UH - Math 3330 - Dr. Heier - Spring 2014
HW 3 - Solutions to *Selected* Homework Problems
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1. (Section 1.7, Problem 8) Prove that

$$xRy \text{ if and only if } x + 3y \text{ is a multiple of } 4$$

is an equivalence relation.

Proof. To prove R is an equivalence relation, we must show that it is reflexive, symmetric, and transitive.

Reflexive: $x + 3x = 4x$, which is a multiple of 4. So xRx, and R is reflexive.
Symmetric: Assume xRy–that is $x + 3y$ is a multiple of 4. This means that $\exists\ k \in \mathbb{Z}$ such that $x + 3y = 4k$. So $x = 4k - 3y$. Thus,

$$y + 3x = y + 3(4k - 3y) = 12k - 8y = 4(4k - 2y)$$

Because $(4k - 2y) \in \mathbb{Z}$, $y + 3x$ is a multiple of 4. Hence, yRx and R is symmetric.
Transitive: Assume xRy and yRz. This means that $\exists\ k, m \in \mathbb{Z}$ such that $x + 3y = 4k$ and $y + 3z = 4m$. Then $x + 3y + y + 3z = 4k + 4m$, which implies

$$x + 3z = 4k + 4m - 4y = 4(k + m - y)$$

Because $4(k + m - y) \in \mathbb{Z}$, $x + 3z$ is a multiple of 4. So xRz and R is transitive. □

5. (Section 2.2, Problem 39) Prove that $1 + 2n < 2^n$, for all integers $n \geq 3$.

Proof. When $n = 3$, the LHS becomes $1 + 2(3) = 7$, and the RHS becomes $2^3 = 8$. Because $7 < 8$, the given statement is true for $n = 3$.

Now assume that the statement holds true for $n = k$— that is, $1 + 2k < 2^k$ holds true $\forall\ k \geq 3$. Now we must prove that the statement holds when $n = k + 1$. We have

$$\begin{aligned} 1 + 2(k+1) = 1 + 2k + 2 &< 2^k + 2 \quad \text{(by our assumption)} \\ &< 2^k + 2^k \quad \text{(because } k \geq 3\text{)} \\ &= 2(2^k) \\ &= 2^{k+1} \end{aligned}$$

Therefore, $1 + 2n < 2^n, \forall\ n \geq 3$. □

6. (Section 2.2, Problem 45) Show that if the statement

$$1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n$$

is assumed to be true for $n = k$, then it can be proved to be true for $n = k + 1$. Is the statement true for all positive integers n? Why?

Solution. Assume that the statement holds true for $n = k$—that is

$$1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k$$

is true. So adding 2^k to both sides of the equation above yields

$$1 + 2 + 2^2 + \cdots + 2^{k-1} + 2^k = 2^k + 2^k = 2(2^k) = 2^{k+1}$$

Thus, the statement is true for $n = k + 1$.

No, the statement is not true for all positive integers n. When $n = 1$, the LHS is $2^{1-1} = 2^0 = 1$, and the RHS is $2^1 = 2$. Because $1 \neq 2$, the statement is not true.

8. (Section 2.3, Problem 14) With $a = -5316$ and $b = 171$, find the q and r that satisfy the conditions in the Division Algorithm.

Solution. Using the Division Algorithm for $a_0 = 5316$ and $b = 171$, we get

$$5316 = 171(31) + 15$$

Now, for $a = -5316$ and $b = 171$, we simply multiply both sides of the equation above by (-1). This gives us

$$-5316 = 171(-31) + (-15)$$

Note that the Division Algorithm requires that $0 \leq r < b$. So, to obtain an expression with positive remainder, we add and subtract 171 to the RHS of the equation. This yields

$$-5316 = 171(-31) + (-15) = 171(-31) + 171(-1) + (-15) + 171 = 171(-32) + 156$$

Therefore, $q = -32$ and $r = 156$ are the values of q and r which satisfy the conditions of the Division Algorithm.

9. (Section 2.3, Problem 25) Let a, b, c be integers. Prove or disprove that $a|bc$ implies $a|b$ or $a|c$.

Solution. This statement is **false**. We will disprove this using a counterexample: Let $a = 10, b = 5$, and $c = 2$. We have that $10 \mid (5 \cdot 2)$, but $10 \nmid 5$ and $10 \nmid 2$.