UH - Math 4377/6308 - Dr. Heier - Spring 2010
Sample Final Exam
Time: 175 min

1. (a) (3 points) Let $z = a + ib$ be a complex number. Prove that $|z|^2 = z\bar{z}$.

(b) (4 points) Solve the equation $z(1 + i) = i$ for z.

(c) (3 points) Is the function $f : (1, 4) \to (1, 2),\ x \mapsto \sqrt{x}$ one-to-one? Onto?

2. (a) (5 points) Determine if the following subset of $\mathbb{R}^2$ is a subspace. Justify your answer carefully:

$$\{(a_1, a_2) \in \mathbb{R}^2 : a_1 \cdot a_2 = 0\}.$$

(b) (5 points) Determine if the following subsets of the vector space of 2×2 matrices with real entries are subspaces. You may assume as true that the set of 2×2 matrices with real entries forms a vector space with the usual addition and scalar multiplication.

(a) $\left\{\begin{pmatrix} a_1 & a_1 + a_2 \\ a_2 & 0 \end{pmatrix} : a_1, a_2 \in \mathbb{R}\right\}$

(b) $\left\{\begin{pmatrix} a_1 & a_1 \cdot a_2 \\ a_2 & a_3 \end{pmatrix} : a_1, a_2, a_3 \in \mathbb{R}\right\}$

3. (a) (5 points) Find bases for the kernel and range of

$$T : \mathbb{R}^5 \to \mathbb{R}^4, (a_1, a_2, a_3, a_4, a_5) \mapsto (a_1 + a_4 + a_5, -a_1 + a_2 + a_4, a_5 - a_4, a_1 + 2a_5).$$

(b) (5 points) Let $G = \{(1, -1, 0, 1), (1, 0, 1, 0), (1, 2, 4, 2), (0, 2, 2, 2)\}$. Let $L = \{(2, -4, -3, 0)\}$. Find a subset $H \subset G$ of cardinality 3 such that $H \cup L$ spans $\mathbb{R}^4$. Prove the spanning property with an explicit computation.

4. (a) (5 points) Find the rank of

$$\begin{pmatrix} 2 & 2 & 0 & 1 \\ 3 & 1 & 3 & 3 \\ 5 & 3 & 3 & 4 \\ 7 & 5 & 3 & 5 \\ 8 & 4 & 6 & 7 \end{pmatrix}.$$

(b) (5 points) Give an example of $A, B \in M_{4\times 4}(\mathbb{R})$ such that both A and B have rank 2, but their product AB has rank 1.

5. (10 points) Find the inverse of

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}.$$

6. (a) (5 points) Let

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 3 & 1 & 3 \\ 5 & 3 & -2 \end{pmatrix}$$

and let

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 3 & 1 & 3 \\ 4 & 2 & 1 \end{pmatrix}.$$

Find $\det(A)$, $\det(B)$ and $\det(AB)$.

(b) (5 points) Compute the determinant of

$$\begin{pmatrix} 5 & -1 & 0 & 1 \\ 4 & 0 & 1 & 0 \\ 5 & 2 & 5 & 3 \\ 4 & -4 & -3 & 0 \end{pmatrix}.$$

7. (a) (5 points) Let $A, B \in M_{n\times n}(\mathbb{R})$ be such that $AB = -BA$. Prove that if n is odd, then at least one of the two matrices A, B is not invertible.

(b) (5 points) Let $A \in M_{n\times n}(\mathbb{R})$ have two distinct eigenvalues λ_1, λ_2. Give a necessary and sufficient criterion in terms of $\dim E_{\lambda_1}$ and $\dim E_{\lambda_2}$ for the diagonalizability of A.

8. (a) (5 points) Find the eigenvalues of

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

(b) (5 points) Find the eigenvectors of A.

(c) (5 points) Find a matrix Q such that $Q^{-1}AQ$ is diagonal.

9. (15 points) Is the matrix

$$A = \begin{pmatrix} 1 & 0 & -8 \\ -4 & 9 & -4 \\ -10 & 0 & -1 \end{pmatrix}$$

diagonalizable? If yes, give a basis of eigenvectors of A for $\mathbb{R}^3$.