UH - Math 3330 - Dr. Heier - Spring 2014
HW 4 - Solutions to *Selected* Homework Problems
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1. (Section 2.4, Problem 3 (d) and (e)) Find the great common divisor (a, b) and integers m and n such that $(a, b) = am + bn$.

(d) $a = 52, b = 124$.
Solution. Note that $a = 52 = 4 \times 13$ and $b = 124 = 4 \times 31$. Thus, $\boxed{(52, 124) = 4}$.
Using the Divsion Algorithm, we have

$$124 = 52(2) + 20, \quad 52 = 20(2) + 12, \quad 20 = 12(1) + 8, \quad 12 = 8(1) + 4, \quad 8 = 4(2)$$

Thus,

$$\begin{aligned} 4 &= 12 - 8 \\ &= 12 - (20 - 12) \\ &= 2(12) - 20 \\ &= 2(52 - 2(20)) - 20 \\ &= 2(52) - 5(20) \\ &= 2(52) - 5(124 - 52(2)) \\ &= 12(52) - 5(124) \end{aligned}$$

Thus, $\boxed{m = 12 \text{ and } n = -5.}$

(e) $a = 414, b = -33$
Solution. Note that $a = 414 = 2(3)(3)(23)$, and $b = -33 = 3(-11)$. Thus, $\boxed{(414, -33) = 3}$
By the Division Algorithm, we have

$$414 = (-33)(-12) + 18, \quad -33 = 18(-2) + 3, \quad 18 = 3(6)$$

Thus,

$$\begin{aligned} 3 &= -33 + 18(2) \\ &= -33 + (414 - 33(12))(2) \\ &= -25(33) + 2(414) \\ &= 25(-33) + 2(414) \end{aligned}$$

Thus, $\boxed{m = 2 \text{ and } n = 25.}$

4. (Section 2.4, Problem 8) Let a, b, and c be integers such that $a \neq 0$. Prove that if $a \mid bc$, then $a \mid c \cdot (a, b)$.

Proof. We know that there exists $m, n \in \mathbb{Z}$ such that $(a, b) = am + bn$. Multiplying both sides by c yields

$$c \cdot (a, b) = cam + cbn = acm + bcn$$

Because $a \mid bc$, $\exists\, k \in \mathbb{Z}$ such that $bc = ak$. After substituting ,we have

$$c \cdot (a, b) = acm + akn = a(cm + kn)$$

Since $cm + kn \in \mathbb{Z}$, $a \mid c \cdot (a, b)$. $\square$

5. (Section 2.4, Problem 11) Prove that if $d = (a, b)$, $a \mid c$ and $b \mid c$, then $ab \mid cd$.

Proof. Assume that $a \mid c$ and $b \mid c$. This means that $\exists\, k, l \in \mathbb{Z}$ such that $c = ak$ and $c = bl$. Also, because $d = (a, b)$, then $\exists\, m, n \in \mathbb{Z}$ such that $d = am + bn$. Therefore,

$$\begin{aligned} cd &= c(am + bn) \\ &= cam + bcn \\ &= bl(am) + ak(bn) \\ &= ab(lm + kn) \end{aligned}$$

Thus, $ab \mid cd$. □

7. (Section 2.4, Problem 21) Let $(a, b) = 1$. Prove $(a^2, b^2) = 1$.

Proof. Let $d = (a^2, b^2)$ and assume $(a, b) = 1$.For sake of contradiction, assume that $d \neq 1$. Because it is not equal to 1, then there exists a prime $p \in \mathbb{Z}$ such that $p \mid d$. Because $p \mid d$ and $d \mid a^2$, then transitivity implies that $p \mid a^2$. Thus, $p \mid a$. Similarly, because $p \mid d$ and $d \mid b^2$, then $p \mid b^2$. So $p \mid b$. Hence, $p \mid a$ and $p \mid b$. Because $1 = (a, b)$, $p \mid 1 \iff p = 1$—but we assumed p is prime. $\lightning$ Thus, $d = (a^2, b^2) = 1$. □

9. (Section 2.5, Problem 7) Find a solution $x \in \mathbb{Z}, 0 \leq x < n$ for the following congruence $ax = b(modn)$. Note that a and n are relatively prime.

$$8x \equiv 1(\text{mod}21)$$

Solution. First note that 8 and 21 are relatively prime, i.e. $(8, 21) = 1$. Thus, $\exists\, m, n \in \mathbb{Z}$ such that $1 = 8m + 21n$. Using the Division algorithm, we have

$$21 = 8(2) + 5, \;\; 8 = 5(1) + 3, \;\; 5 = 3(1) + 2, \;\; 3 = 2(1) + 1, \;\; 2 = 1(2)$$

Solving for the remainders yields

$$5 = 21 - 8(2), \;\; 3 = 8 - 5(1), \;\; 2 = 5 - 3(1), \;\; 1 = 3 - 2(1)$$

Therefore, we get

$$\begin{aligned} 1 &= 3 - 2(1) \\ &= 3 - [5 - 3(1)](1) \\ &= 3(2) + 5(-1) \\ &= [8 - 5(1)](2) + 5(-1) \\ &= 8(2) + 5(-3) \\ &= 8(2) + [21 - 8(2)](-3) \\ &= 8(8) + 21(-3) \end{aligned}$$

Hence, $21 \mid (1 - 8(8))$, and $1 \equiv 8(8)(\text{mod}21)$. So $\boxed{x = 8}$.