UH - Math 4377/6308 - Dr. Heier - Fall 2012
Sample Midterm Exam I
Time: 78 min

1. (a) (5 points) Let $X = \{1, 2, 3, 4\}$ and $Y = \{3, 4\}$. Call two subsets A, B of X equivalent if $A \cup Y = B \cup Y$. Prove that this defines an equivalence relation on the set of subsets of X.

(b) (5 points) Let $z = 1+4i$ and $w = -4-3i$. Find $|w|$. Write zw and $\frac{z}{w}$ in the form $a+bi$.

2. (a) (10 points) Let $W_1 = \{(a_1, a_2, a_1 - a_2) | a_1, a_2 \in \mathbb{R}\} \subset \mathbb{R}^3$. Let $W_2 = \{(b, -b, 0) | b \in \mathbb{R}\} \subset \mathbb{R}^3$. Is $W_1 \oplus W_2 = \mathbb{R}^3$? Justify your answer.

(b) (5 points) Let $V = \{f : \mathbb{R} \to \mathbb{R}\}$ be the (infinite dimensional) vector space of all functions $\mathbb{R} \to \mathbb{R}$. Let $W = \{f : \mathbb{R} \to \mathbb{R} | f(1) = f(2) = 0\}$. Is W a subspace of V? Prove your answer.

3. (a) (5 points) Find the condition on a, b, c so that

$$(a, b, c) \in \text{span}\{(1, 1, 2), (3, 0, 3), (-1, 1, 0)\}.$$

(b) (10 points) Find a basis for the following subspace W of $\mathbb{R}^5$:

$$W = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 : a_1 + a_2 + a_3 + a_4 + a_5 = 0, a_2 = 2a_3 = -a_5\}.$$

4. (a) (15 points) Find bases for the kernel and range of $T : \mathbb{R}^5 \to \mathbb{R}^4$,
$(a_1, a_2, a_3, a_4, a_5) \mapsto (a_1 + a_3 - a_4 + a_5, -a_1 + a_2 + a_4, -a_1 + 2a_4, -a_1 + a_2 + a_3 + 2a_4 + a_5)$.

(b) (10 points) Give a complete and explicit list of all linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying both $T(1, 1) = (1, 2)$ and $T(1, 0) = (3, 1)$. For every T on your list, compute $T(0, 1)$.

5. (15 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(a_1, a_2) = (a_1 + 3a_2, -a_1 - a_2)$. Let $\beta = \{(1, 2), (-1, 1)\}$ and $\gamma = \{(2, 1), (2, 0)\}$. Compute $[T]_\beta^\gamma$ and $[T^{-1}]_\gamma^\beta$.

6. (a) (10 points) Let $T : V \to W$ be a linear transformation. Assume that T is one-to-one. Let S be a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

(b) (10 points) Let $T : V \to W$ be a linear transformation. Suppose $\beta = \{v_1, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ is a basis for W.