UH - Math 6303 - Dr. Heier - Spring 2016
HW 5
Due no later than Wednesday, May 04, 3pm, at my office PGH 666
(if I am not in, please slide your solution under my office door)
or by email to heier@math.uh.edu.

Use regular sheets of paper, stapled together.
Don't forget to write your name on page 1.

1. (1 point) Let $k = \mathbb{Z}_2$ and $V = \{(0,0),(1,1)\} \subset \mathbb{A}^2$. Prove that $\mathcal{I}(V)$ is the product ideal $(x,y) \cdot (x-1, y-1)$.

2. (2 points) Let $V = \mathcal{Z}(xy - z) \subset \mathbb{A}^3$. Prove that V is isomorphic to $\mathbb{A}^2$ and provide an explicit isomorphism φ and associated k-algebra isomorphism $\tilde{\varphi} : k[V] \to k[\mathbb{A}^2]$, along with their inverses. Is $V = \mathcal{Z}(xy - z^2)$ isomorphic to $\mathbb{A}^2$? Prove your answer.

3. (0.5 points for each item)

(a) Let V be an affine algebraic set in $\mathbb{A}^n_{\mathbb{R}}$. Prove that there is a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ such that $V = \mathcal{Z}(f)$.

(b) Does the same hold with $\mathbb{R}$ replaced by $\mathbb{C}$? Prove your answer.

4. (1 point) Prove that $GL_n(k)$ is a Zariski-open subset of $\mathbb{A}^{n^2}$ and can be embedded as an affine algebraic set in $\mathbb{A}^{n^2+1}$.

5. (2 points) Let I, J be ideals in the ring R. Prove the following statements:

(a) If $I^k \subseteq J$ for some $k \geq 1$ then $\operatorname{rad} I \subseteq \operatorname{rad} J$.

(b) If $I^k \subseteq J \subseteq I$ for some $k \geq 1$ then $\operatorname{rad} I = \operatorname{rad} J$.

(c) $\operatorname{rad}(IJ) = \operatorname{rad}(I \cap J) = \operatorname{rad} I \cap \operatorname{rad} J$.

6. (1 point) Prove that for k a finite field the Zariski topology is the same as the discrete topology, i.e., every subset is closed and open.

7. (2 points) Let k be an algebraically closed field. Prove that every proper radical ideal in $k[x_1, \ldots, x_n]$ is the intersection of maximal ideals. Hint: Use Hilbert's Nullstellensatz.