UH - Math 4377/6308 - Dr. Heier - Fall 2012
Sample Midterm Exam II
Time: 78 min

1. (a) (5 points) Let X, Y, Z be sets. Prove that $(X \cup Y) \backslash Z = (X \backslash Z) \cup (Y \backslash Z)$.

(b) (5 points) Let A, B, C be sets. Let $f : A \to B$ and $g : B \to C$ be functions. Assume that f and g are one-to-one. Prove that the composition $g \circ f$ is one-to-one.

2. (a) (5 points) Determine if the following subset of $\mathbb{R}^3$ is a subspace. Justify your answer carefully:

$$\{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 2a_2 + a_3^2 = 0\}.$$

(b) (5 points) Let W_1, W_2 be two subspaces of a vector space V. Prove that the intersection $W_1 \cap W_2$ is also a subspace of V.

3. (15 points) Find nullity and rank of

$$T : \mathbb{R}^5 \to \mathbb{R}^3, (a_1, a_2, a_3, a_4, a_5) \mapsto (a_1 + a_5, -a_1 + a_2 + a_3, 3a_1 - a_2 - a_3 + 2a_5).$$

4. (a) Let $G = \{(1, 2, 3), (-1, 1, 1), (2, -2, 0)\}$. Let $L = \{(1, 0, 1), (-1, 2, 1)\}$.

(i) (10 points) Show that G is a basis for $\mathbb{R}^3$.

(ii) (10 points) Find a vector $v \in G$ such that $\{v\} \cup L$ is a basis for $\mathbb{R}^3$. Prove the basis property.

5. (15 points) Let $T : V \to V$ be a linear transformation. Let $T \circ T : V \to V$ be the zero transformation, i.e., for all $v \in V$, $T(T(v)) = \vec{0}$. Prove that T is not invertible

(b) (10 points) Let V be a finite-dimensional vector space. Let $T : V \to W$ be a one-to-one linear transformation. Let V_0 be a subspace of V. Prove that $\dim V_0 = \dim T(V_0)$.

(c) (10 points) Let $\{v_1, v_2\}$ be a basis for $\mathbb{R}^2$. Is $\{v_1 + v_2, v_1 - v_2\}$ is basis for $\mathbb{R}^2$? Justify your answer carefully.

6. (10 points) For the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

and the ordered basis $\beta = \{(1, 1), (1, -1)\}$, find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$. Then use the formula to find $[L_A]_\beta$ explicitly.