Solutions to Homework 9

FALL 2017
Dr. Heier - Adv. Linear Algebra

#3 §3.2 Problem 5.

For this problem we want to find the rank of a matrix and its inverse if possible. Look for dependency in the columns or rows. If they appear to be independent then transform the augmented matrix $(A|I_n)$ into $(I_n|A^{-1})$.

(a) $A=\begin{pmatrix}1&2\\1&1\end{pmatrix}$; we can see immediately that rank = 2.

$$\left(\begin{array}{cc|cc}1&2&1&0\\1&1&0&1\end{array}\right)\to\left(\begin{array}{cc|cc}1&1&0&1\\1&2&1&0\end{array}\right)\to\left(\begin{array}{cc|cc}1&1&0&1\\0&1&1&-1\end{array}\right)\to\left(\begin{array}{cc|cc}1&0&-1&2\\0&1&1&-1\end{array}\right)$$

$$(A\,\vdots\,I_2)\longrightarrow(I_2\,\vdots\,A^{-1}).$$

$$A^{-1}=\begin{pmatrix}-1&2\\1&-1\end{pmatrix}$$

(b) $A=\begin{pmatrix}1&2\\2&4\end{pmatrix}$; $2\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}$ so rank = 1 and there exists no inverse.

(c) $A=\begin{pmatrix}1&2&1\\1&3&4\\2&3&-1\end{pmatrix}$; $\begin{pmatrix}1\\4\\-1\end{pmatrix}=-5\begin{pmatrix}1\\1\\2\end{pmatrix}+3\begin{pmatrix}2\\3\\3\end{pmatrix}$, and $\begin{pmatrix}1\\1\\2\end{pmatrix},\begin{pmatrix}2\\3\\3\end{pmatrix}$ are linearly independent.

So, the rank(A) = 2 and there is no inverse.

#3 cont... (d) $A = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{pmatrix}$; the rank(A) = 3 by inspection, so, we reduce.

$$\left(\begin{array}{ccc|ccc} 0 & -2 & 4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 4 & -5 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 0 & 0 \\ 2 & 4 & -5 & 0 & 0 & 1 \end{array}\right) \to \cdots \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -3 & 8 & -4 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 3 & -1 \\ 0 & -2 & 0 & -3 & 8 & -4 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 3 & -1 \\ 0 & 1 & 0 & 3/2 & -4 & 2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right); \quad A^{-1} = \begin{pmatrix} -1/2 & 3 & -1 \\ 3/2 & -4 & 2 \\ 1 & -2 & 1 \end{pmatrix}$$

(e) $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$; the rank(A) = 3.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\cdots} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/6 & 1/3 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \end{array}\right) \to \cdots \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/6 & -1/3 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -1/6 & 1/3 & 1/2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1/6 & -1/3 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/6 & 1/3 & 1/2 \end{pmatrix}.$$

(f) $A = \begin{pmatrix} v_1 & v_2 & v_3 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$; $v_1 = v_3$ and v_2, v_3 are linearly independent so rank(A) = 2

There is NO Inverse.

(g) $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & 2 & -3 \end{pmatrix}$; These columns are indepedent by inspection, so rank(A) = 4.

Modern times

$$A^{-1} = \begin{pmatrix} -51 & 15 & 7 & 12 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{pmatrix}.$$

(h) $A = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix}$; $v_4 = -1v_1 + 5v_2 + 2v_3$.

rank(A) = 4

No Inverse.

Solutions to HW 9

#5 §3.2 Problem 21: Let $A \in M_{m\times n}(F)$. Rank$(A) = m$.

Prove that $\exists$ $B \in M_{n\times m}$ s.t. $AB = I_m$. This proof uses theorem 3.6 (pg 155) and its corollaries that follow. Corollary 1 is especially important and really the only thing we need. I recommend reading through the proof, it should help you to understand this section. Also see theorem 3.1 (pg 149).

proof: Given corollary 1 to theorem 3.6 and rank$(A) = m$ $(m=r)$ our matrix D will be $D = (I_m \vdots 0) = EAG$. Where E, G are $m\times m$ and $n\times n$ invertible matrices which are actually just the product of elementary row/column operations. (see proof as mentioned)

Since $r = m$ we can take $E = I_m$ so that $D = AG$ where now $(I_m \vdots 0) = AG$. See that $(AG)^T = \begin{pmatrix} I_m \\ \cdots \\ 0 \end{pmatrix}$ and that $AG(AG)^T = (I_m \vdots 0)\begin{pmatrix} I_m \\ \cdots \\ 0 \end{pmatrix} = I_m$.

$$I_m = AG(AG)^T$$

$$I_m = AG_1G_2\cdots G_q G_q^T G_{q-1}^T \cdots G_2^T G_1^T A^T$$; see that $(G_i G_i^T)$ for $i = 1, \ldots, q$ is a computable $n\times n$ matrix with full rank (elementary matrices are invertible)

$$I_m = AGG^TA^T$$

where $B = GG^TA^T$. $\therefore$ $\underline{\underline{I_m = AB}}$.

• Just to check dimensionality. GG^T has dimension $(n\times n)$ since and A^T has dim. $(n\times m)$. So GG^TA^T has dimension $(n\times m)$ as desired. ∎

Solutions to HW9

#6 §3.3 Problem 5: Give an example of a system of n linear equations in n unknowns with ∞-many solutions.

Many of you said the (n×n) zero matrix. It's technically correct, like a rhyme bereft, or how not being right makes you feel left. But think of a truth using more than naught, so that the next pair of eyes have food for thought.

Theorem 3.10 (pg 174) tips us off to the fact that an (n×n) which is invertible will have only one sol'n to $Ax=b$. We want ∞-many so we just need the matrix to have less than full rank. Having two (or more) columns/row which are dependent (rank A = rank A^T).