.9919 [0.76]
110	0.16	25.3902	26.9789 [0.68]
110	0.40	34.6676	34.7105 [0.57]

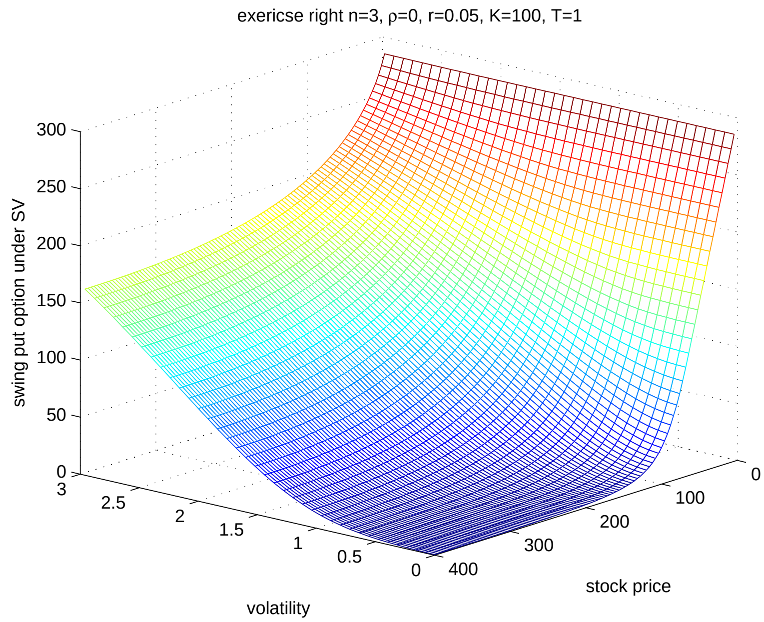

Figure 7:

swing price for a single (S, Y) point. The whole (S, Y) plane has 10,000 points, so the FEM-based method is substantially faster than Monte Carlo simulations. In Figure 7, we choose two specific σ values and compare the swing option values for the SV model and the constant volatility model respectively.

In the case of $\sigma = 0.32$, the prices of the two models exhibit similar behavior. There are some differences around the strike price. When $S > 2K$, as the stock price increases, the difference between these two approaches becomes negligible. When $\sigma = 0.96$, the asymptotic behaviors of these two models are different. From these two cases, we can see that the stochastic volatility model can capture more dynamic changes of the pricing behavior, while the constant volatility model only provides a coarse approximation and would cause mispricing.

6 Conclusion

The notion of the stochastic volatility was first included the study of European options and later extended to that of American options. This enhancement captures the financial market behavior more closely than that under the simplifying assumption of the constant volatility. In this paper, we include stochastic volatility in the swing option in order to make it more reflective of the real-world price movement. By transforming the solution process for the swing option to a sequence of single stopping time problems, we reduce the problem to a series of problems involving the valuations of European/American options under the stochastic volatility. In this paper, we develop an algorithm for pricing the swing option under the Stein-Stein stochastic volatility model. The algorithm is flexible with respect to different payoff functions. We explore the behavior of the swing option under SV, as well as two special cases. We compare the results with Monte Carlo simulations. The numerical results show that the finite element method is a fast and accurate. Future work could be the study of greeks for the swing option under SV, or a model including Lévy process.

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