UH - Math 4378/6309 - Dr. Heier - Spring 2011
Sample Final Exam
Time: 175 min

1. (a) (5 points) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V. Prove that $N(T)$ and $R(T)$ are T-invariant.

(b) (5 points) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V. Let $v \in V \setminus \{\vec{0}\}$, and let W be the T-cyclic subspace generated by v. Prove that for every $w \in W$, there exists a polynomial $g(t)$ such that $w = g(T)(v)$.

(c) (5 points) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V of dimension n. Let $v \in V \setminus \{\vec{0}\}$ and let W_1 be the T-cyclic subspace generated by v. Let W_2 be the T-cyclic subspace generated by $T(v)$. Give concrete examples of the above data where i) $W_1 = W_2$ and ii) $W_1 \neq W_2$.

2. (a) (5 points) Let V be an inner product vector space such that $\|T(x)\| = \|x\|$ for all $x \in V$. Prove that T is one-to-one.

(b) (5 points) Let V be an inner product vector space over the reals. Prove the polar identity for all $x, y \in V$:

$$\langle x, y \rangle = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2.$$

3. (a) (5 points) Prove that a matrix which is both unitary and lower triangular is diagonal.

(b) (5 points) Let V be a real inner product space. Let $f : V \to V$ be a function. Define what it means for f to be a rigid motion.

(c) (5 points) Prove that the composition of any two rigid motions is a rigid motion.

4. (a) (5 points) Let V be a finite-dimensional inner product vector space. Let W be a subspace of V. Let T be the orthogonal projection of V on W. Prove that $\|T(x)\| \leq \|x\|$ for all $x \in V$.

(b) (5 points) Let V be a finite-dimensional vector space over the complex numbers. Let P be an orthogonal projection. Prove that $2P - I$ is unitary. (Hint: Use a theorem from class.)

5. (a) (5 points) Prove directly, i.e., without resorting to any theorems from class, that a cycle of generalized eigenvectors is linearly independent.

(b) (5 points) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V. Let λ be an eigenvalue of T. State the definition of the generalized eigenspace K_λ.

(c) (5 points) In the situation of (b) above, prove that K_λ is T-invariant.

6. (20 points) Find a Jordan canonical form J and a Jordan canonical basis β for the operator $T : \mathbb{R}^4 \to \mathbb{R}^4$ given in standard coordinates by the matrix

$$A = \begin{pmatrix} 2 & -2 & -2 & -2 \\ -4 & 0 & -2 & -6 \\ 2 & 1 & 3 & 3 \\ 2 & 3 & 3 & 7 \end{pmatrix}.$$

You may use without proof that the characteristic polynomial of A is $(t-2)^2(t-4)^2$.

7. (a) (10 points) Let $T : V \to V$ be a linear operator on a n-dimensional vector space V. Let T have n distinct eigenvalues. Cite results from class to prove that the minimal polynomial and the characteristic polynomial are identical up to a factor of ± 1.

(b) (5 points) Find the minimal polynomial of

$$A = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$