

# Answers to Odd-Numbered Problems

## CHAPTER 1

### Exercises 1.1

- (a) ordinary, first order  
(c) partial, second order  
(e) ordinary, third order  
(g) ordinary, second order
- Both  $y$  and  $z$  are solutions.
- Both  $y$  and  $z$  are solutions.
- Both  $u_1$  and  $u_2$  are solutions.
- $u_1$  is a solution;  $u_2$  is not a solution.
- $y = 16x^2 + C_1x + C_2$
- $y = Ce^{3x}$ .
- $r = 2, -2$ ;  $y_1(x) = e^{2x}$  and  $y_2(x) = e^{-2x}$  are solutions.
- $r = 3$ ;  $y(x) = e^{3x}$  is a solution.
- No real values of  $r$ ;  $r = 1 \pm 2i$  are complex values.
- $r = 3, -3$ ;  $y_1(x) = x^3$  and  $y_2(x) = x^{-3}$  are solutions.

### Exercises 1.3

- (b)  $y = 2e^{5x}$ .
- (b)  $y = \frac{e}{e - 2e^x}$ .
- (b)  $y = -\sin 3x + \frac{1}{3} \cos 3x$ .
- (b)  $y = -\frac{17}{4}x + 9\sqrt{x}$ .  
(c)  $y'$  is not defined at  $x = 0$ ; there is no solution to  $y'(0) = 2$ .
- $xy' - 3y + 3 = 0$ .
- $y' - 2y = -4e^{-2x}$ .
- $y'^2 + xy' - y = 0$ .
- $y'' - 4y' + 4y = 0$ .

19.  $x^2y'' + xy' - y = 0$ .

21.  $y'' + 9y = 0$ .

23.  $y''' = 0$ .

## CHAPTER 2

### Exercises 2.1

1.  $y = -\frac{1}{2} + Ce^{2x}$ .

3.  $y = 1 + Ce^{-x^2}$ .

5.  $y = e^{-x} + Ce^x$ .

7.  $y = x^{-2} \sin x + Cx^{-2}$ .

9.  $y = \frac{2}{9}(x+1)^{5/2} + C(x+1)^{-2}$ .

11.  $y = \sin x \cos x + C \cos x = \frac{1}{2} \sin 2x + C \cos x$ .

13.  $y = e^x + \frac{C}{x}$ .

15.  $y = x(\ln x)^2 + Cx$ .

17.  $y = 1 + Ce^{-e^x}$ .

19.  $y = x - 1 + 2e^{-x}$ .

21.  $y = \frac{\ln(1+e^x)}{e^x} + (e - \ln 2)e^{-x}$ .

23.  $y = \frac{5 - \cos 2x}{2 \sin x}$ .

25.  $y = \frac{2}{Cx - 3x^3}$ .

27.  $y = (Ce^{2x} - e^x)^2$ .

29.  $y = \frac{1}{\sqrt[3]{Cx^3 - 2x^3 \ln x}}$ .

### Exercises 2.2

1.  $y = \left(\frac{x^2}{4} + C\right)^2$ .

3.  $\tan^{-1} y = x^3 + C$  or  $y = \tan(x^3 + C)$ .

5.  $\cot y = \ln \sqrt{\frac{1-x}{1+x}} + C$ .

7.  $e^{-y} = e^x - xe^x + C.$
9.  $y = \frac{x + C}{1 - Cx}.$
11.  $y^2 = C(\ln x)^2 - 1.$
13.  $\ln |y| = -\ln |x| - \frac{1}{x} - 1.$
15.  $y = xe^{x^2-1}.$
17.  $y + \ln |y| = \frac{1}{3}x^3 - x - 5.$
19.  $y^2 = \frac{C}{1 + x^2} - 1.$
21.  $y = \frac{\ln |\sec x + \tan x|}{x} + \frac{C}{x}.$
23.  $y = 1 + Ce^{-x^2}.$
25.  $y = C(3x^2 + 1)^{1/3} - 3.$
27.  $y = \frac{1}{Cx + 1 + \ln x}.$
29.  $y = \sqrt{x^2 + Cx}.$
31.  $x \ln x + \frac{x + y}{e^{y/x}} = Cx.$
33.  $\csc(y/x) - \cot(y/x) = Cx.$

### Exercises 2.3.1

1.  $x^2 + 3y^2 = C.$
3.  $\frac{x^2}{2} + y^2 - 4y = C.$
5.  $\frac{x^2}{2} + y^2 = C;$  ellipses, center at the origin, major axis horizontal.
7.  $y = C(x - a)$

### Exercises 2.3.2

1. (a)  $A(t) = 50 \left(\frac{9}{10}\right)^{t/2} \approx 50e^{-0.05268t}.$  (b)  $A(4) = 50 \left(\frac{9}{10}\right)^2 = 40.5$  grams.  
 (c)  $T \approx 13.16$  hours.
3. (a)  $P(t) \approx 0.25e^{0.0421t}$  (b)  $\approx 1.6573$  square centimeters (c)  $\approx 16.464$  hours
5. (a)  $P(t) \approx 4.5e^{0.01438t}.$  (b) 48.19 years (c)  $\approx 6.17$  billion.

**Exercises 2.3.3**

1. (a)  $40.1^\circ$ .      (b) 1.62 minutes.

3. 8:52 pm.

**Exercises 2.3.4**

1. (a)  $v = \left(v_0 + \frac{g}{r}\right)e^{-rt} - \frac{g}{r}$       (b)  $\lim_{t \rightarrow \infty} v = -\frac{g}{r}$ .

(c)  $y = y_0 + \frac{1}{r} \left(v_0 + \frac{g}{r}\right) (1 - e^{-rt}) - \frac{g}{r}t$

3.  $k \approx 17.8$

**Exercises 2.3.5**

1. (a)  $A(t) = 10,000(1 - e^{-t/200})$       (b)  $t = 200 \ln 5 \approx 322$  minutes

3. (a)  $A(t) = \frac{9}{2}(1 - e^{-t/150})$       (b)  $t = 150 \ln 3 \approx 165$  minutes

5. (a)  $A(t) = \frac{3}{20}t(100 - t)$       (b)  $\max = A(50) = 375$

**Exercises 2.3.6**

1. (a) 3259 people.      (b)  $\approx 6.89$  days.

## CHAPTER 3

### Exercises 3.2

1. Yes

3. Yes

5. Yes

7. (a)  $r = -1, r = 4$ .

(b) Fundamental set:  $y_1(x) = x^{-1}, y_2(x) = x^4$ ; general solution:  $y = C_1x^{-1} + C_2x^4$ .

(c)  $y = \frac{9}{5}x^{-1} + \frac{1}{5}x^4$ .

(d) The trivial solution:  $y \equiv 0$ .

9.  $y'' - 2y' - 3y = 0$ .

11.  $y'' = 0$ .

13.  $x^2 y'' - 2x y' + 2y = 0$ .

15.  $W[y_1, y_2](x) = e^{-\int_a^x p(t) dt} \neq 0$  for all  $x$ .

17.  $\{y_1(x) = x, y_2(x) = x^2\}$ .

19.  $\{y_1(x) = e^{x^2}, y_2(x) = e^{-x^2}\}$ .

21.  $\alpha\delta - \beta\gamma \neq 0$ .

23.  $W[y_1 + y_2, y_1 - y_2] = -2W[y_1, y_2]$ .

25. Set  $u(x) = \frac{y_2(x)}{y_1(x)}$ . Then

$$u'(x) = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W[y_1, y_2]}{y_1^2} \equiv 0.$$

Therefore,  $u \equiv \lambda$  constant, which implies that  $y_2 = \lambda y_1$ .

### Exercises 3.3

1.  $y = C_1 e^{2x} + C_2 e^{-4x}$ .

3.  $y = C_1 e^{5x} + C_2 x e^{5x}$ .

5.  $y = e^{-2x} [C_1 \cos 3x + C_2 \sin 3x]$ .

7.  $y = C_1 + C_2 e^{-2x}$ .

9.  $y = C_1 e^{2\sqrt{3}x} + C_2 e^{-2\sqrt{3}x}$ .

11.  $y = e^x [C_1 \cos x + C_2 \sin x]$ .

13.  $y = C_1 e^{6x} + C_2 e^{-5x}$ .

15.  $y = e^{-x/2} [C_1 \cos x/2 + C_2 \sin x/2]$ .

17.  $y = C_1 e^{4x} + C_2 x e^{4x}$ .

19.  $y = 2e^{2x} - e^{3x}$ .

21.  $y = -3e^{-x} - 2xe^{-x}$ .

23.  $y = -e^x \cos x$ .

25.  $y'' + 3y' - 10y = 0$ .

27.  $y'' + 4y = 0$ .

29.  $y'' - \frac{5}{2}y' + y = 0$ .

31.  $y'' + 2y' + 10y = 0$ .

33.  $y'' + 16y = 0$ .

35.  $y = (1 + \beta)e^{x/2} + (1 - \beta)e^{-x/2}; \beta = -1$ .

37. If the roots of  $r^2 + ar + b = 0$  are real (real and unequal, or real and equal), then they are negative;  $r$  negative implies  $e^{rx} \rightarrow 0$  and  $xe^{rx} \rightarrow 0$  as  $x \rightarrow \infty$ . If the roots are complex conjugates, then they have negative real part and  $\alpha$  negative implies  $e^{\alpha x} \cos \beta x \rightarrow 0$  and  $e^{\alpha x} \sin \beta x \rightarrow 0$  as  $x \rightarrow \infty$ .

39. Suppose that  $a > 0$  and  $b = 0$ . Then the general solution of the differential equation is

$$y = C_1 + C_2 e^{-ax} \quad \text{and} \quad \lim_{x \rightarrow \infty} y = C_1.$$

The solution that satisfies the initial conditions is:  $y = \left(\alpha + \frac{\beta}{a}\right) - \frac{\beta}{a} e^{-ax}; \quad k = \alpha + \frac{\beta}{a}$ .

41.  $r_1, r_2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{-a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2} = \alpha \pm \beta$ .

General solution:

$$\begin{aligned} y = C_1 e^{(\alpha+\beta)x} + C_2 e^{(\alpha-\beta)x} &= C_1 e^{\alpha x} e^{\beta x} + C_2 e^{\alpha x} e^{-\beta x} \\ &= e^{\alpha x} \left[ (C_1 + C_2) \frac{e^{\beta x} + e^{-\beta x}}{2} + (C_1 - C_2) \frac{e^{\beta x} - e^{-\beta x}}{2} \right] \\ &= e^{\alpha x} (K_1 \cosh \beta x + K_2 \sinh \beta x). \end{aligned}$$

43.  $y = C_1 x^{-2} + C_2 x^4$ .

45.  $y = C_1 x^2 + C_2 x^2 \ln x$ .

### Exercises 3.4

1.  $z(x) = x^2 \ln x + \frac{1}{2}$ ;  $y = C_1 x^2 + C_2 x^{-1} + x^2 \ln x + \frac{1}{2}$ .
3.  $z(x) = -x^2 \ln x + \frac{1}{2} x^2 (\ln x)^2$ ;  $y = C_1 x + C_2 x^2 - x^2 \ln x + \frac{1}{2} x^2 (\ln x)^2$ .
5.  $z(x) = -(1 + x^2)$ ;  $y = C_1 x + C_2 e^x - (1 + x^2)$ .
7.  $y = C_1 e^{-x} + C_2 e^{2x} - \frac{2}{3} x e^{-x}$ .
9.  $y = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4} \cos 2x \ln(\cos 2x) + \frac{1}{2} x \sin 2x$ .
11.  $y = C_1 e^x + C_2 x e^x - e^x \cos x$ .
13.  $y = C_1 e^{-2x} + C_2 x e^{-2x} - e^{-2x} \ln x$ .
15.  $y = C_1 \cos 3x + C_2 \sin 3x + \sin 3x \ln(\sec 3x + \tan 3x) - 1$ .
17.  $y = C_1 x + C_2 x^{-1} + x \ln x$ .
19.  $y = C_1 x + C_2 x \ln x + x^2$ .

### Exercises 3.5

1.  $y = C_1 e^{-x} + C_2 e^{3x} - e^{2x}$ .
3.  $y = C_1 e^{-3x} + C_2 x e^{-3x} + \frac{1}{4} e^{3x}$ .
5.  $y = C_1 e^{-2x} + C_2 - \frac{1}{2} \cos 2x - \frac{1}{2} \sin 2x$ .
7.  $y = C_1 e^{-x/2} + C_2 e^{-x} + x^2 - 6x + 14 - \frac{9}{10} \cos x - \frac{3}{10} \sin x$ .
9.  $y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{1}{2} x + \frac{1}{4}$ .
11.  $y = C_1 e^{-2x} + C_2 e^{-4x} + \frac{3}{2} x e^{-2x}$ .
13.  $y = C_1 \cos 3x + C_2 \sin 3x + \frac{2}{3} + \frac{1}{162} (9x^2 - 6x + 1) e^{3x}$ .
15.  $y = e^x (C_1 \cos 2x + C_2 \sin 2x) - \frac{1}{10} e^{-x} \cos 2x - \frac{1}{20} e^{-x} \sin 2x$ .
17.  $y = e^x - \frac{1}{2} e^{-2x} - x - \frac{1}{2}$ .
19.  $y = \frac{13}{15} e^{-x} + \frac{1}{12} e^{2x} + \frac{1}{20} \cos 2x - \frac{3}{20} \sin 2x$ .
21.  $z = A + (Bx^2 + Cx)e^{-x} + D \cos 3x + E \sin 3x$ .
23.  $z = Ax^2 + Bx + C + Dx \cos x + Ex \sin x$ .
25.  $z = (Ax^3 + Bx^2)e^{2x} + Cx^2 + Dx + E + (Fx + G) \cos 2x + (Hx + I) \sin 2x$ .
27.  $z = Ae^{-x} + Bxe^{-x} \cos x + Cxe^{-x} \sin x + D$ .
29.  $y = C_1 e^{2x} + C_2 x e^{2x} + \frac{8}{25} \cos x + \frac{6}{25} \sin x + 3x e^{2x} \ln x$ .

31.  $y = C_1 \cos 3x + C_2 \sin 3x + \frac{3}{8} \cos x - \sin 3x \ln(\sec 3x + \tan 3x) + 1.$
33.  $y_1 - y_2$  is a solution of the reduced equation  $y'' + ay' + by = 0$  with  $a, b > 0$ . As shown in Exercises 3.4, Problem 37,  $y_1 - y_2 \rightarrow 0$  as  $x \rightarrow \infty$ . If  $a = 0, b > 0$ , then all solutions of the reduced equation are bounded (Problem 38, Exercises 3.4).

### Exercises 3.6

1. The equation of motion is  $y(t) = \sin\left(8t + \frac{1}{2}\pi\right)$ . The amplitude is 1 and the frequency is  $8/2\pi = 4/\pi$ .
3. The velocity at the equilibrium point is:  $\pm 2\pi A/T$ .
5. (a)  $A \sin(\omega t + \phi_0) = A \cos(\omega t + \phi_0 - \frac{\pi}{2})$ ; take  $\phi_1 = \phi_0 - \frac{1}{2}\pi$ .
- (b)  $A \sin(\omega t + \phi_0) = A \cos \phi_0 \sin \omega t + A \sin \phi_0 \cos \omega t = B \sin \omega t + C \cos \omega t.$
7. Assume that  $r_1 > r_2$ . If  $C_1 = 0$  or  $C_2 = 0$ , then  $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$  can never be zero. If both  $C_1$  and  $C_2$  are nonzero, then  $C_1 e^{r_1 t} + C_2 e^{r_2 t} = 0$  implies  $e^{(r_1 - r_2)t} = -\frac{C_2}{C_1}$ . Since  $e^{(r_1 - r_2)t}$  is an increasing function ( $r_1 > r_2$ ), it can take the value  $\frac{C_2}{C_1}$  at most once. By the same reasoning,  $x'(t) = C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t}$  can be zero at most once. Therefore the motion can change direction at most once.
9. If  $\gamma \neq \omega$ , we try  $z = A \cos \gamma t + B \sin \gamma t$  as a particular solution of  $y'' + \omega^2 y = \frac{F_0}{m} \cos \gamma t$ . Substituting  $z$  into the equation, we get  $-\gamma^2 z + \omega^2 z = \frac{F_0}{m} \cos \gamma t$ , giving

$$z = \frac{F_0/m}{\omega^2 - \gamma^2} \cos \gamma t.$$

11. If  $\gamma = \omega$ , we try  $z = At \cos \omega t + Bt \sin \omega t$  as a particular solution of

$$y'' + \omega^2 y = \frac{F_0}{m} \cos \omega t.$$

Substituting  $z$  into the equation, we have

$$(2B\omega - A\omega^2 t) \cos \omega t - (2A\omega + B\omega^2 t) \sin \omega t + \omega^2 (At \cos \omega t + Bt \sin \omega t) = \frac{F_0}{m} \cos \omega t,$$

which gives  $A = 0, B = \frac{F_0}{2\omega m}$ , as required.

## Chapter 4

### Exercises 4.1

1.  $\frac{1}{s^2}$ .

$$3. \frac{1}{s^2 + 1}.$$

$$5. \frac{1}{2(s-1)} - \frac{1}{2(s+1)}.$$

$$7. \frac{s-a}{(s-a)^2 + b^2}.$$

### Exercises 4.2

$$1. \frac{3}{s} - \frac{2}{s^2} + \frac{2}{s^3}.$$

$$3. \frac{3}{s} + \frac{4}{s-3} - \frac{2s}{s^2+4}.$$

$$5. \frac{10}{s^3} - \frac{4}{(s+3)^2+4}.$$

$$7. \frac{2s}{(s^2+1)^2} + \frac{2(s^2-4)}{(s^2+4)^2}.$$

$$9. \sinh \beta x = \frac{e^{\beta x} - e^{-\beta x}}{2}.$$

$$11. \frac{1}{2} \left[ \frac{1}{s-3} + \frac{1}{s-2} - \frac{1}{s-1} + \frac{1}{s-4} \right].$$

$$15. Y(s) = \frac{1}{s-2}.$$

$$17. Y(s) = \frac{2}{(s-2)(s+4)} - \frac{9}{(s^2+9)(s+4)} - \frac{3}{s+4}.$$

$$19. Y(s) = \frac{2}{(s+3)^2}.$$

$$21. Y(s) = \frac{3}{s(s-5)(s+3)} + \frac{4}{(s-5)(s+3)^2} + \frac{s-5}{(s-5)(s+3)}.$$

$$23. \text{ Set } g(x) = \int_0^x f(t) dt. \text{ Then } g'(x) = f(x) \text{ and } g(0) = 0.$$

$$F(s) = \mathcal{L}[f(x)] = \mathcal{L}[g'(x)] = s\mathcal{L}[g(x)] - g(0) = s\mathcal{L}[g(x)].$$

$$\text{Therefore, } \mathcal{L}[g(x)] = \frac{1}{s}F(s).$$

### Exercises 4.3

$$1. 6e^{-7x}.$$

$$3. \frac{1}{5} \sin 5x.$$

$$5. e^{-4x} \cos x.$$

7.  $e^{-2x} \cos 2x + e^{-2x} \sin 2x$ .
9.  $2xe^{-2x} - e^x \cos x - e^x \sin x$ .
11.  $\frac{1}{2} e^{-x} - \frac{1}{2} \cos x + \frac{1}{2} \sin x$ .
13.  $\frac{1}{4} - \frac{1}{4} \cos 2x$ .
15.  $\frac{1}{2} - e^x + \frac{3}{2} e^{-2x}$ .
17.  $e^{2x} - 4e^x + 2x + 3$ .
19.  $\frac{2}{3} e^{-2x} + \frac{1}{3} e^x$ .
21.  $\frac{3}{2} e^{-x} - \frac{1}{2} \cos x + \frac{1}{2} \sin x$ .
23.  $e^x \sin x$ .
25.  $\frac{3}{4} e^{-x} + \frac{1}{4} e^x + \frac{1}{2} x e^x$ .
27.  $\frac{1}{4} e^x + x e^{-x} + x - 2$ .
29.  $e^{-2x} + e^x$ .
31.  $\alpha = \frac{1}{4}$ .
33.  $\beta = -\frac{26}{5}$ .
35.  $\frac{7}{4} e^{2(x-1)} - 3e^{x-1} + \frac{1}{2} x + \frac{3}{4}$ .

#### Exercises 4.4

1.  $\mathcal{L}[f(x)] = \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2}$ .
3.  $\mathcal{L}[f(x)] = \frac{2e^{-5s}}{s}$ .
5.  $\mathcal{L}[f(x)] = \frac{1}{s} + e^{-2s} \left( \frac{1}{s^2} - \frac{1}{s} \right) + e^{-4s} \left( \frac{1}{s+1} - \frac{2}{s} - \frac{1}{s^2} \right)$ .
7.  $\frac{\frac{2}{s^2} - 2e^{-2s} \frac{1}{s} - 2e^{-2s} \frac{1}{s^2} - 2e^{-4s} \frac{1}{s}}{1 - e^{-4s}}$ .
9.  $\frac{2}{s^3} - e^{-3s} \frac{2}{s^3} - 3e^{-3s} \frac{1}{s^2}$ .
11.  $\frac{1}{s(s^2 + 1)} (e^{-\pi s} - e^{-2\pi s})$ .
13.  $\frac{1}{s^2} (1 - 2e^{-s} + 2e^{-3s} - e^{-4s})$ .
15.  $\sin x - \sin x u(x - \pi)$ .

17.  $\cos x - \cos x u(x - \pi) + \sin x u(x - \pi)$ .

19.  $u(x - 2) - e^{-(x-2)}u(x - 2)$ .

### Exercises 4.5

1.  $y = -\frac{1}{2} + \frac{5}{2}e^{2x} + u(x - 1) \left[-\frac{1}{2} + \frac{1}{2}e^{2(x-1)}\right]$ .

3.  $y = 1 - \cos x + \sin x - u(x - 1)[\cos(x - 1) - 1]$ .

5.  $y = 1 - e^{-x} - xe^{-x} + u(x - 2) \left[x - 4 + xe^{-(x-2)}\right]$ .

7.  $y = -\frac{1}{3} - \frac{1}{6}e^{3x} + \frac{1}{2}e^x + u(x - 1) \left[\frac{1}{3} + \frac{1}{6}e^{3(x-1)} - \frac{1}{2}e^{x-1}\right]$ .

9.  $y = \frac{1}{4}e^x + \frac{11}{4}e^{-x} + \frac{3}{2}xe^{-x} + u(x - 1) \left[xe^{-(x-1)} - 1\right]$ .

### Exercises 5.2

1.  $x = 4, y = 1$ .

3.  $x = 4 - 2a, y = a, a$  any real number.

5.  $x = -3, y = 1$ .

7. No solution.

9.  $x = 2a - 3, y = a, a$  any real number.

11.  $x = \frac{3}{7}a + 1, y = \frac{5}{7}a - 1, z = a, a$  any real number.

13. No solution.

15.  $x = 2, y = 1, z = 1$ .

### Exercises 5.3

1. matrix of coefficients: 3; augmented matrix: 3;  $x = 5, y = 3, z = -1$ .

3. matrix of coefficients: 2; augmented matrix: 2;  $x = 4 - 2a, y = a, z = -2, a$  an real number.

5. matrix of coefficients: 3; augmented matrix: 3;  $x_1 = -1, x_2 = -1 - 2a, x_3 = 3 + a, x_4 = a, a$  any real number.

7. matrix of coefficients: 3; augmented matrix: 3;  $x_1 = 8 + 2a - 3b, x_2 = a, x_3 = 3 - 1 - 2b, x_4 = b, x_5 = -3, a, b$  any real numbers.

9.  $x = 2, y = 5$ .

11.  $x = -3 - a, y = 2 + 2a, z = a, a$  any real number.

13.  $x = \frac{10}{7}, y = \frac{2}{7}, z = \frac{3}{2}$ .

15.  $x_1 = 11 - 2a + b$ ,  $x_2 = a$ ,  $x_3 = 3 - b$ ,  $x_4 = b$ ,  $a$ ,  $b$  any real numbers.
17.  $x_1 = -2$ ,  $x_2 = -5$ ,  $x_3 = -1$ ,  $x_4 = 5$ .
19.  $x_1 = 3 - 2a$ ,  $x_2 = a$ ,  $x_3 = 2$ ,  $x_4 = 1$ .
21. No solution.
23. (i)  $k \neq -3, 2$  (ii)  $k = -3$  (iii)  $k = 2$ .
25. (a) No (b) No (c) Yes

#### Exercises 5.4

1. Yes
3. Yes
5. No. The leading 1 in the last column is not the only nonzero in its column.
7. Yes
9.  $x = 10$ ,  $y = -9$ ,  $z = -7$
11.  $x = -3 - a$ ,  $y = 2 + 2a$ ,  $z = a$ ,  $a$  any real number.
13.  $x_1 = 11 - 2a + b$ ,  $x_2 = a$ ,  $x_3 = 3 - b$ ,  $x_4 = b$ ,  $a$ ,  $b$  any real numbers.
15.  $x_1 = 7 - 2a - b$ ,  $x_2 = 1 + 3a - 4b$ ,  $x_3 = a$ ,  $x_4 = b$ ,  $a$ ,  $b$  any real numbers.
17.  $x = y = 0$ .
19.  $x = y = z = 0$ .
21.  $x_1 = 2a - b$ ,  $x_2 = -a + 4b$ ,  $x_3 = a$ ,  $x_4 = b$ ,  $a$ ,  $b$  any real numbers.
23.  $x_1 = x_2 = x_3 = x_4 = 0$ .
25. Consider the system

$$\begin{aligned}x + y &= 0 \\2x + 2y &= 0 \\3x + 3y &= 0\end{aligned}$$

This system has the solutions  $x = -a$ ,  $y = a$ ,  $a$  any real number.

27.  $b = -3a$ ,  $a$  any real number.
29.  $a = 4$ .

### Exercises 5.5

1. (a)  $\begin{pmatrix} 0 & 4 \\ 3 & 5 \\ 1 & -1 \end{pmatrix}$ .

(c)  $\begin{pmatrix} 8 & -4 \\ 1 & 4 \end{pmatrix}$ .

(e)  $\begin{pmatrix} 2 & 8 \\ 2 & 8 \\ 5 & -3 \end{pmatrix}$ .

3. (a)  $\begin{pmatrix} -4 & -3 \\ 28 & -6 \\ -20 & 24 \end{pmatrix}$ .

(c) Not defined.

(e)  $\begin{pmatrix} 1 & 3 \\ -3 & -12 \\ -41 & 21 \end{pmatrix}$ .

5. (a)  $c_{32} = 2$     (b)  $c_{13} = 34$     (c)  $d_{21} = 5$     (d)  $d_{22} = 1$ .

7. (a)  $d_{22} = 6$     (b)  $d_{12} = -4$     (c)  $d_{23} = -18$ .

11. (a)  $AB = \begin{pmatrix} 4 & 7 & 10 \\ 0 & -5 & -14 \end{pmatrix}$ ,  $BA$  not defined.

(b)  $AC = \begin{pmatrix} 14 & 5 \\ -2 & -3 \end{pmatrix}$ ,  $CA = \begin{pmatrix} -1 & 14 \\ 5 & 12 \end{pmatrix}$ .

(c)  $AD = DA = \begin{pmatrix} 4 & 4 \\ -2 & 2 \end{pmatrix}$ .

13.  $A(BD) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -6 & 0 & 8 \\ 9 & 9 & -26 \end{pmatrix} = \begin{pmatrix} -6 & 0 & 8 \\ 9 & 9 & -26 \end{pmatrix}$

$(AB)D = \begin{pmatrix} 0 & -2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -2 & 3 & -2 \\ 3 & 0 & -4 \end{pmatrix} = \begin{pmatrix} -6 & 0 & 8 \\ 9 & 9 & -26 \end{pmatrix}$ .

15. (a)  $3 \times 3$     (c) Does not exist    (e)  $2 \times 3$ .

### Exercises 5.6

1.  $A^{-1} = \begin{pmatrix} 1/2 & 0 \\ -3/2 & 1 \end{pmatrix}$ .

3.  $A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$ .

5.  $A^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$ .

7. No inverse.

9. No inverse.

11.  $\det A = \pm 1$ .

13.  $x = 5, y = 0$ .

15.  $x = \frac{9}{2}, y = -5$ .

17.  $x = \frac{7}{9}, y = \frac{1}{3}, z = -\frac{5}{9}$ .

19.  $-31$ .

21.  $-45$ .

23.  $30$ .

25.  $-21$ .

27.  $-18$ .

29.  $26$ .

31.  $x = 0, 1, -3$ .

33.  $y = -\frac{25}{37}$ .

35. Cramer's rule does not apply.

37.  $x = 0$ .

39.  $\lambda = -4, 7$ .

### Exercises 5.7

3. Dependent;  $(-4, 8, 9) = 2(1, -2, 3) + 3(-2, 4, 1)$ .

5. Dependent;  $(-2, 6, 3) = (1, -1, 3) + 2(0, 2, 3) - 3(1, -1, 2)$ .

7. Dependent;  $(7, -4, 1) = 3(1, -2, 1) + 2(2, 1, -1)$ .

9. Dependent;  $(4, -2, 0, 2) = 2(2, -1, 0, 1)$ .

11.  $b \neq -\frac{1}{3}$ .

13.  $b = 0, -7$ .

17. No; a linearly dependent set can have linearly independent subsets. For example,  $\{(1, -2, 3), (-2, 4, 1)\}$  is a linearly independent subset of  $\{(1, -2, 3), (-2, 4, 1)\}, (-4, 8, 9)$ .

19.  $W(x) = -a$ ; linearly independent.

21.  $W(x) = -2x^{-6}$ ; linearly independent.

23.  $W(x) = e^{2x}(x - 2)$ ; linearly independent.

25. (a) False      (b) True      (c) True.

### Exercises 5.8

1.  $2, \begin{pmatrix} 1 \\ 0 \end{pmatrix}; 3, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

3.  $-1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}; 4, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ .

5.  $1, 1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

7.  $2, 2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

9.  $2 + i, \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}; 2 - i, \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ .

11.  $8, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; 1, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}; 2, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ .

13.  $1, 1, 1, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ .

15.  $1 + i, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}; 1 - i, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}; 0, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ .

17.  $1, 1, 1, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ .

19.  $2 + 3i, \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}; 2 - 3i, \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}; 2, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ .

$$21. \quad 2, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}; \quad 2, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}; \quad 6, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \quad 4, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

## Chapter 6

### Exercises 6.1

$$1. \quad y = C_1 e^x + C_2 e^{2x} + C_3 e^{3x}.$$

$$3. \quad y = C_1 e^{2x} + C_2 e^{-2x} + e^x [C_3 \cos 2x + C_4 \sin 2x].$$

$$5. \quad y = C_1 \cos x + C_2 \sin x + e^{2x} [C_3 \cos 3x + C_4 \sin 3x].$$

$$7. \quad y = C_1 + C_2 x + C_3 e^x + C_4 e^{-x} + C_5 \cos x + C_6 \sin x.$$

$$9. \quad y = 2x.$$

$$11. \quad y = \frac{1}{5} e^x - \frac{1}{5} \cos 3x - \frac{1}{15} \sin 3x.$$

$$13. \quad y^{(4)} - 8y''' + 31y'' - 78y' + 90y = 0.$$

$$15. \quad y^{(5)} - 2y^{(4)} - 2y''' - 2y'' - 3y' = 0.$$

$$17. \quad y^{(5)} - 2y^{(4)} + y''' - 2y'' = 0.$$

$$19. \quad y^{(4)} - y'' = 0.$$

$$21. \quad y = C_1 e^{-x} + C_2 \cos x + C_3 \sin x + \frac{1}{4} e^x + 4.$$

$$23. \quad y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x + 6 + \frac{1}{9} \cos 2x.$$

$$25. \quad y = \frac{1}{72} e^{-x} [e^{3x}(2x-1) + 3 \cos \sqrt{3}x + \sqrt{3} \sin \sqrt{3}x].$$

### Exercises 6.2

$$1. \quad \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin 2t \end{pmatrix}.$$

$$3. \quad \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix}.$$

$$5. \quad \begin{aligned} x'_1 &= 2x_1 - x_2 + e^{2t} \\ x'_2 &= 3x_1 + 2e^{-t} \end{aligned}.$$

$$7. \quad \begin{aligned} x'_1 &= 2x_1 + 3x_2 - x_3 + e^t \\ x'_2 &= -2x_1 + x_3 + 2e^{-t} \\ x'_3 &= 2x_1 + 3x_2 + e^{2t} \end{aligned}.$$

$$9. \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \sin t \\ -2 \cos t \end{pmatrix}.$$

$$1. 1 \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -3 & 0 \\ 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 3e^{2t} \\ -2 \cos t \\ t \end{pmatrix}.$$

### Exercises 6.3

1. Independent

3. Independent

5. Dependent

7. Dependent

9. Dependent

1. 1 (c)  $\mathbf{x}(t) = c_1 \mathbf{u} + c_2 \mathbf{v}$ , where  $c_1, c_2$  are arbitrary constants.

$$(d) \mathbf{x}(t) = -2 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + \begin{pmatrix} 3e^{3t} \\ 2e^{3t} \end{pmatrix}.$$

$$1. 3 (b) \mathbf{x}(t) = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 4te^{-t} \\ e^{-t} \\ 0 \end{pmatrix}.$$

### Exercises 6.4.1

$$1. \mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

$$3. \mathbf{x}(t) = c_1 \left[ \cos 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin 2t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] + c_2 \left[ \cos 2t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right];$$

$$\mathbf{x}(t) = \cos 2t \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \sin 2t \begin{pmatrix} 7 \\ -4 \end{pmatrix}.$$

$$5. \mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$7. \mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

$$9. \mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}.$$

$$11. \mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \left[ \cos t \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} + \sin t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] + c_3 e^{2t} \left[ \cos t \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} \right].$$

### Exercises 6.4.2

$$1. \mathbf{v}_1 = e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$$3. \mathbf{v}_1 = e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$5. \mathbf{v}_1 = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{v}_2 = e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

$$7. \mathbf{v}_1 = e^{4t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = e^{-2t} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + t e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

$$9. \mathbf{v}_1 = e^{-2t} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = e^{-2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t e^{-2t} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

$$11. \mathbf{v}_1 = e^{-t} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + t e^{-2t} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$