4/22/2014 Abstract Algebra

* Possible quiz in class on Thursday, April 24^{th}

* HW # 11 will be due in class on Thurs, April 24^{th}

Chapter 5 Section 1 cont.

Recall: Defn 5.2 $\Rightarrow$ Let R_2 be a ring. If $R_1 \subset R_2$ is also a ring with respect to "+" and "$\cdot$", then R_1 is called a subring

* To show $R_1 \subset R_2$ is a subring, we don't need to go through the LONG definition of a ring. Instead we will use the following,

Theorem 5.3 Equivalent Defn #1 for Subrings

$R_1 \subset R_2$ is a subring iff

(1) $R_1 \neq \phi$

(2) $x, y \in R_1 \Rightarrow x+y \in R_1$ (checking closedness
$xy \in R_1$ under "+" and "$\cdot$")

(3) $x \in R_1 \Rightarrow -x \in R_1$ (it's additive inverse is in R_1)

Theorem 5.4 Equivalent Defn #2 for Subrings

$R_1 \subset R_2 \Longleftrightarrow$

(1) $R_1 \neq \phi$

(2) $x, y \in R_1 \Rightarrow x-y \in R_1$ and $xy \in R_1$

Ex: $R_1 = \{m + n\sqrt{2} \mid m,n \in \mathbb{Z}\} \subset \mathbb{R}$

Is this a subring?

$\Rightarrow$ Using Thm 5.4,

(1) $0 + 0\sqrt{2} \in R_1 \Rightarrow R_1 \neq \phi$

(2) Let $a + b\sqrt{2}$, $c + d\sqrt{2} \in R_1$, where $a-c, b-d \in \mathbb{Z}$

then, $(a+b\sqrt{2}) - (c+d\sqrt{2})$
$= (a-c) + (\sqrt{2})(b-d) \in R_1$ ✓

then, $(a+b\sqrt{2})(c+d\sqrt{2})$
$= \underbrace{ac}_{\in \mathbb{Z}} + (\sqrt{2})\underbrace{(ad+bc)}_{\in \mathbb{Z}} + \underbrace{2bd}_{\in \mathbb{Z}}$ ✓

$\boxed{\therefore R_1 \text{ is a subring}}$

Note: "$\cdot$" does not need to be commutative

Recall: For R to be a ring, R need not have a multiplicative identity nor does it have to be commutative. But if R has these properties.... then NICE.

Defn: Let R be a ring. If $\exists e \in R$ such that $e \cdot x = x \cdot e = x$, then e is called a unity. If R has unity, R is a ring with unity.

Note: If "$\circ$" ~~x~~ in R is commutative, then R is called a commutative ring.

Note: "e" is unique

ex: $\mathbb{Z}$ has unity: 1
$\mathbb{Z}_n$ has unity: $[1]$

ex: $\mathbb{E}$, set of all even integers : No unity; it is commutative

ex: $M_{2\times2}(\mathbb{R})$; unity $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Not commutative

ex: $M_{2\times2}(\mathbb{E})$; unity? There is no unity present because $1 \notin \mathbb{E}$.

Not commutative

Note: If R has unity $\Longrightarrow$ question if inverses exist

Defn: Let R be a ring with unity
5.7 Let $x \in R$. If $\exists a \in R$ such that $xa = ax = e$, then a is called the multiplicative inverse of x. We write $a = x^{-1}$

If x has a multiplicative inverse, we call

x as "a unit".

<u>Note:</u> unit = invertible element in R

* Multiplicative inverses are unique.

Relatively Prime

$\Downarrow$

$\gcd(x,n) = 1$

<u>Ex:</u> In $\mathbb{Z}_n$, the units (invertible elements) are elements $[x] \in \mathbb{Z}_n$ such that "x" and "n" are <u>relatively prime</u>.

<u>Ex:</u> In $\mathbb{Z}_{10}$: $[1], [3], [7], [9]$ are the units.

$\mathbb{Z}_{10} = \{[0],[2],[4],[6],[8]\}$

<u>Note:</u>

0 = additive identity

<u>Theorem 5.9</u> - Let R be a ring. Then $a \cdot 0 = 0 \cdot a = 0$, $\forall a \in R$.

<u>Proof:</u>

$$\begin{aligned} 0 \cdot a &= 0 \cdot a + 0 \quad \text{(additive identity)} \\ &= 0 \cdot a + (0 \cdot a - 0 \cdot a) \\ &= (0 \cdot a + 0 \cdot a) - 0 \cdot a \quad \text{(associative)} \\ &= (0 + 0) \cdot a - 0 \cdot a \\ &= 0 \cdot a - 0 \cdot a \\ &= 0 \end{aligned}$$

Similarly, $a \cdot 0 = 0$ $\boxed{\text{QED}}$

Theorem 5.9 $\Longrightarrow$ if one of the factors is 0 $\Longrightarrow$ product is 0

$\not\Longleftarrow$

converse may ~~or~~ may not be true.

Defn 5.10:

Let R be a ring. If $a \neq 0 \in R$, and $\exists b \neq 0 \in R$ such that $ab = 0$ or $ba = 0$, we call "a" a zero divisor of R.

Same could be said of "b"

Ex: In $\mathbb{Z}_{10}$: $[2][5]$

$\not\equiv \quad \not\equiv$

$[0]\ [0]$

notation indicates that these are not equivalent to the equivalence class of 0 (aka $[0]$).

so, $[2][5] = [2 \cdot 5]$
$= [10] = [0] \Longrightarrow$ $[2], [5]$ are zero divisors

Theorem (Additive Inverses and Products)

5.11

Let R be a ring. Then $\forall x, y, z \in R$

(1) $(-x)(y) = -(xy)$

(2) $x(-y) = -(xy)$

(3) $(-x)(-y) = xy$

(4) $x(y-z) = xy - xz$

(5) $(x-y)z = xz - yz$

Theorem: Let R be a ring. If $x \in R$ is a unit (x is invertible), then X cannot be a zero divisor.

Chapter 5 Section 2 - Integral Domains and Fields
(These are the NICEST types of rings)

Defn: Let D be a ring. Then D is an integral domain if
(1) D is a commutative ring
$e \neq 0 \rightarrow$ (2) D has unity (has multiplicative identity)
(3) D has no zero divisors
($ab=0, ba=0 \Rightarrow$ either $a=0$ or $b=0$)

ex: $\mathbb{Z}_{10}$ is not an integral domain because it has zero divisors.

Theorem 5.15 - For $n>1$, $\mathbb{Z}_n$ is an integral domain $\Longleftrightarrow n$ is prime.

Proof: "SideNote": $\mathbb{Z}_n$ is a commutative ring with unity of [1]

"$\Rightarrow$" Assume $\mathbb{Z}_n$ is an integral domain
For sake of contradiction, assume "n" is not prime.
so, n not prime $\Rightarrow \exists a, b \in \mathbb{Z}$, $1 < a < n$
$1 < b < n$
such that $n = ab$
so, $[n] = [ab] = [a][b]$

since n is not prime $\Rightarrow [a] \neq [0], [b] \neq [0]$

so, $[n] = [a][b] = [0]$

but $\mathbb{Z}_{10}$ is an integral domain, so $\mathbb{Z}_{10}$ should not have zero divisors. ($[a], [b]$ are the zero divisors as noted above)

Contradiction

hence, n must be prime $\boxed{\text{QED of "}\Rightarrow\text{"}}$

"$\Leftarrow$" Now assume "n" is prime and $[a][b] = [0]$.

Without loss of generality, $[a] \neq 0$

n is prime $\Rightarrow n \mid (ab) \Rightarrow n \mid a$ or $n \mid b$

Defn of being "Prime"

because, $[a] \neq [0], [a] \neq [n]$, $\underbrace{n \nmid a}_{\text{"}n\text{" does not divide "}a\text{"}}$

then, $n \mid b \Rightarrow [b] = 0$

so, $\mathbb{Z}_n$ has no zero divisors $\boxed{\text{QED of "}\Leftarrow\text{"}}$

$\boxed{\text{QED}}$

<u>Theorem:</u> Cancellation Law for Multiplication

Let D be an integral domain. Let $a, b, c \in D$ such that $a \neq 0$. If $ab = ac$, then $b = c$.

We can not assume multiplicative

inverses, because we solving within an integral domain.

Proof: $ab = ac \Rightarrow ab - ac = 0$

$a(b-c) = 0$ Not for this case.

$\Rightarrow$ either "$\cancel{a=0}$" or "$b-c=0$"

$\Rightarrow b-c=0 \Rightarrow b=c$ $\boxed{QED}$

Defn: Let F be a ring. Then F is a field if,

(1) F is a commutative ring
(2) F has unity (i.e. has multiplicative identity), $e \neq 0$
(3) Every nonzero element of F has a multiplicative inverse. (i.e. every nonzero element is a unit)

ex: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ } All of these are fields

Theorem: Every field is an integral domain.

Possible Exam #2 questions

Proof: Assume F is a field. All you need to show is F has no zero divisors.

Let $a, b \in F$ such that $ab = 0$.

Without loss of generality, let $a \neq 0$

so, F is a field $\Rightarrow \exists a^{-1}$

then, $ab = 0$

$\Rightarrow a^{-1} \cdot ab = a^{-1} \cdot 0$

$\Rightarrow (a^{-1}a)b = 0 \Rightarrow e \cdot b = 0$

$\Rightarrow b = 0 \Rightarrow F$ has no zero divisors $\boxed{\text{QED}}$

Theorem: Every finite integral domain is a field

Corollary: $\mathbb{Z}_n$ is a field $\Longleftrightarrow$ n is prime

This holds because $\mathbb{Z}_n$ is finite and $\mathbb{Z}_n$ is an integral when "n is prime".

4/22/2014 Vector Analysis

* HW#7 will be due in class on Thursday, April 24th
* Exam#2 will be during class on Tuesday, April 29th

Divergence Theorem and Laplace's Equation cont.

$$\iiint_D \nabla \cdot F \, dV = \iint_S F \cdot n \, dS \quad \left\} \begin{array}{l} D \text{ is the domain} \\ S \text{ is the boundary of } D \end{array} \right.$$

Possible Exam #2 questions

Find f such that $f(x,y,z)$

$\nabla^2 f = \rho$, where ρ is given in some domain D

and we also need some info on the boundary

$$\underset{\substack{\text{inside} \\ D}}{\rho(x,y,z)} = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f \quad \Big\} \text{ D is simply connected.}$$

Consider $F_1 = \phi \nabla \psi$; $\phi_1 \psi$ - scalar functions

$F_2 = \psi \nabla \phi$