UH - Math 3330 - Dr. Heier - Spring 2014
HW 8 - Solutions to *Selected* Homework Problems
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3. (Section 3.5, Problem 26) Prove that any infinite cyclic group is isomorphic to $\mathbb{Z}$ under addition.

Proof. Let $G = \langle a \rangle$ be any infinite cyclic group. Then $G = \{a^n \mid n \in \mathbb{Z}\}$. Define a map

$$\varphi : \mathbb{Z} \to G, \quad \varphi(n) = a^n$$

We claim that this map is an isomorphism.
Let $n, m \in \mathbb{Z}$. Then, $\varphi(m+n) = a^{m+n} = a^m \cdot a^n = \varphi(m) \cdot \varphi(n)$. Thus, φ is a homomorphism.

Now let $a^k \in G$. Then, a^k has a pre-image in $\mathbb{Z}$—namely $k \in \mathbb{Z}$. So, φ is onto. Lastly, assume $\varphi(m) = \varphi(n)$. Then, $a^m = a^n$. Taking the inverse of a^n on both sides, we have $a^m \cdot a^{-n} = e \iff a^{m-n} = a^0$. Thus, $m - n = 0 \iff m = n$. So, φ is one-to-one. Hence, φ is an isomorphism and $G \cong \mathbb{Z}$. □

4. (Section 3.5, Problem 33) if G and H are groups and $\varphi : G \to H$ is an isomorphism, prove that a and $\varphi(a)$ have the same order, for any $a \in G$.

Proof. Let e_G and e_H denote the identity of G and H, respectively. Let $a \in G$ and let $order(a) = n$. This means that $a^n = e_G$. Then because φ is an isomorphism, $\varphi(a^n) = \varphi(e_G) = e_H$. Also, we have that $\varphi(a^n) = [\varphi(a)]^n$. Thus, $[\varphi(a)]^n = e_H$.

Now we must show that n is the *smallest* integer n such that $[\varphi(a)]^n = e_H$. For sake of contradiction, assume that there exists another integer $m \in \mathbb{Z}^+$ such that $m < n$ and $[\varphi(a)]^m = e_H$. Then $[\varphi(a)]^m = \varphi(a^m) = \varphi(e_G)$. Because φ is injective, we must have that $a^m = e_G$. But we already assumed that n is the least positive integer such that $a^n = e_G$. Thus, we have a contradiction. $\lightning$ □

5. (Section 3.6, Problem 4) Consider the additive group $\mathbb{Z}$ and the multiplicative group $G = \{1, i, -1, -i\}$ and define $\varphi : \mathbb{Z} \to G$ by $\varphi(n) = i^n$. Prove that φ is a homomorphism and find $\ker \varphi$. Is φ an epimorphism? Is φ a monomorphism?

Solution. Let $m, n \in \mathbb{Z}$. Then $\varphi(m+n) = i^{m+n} = i^m \cdot i^n = \varphi(m) \cdot \varphi(n)$. So, φ is a homomorphism.

By definition, $\ker \varphi = \{n \in \mathbb{Z} \mid \varphi(n) = 1\}$. Recall that $i^4 = 1$. Thus, $\boxed{\ker \varphi = \{4k \mid k \in \mathbb{Z}\}}$. We verify this as follows: $\varphi(4k) = i^{4k} = (i^4)^k = 1^k = 1$.

Let $i^m \in G$. Then there exists a pre-image in $\mathbb{Z}$—namely $m \in \mathbb{Z}$, such that $\varphi(m) = i^m$. Thus, φ is onto and **is an epimorphism**.

Note that φ is not one-to one because $\varphi(4) = 1 = \varphi(8)$, but $4 \neq 8$. So φ is **not a monomorphism.**

8. (Section 4.1, Problem 1e) Express the following permutation as a product of disjoint cycles and the the orbits of each permutation.

Solution: Given

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 1 & 2 & 7 \end{bmatrix}$$

The permutation can be written as: $\boxed{(135)(246)(7)}$.

The orbits of the permutation are $\boxed{\{135\}\{246\}\{7\}}$

9. Express the permutation in the previous problem as a product of transpositions.

Solution. The permutation in Problem 1e can be written as the following product of transpositions: $\boxed{(15)(13)(26)(24)}$

10. (Section 4.1, Problem 9b) Compute f^2, f^3, f^{-1} for

$$f = (2, 7, 4, 3, 5)$$

Solution.

$$\begin{aligned} f^2 &= (27435)(27435) \\ &= (24573) \\ f^3 &= (37542) \\ f^{-1} &= (53472) \end{aligned}$$