Selected Solutions to HW #6

§2.1 #3: (a) Prove that T is a linear Transformation.
(b) Find $N(T)$ and $R(T)$. Compute the Nullity and rank(T).
(c) Is T onto? Is T one-to-one?

$T: \mathbb{R}^2 \to \mathbb{R}^3$. $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$

(a) By the definition on pg 65, we want to show $\forall x, y \in V$, $c \in F$ that

- $T(x+y) = T(x) + T(y)$
- $T(cx) = cT(x)$.

- $T((a_1, a_2) + (b_1, b_2)) = T((a_1 + b_1), (a_2 + b_2))$ ← plug this into the given transformation.

$$= ((a_1 + a_2 + b_1 + b_2), 0, (2a_1 + 2b_1 - a_2 - b_2))$$ ← Now manipulate to a form that we endeavour.

$$= (((a_1 + a_2), 0, (2a_1 - a_2)) + ((b_1 + b_2), 0, (2b_1 - b_2))).$$

$$T((a_1, a_2) + (b_1, b_2)) = T(a_1, a_2) + T(b_1, b_2) \quad ✓$$

- $T(c(a_1, a_2)) = T(ca_1, ca_2)$

$$= (ca_1 + ca_2, 0, c2a_1 - ca_2)$$

$$= c(a_1 + a_2, 0, 2a_1 - a_2)$$

$$T(c(a_1, a_2)) = cT(a_1, a_2) \quad ✓ \qquad \therefore T \text{ is a linear Transformation.}$$

(b) For finding $N(T)$, we ask if there are non-zero values for a_1, a_2 that yield the zero transformation?

we need $a_1 + a_2 = 0 \qquad a_1 = -a_2$

$2a_1 - a_2 = 0 \qquad -2a_2 - a_2 = 0$

$-3a_2 = 0 \Rightarrow a_2 = 0 \Rightarrow a_1 = 0 \quad \therefore N(T) = \{\vec{0}\}$

§2.1 #3 (b) cont...

$R(T) = \{(a_1 + a_2, 0, 2a_1 - a_2) : a_1, a_2 \in V\}$

Theorem 2.2 pg 68.

$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(1,0), T(0,1)\})$ → I find this a cleaner and easier way to think a/b $R(T)$. Choose the standard ordered basis for $\mathbb{R}^2$, which is $\{(0,1), (1,0)\}$

$R(T) = \{(1,0,2), (1,0,-1)\}$

* other ways, but all the same.

Now we have that $\dim(N(T)) = \text{Nullity}(T) = 0$, and
$\dim(R(T)) = \text{rank}(T) = 2$

(c)

T is one-to-one (injective) since $N(T) = \{\vec{0}\}$

T is <u>NOT</u> onto (surjective) since $\dim(R(T)) < \dim \mathbb{R}^3$, i.e. $2 < 3$.

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§2.1 #11: Prove that there exists a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T_1(1,1) = (1,0,2)$ and $T_2(2,3) = (1,-1,4)$. Calculate $T(8,11)$.

* There are several ways to go about this, so I'll share the shortest.
We want to be able to represent $T(\beta)$ in terms of T_1, T_2. Let $\beta = \{(1,0),(0,1)\}$.

$T(1,0) = 3T_1 - T_2 = (3,0,6) - (1,-1,4) = (2,1,2)$

$T(0,1) = T_2 - 2T_1 = (1,-1,4) - (2,0,4) = (-1,-1,0)$

Then $T(x,y) = T(x,0) + T(0,y) = xT(1,0) + yT(0,1) = x(2,1,2) + y(-1,-1,0)$

$\boxed{T(x,y) = (2x - y, x - y, 2x)}$ This proves that there exists a linear Transformation $T: V \to W$,
$T: \mathbb{R}^2 \to \mathbb{R}^3$

Compute $T(8,11) = (5, -3, 16)$.

§2.1 #14 : Let V, W be vector spaces and $T: V \to W$ be linear.

(a) ($\Rightarrow$) Assume T is one-to-one. Suppose that $\{v_1, \ldots, v_k\}$ is a linearly independent subset of V. We w.t.s. that $\{T(v_1), \ldots, T(v_k)\}$ is also linearly independent and subset of W.

Note by assumption. $\sum_{i=1}^{k} a_i v_i = 0 \Rightarrow a_i = 0 \ \forall i = 1..k$

looking at $b_1 T(v_1) + \ldots + b_k T(v_k) = 0$ (*)

$T(b_1 v_1 + \ldots + b_k v_k) = 0 \Rightarrow b_1 v_1 + \ldots b_k v_k = 0$ by injective assumption.

$\Rightarrow b_1 = \ldots = b_k = 0$ by assumption

then we see that (*) is linearly independent. ■

($\Leftarrow$) Sps T carries independent sets in V to independent sets in W

looking at $T(a_1 v_1 + a_2 v_2 + \ldots a_k v_k) = 0 = a_1 T(v_1) + \ldots + a_k T(v_k)$. This implies that $a_1 = \ldots = a_k = 0 \Rightarrow a_1 v_1 + \ldots + a_k v_k = 0$.

Then for any ind. subset of V, vectors in that subset can be written in the form $x = \sum_{i=1}^{k} a_i \cancel{\ } v_i \Rightarrow x = 0$ when $\sum_{i=1}^{k} a_i v_i = 0$, as shown above.

$\therefore T$ is one-to-one. ■

(b) ($\Rightarrow$) same proof as (a)($\Rightarrow$).

($\Leftarrow$) Assume $T(S)$ is linearly independent. Let $S = \{v_1, \ldots, v_k\}$

looking at $a_1 v_1 + \ldots + a_k v_k = 0$ * Then take the transformation

$T(a_1 v_1 + \ldots + a_k v_k) = 0$

$a_1 T(v_1) + \ldots + a_k T(v_k) = 0 \Rightarrow a_1 = \ldots = a_k = 0$ by assumption.

These are the same coefficients in $a_1 v_1 + \ldots + a_k v_k = 0$ that we started w/

$\therefore S$ is linearly independent. ■

§2.1 #14 (c)

$\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is bijective.

Prove that $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W

- β is linearly independent
- $T(\beta)$ is linearly independent by part (b)

T onto means that $\forall\, w \in W\ \exists\, v \in V$ s.t. $T(v) = w$.

β is a basis so by theorem 1.8 (pg 43) $v = \sum_{i=1}^{n} a_i v_i \quad \forall v \in V$

and $w = T(v)$, or $w = \sum_{i=1}^{n} T(a_i v_i) \Rightarrow w \in \text{span}(T(\beta))$ and $W = \text{span}(T(\beta))$

since T is one-to-one

$\therefore T(\beta)$ is a basis for W

$\uparrow$

If this doesn't yet seem immediate then refer to Corollary 2 (a) on pg 47:

"Let V be a V.S. with $\dim(V) = n$. Then any finite generating set for V contains at least n vectors. <u>AND</u> <u>a generating set for V that contains exactly n vectors is a basis for V.</u>"