UH - Math 6303 - Dr. Heier - Spring 2014
HW 4
Due 04/17, at the beginning of class.

Use regular sheets of paper, stapled together.
Don't forget to write your name on page 1.

1. (1 point) Suppose V is a finite algebraic set in $\mathbb{A}^n$. If V has m points, prove that $k[V]$ is isomorphic as a k-algebra to k^m. Hint: Use the Chinese Remainder Theorem.

2. (1 point) Let k be a finite field. Prove that every subset of $\mathbb{A}^n$ is an affine algebraic set.

3. (1 point) Let k be a field. Identify the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with entries in k with the point (a, b, c, d) in $\mathbb{A}^4$. Show that the group $SL_2(k)$ of matrices of determinant 1 is an algebraic set in $\mathbb{A}^4$.

4. (3 points) Let $V = \mathcal{Z}(xy - z) \subset \mathbb{A}^3$. Prove that V is isomorphic to $\mathbb{A}^2$ and provide an explicit isomorphism φ and associated k-algebra isomorphism $\tilde{\varphi} : k[V] \to k[\mathbb{A}^2]$, along with their inverses. Is $V = \mathcal{Z}(xy - z^2)$ isomorphic to $\mathbb{A}^2$?

5. (2 points) Let I, J be ideals in the ring R. Prove the following statements:

(a) If $I^k \subseteq J$ for some $k \geq 1$ then $\operatorname{rad} I \subseteq \operatorname{rad} J$.

(b) If $I^k \subseteq J \subseteq I$ for some $k \geq 1$ then $\operatorname{rad} I = \operatorname{rad} J$.

(c) $\operatorname{rad}(IJ) = \operatorname{rad}(I \cap J) = \operatorname{rad} I \cap \operatorname{rad} J$.

6. (2 points) Prove that for k a finite field the Zariski topology is the same as the discrete topology, i.e., every subset is closed and open.