UH Math 3330-01 Dr.Heier-Spring 2017
HW4 Answer Key

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Problem1. (a)

Proof. $G = \langle x \rangle$ so $x^{-1} \in G$. So $\langle x^{-1} \rangle \leq G$ by definition of $\langle x^{-1} \rangle$. Since $x = (x^{-1})^{-1}$, we have $x \in \langle x^{-1} \rangle$, hence $G \leq \langle x^{-1} \rangle$ by definition of $\langle x \rangle$. □

(b) omitted.

Problem2.

(a)omitted.

(b)

Proof. ($\Leftarrow$) Obvious. ($\Rightarrow$) Assume it is not true. Then we have $H \cup K \leq G$ with $H \not\subset G$ and $G \not\subset H$. So we can find x, y such that $x \in H, x \notin G$ and $y \in G, y \notin H$. Then $xy \in H \cup G$ because $H \cup G$ is a subgroup. If $xy \in H$, then $y = x^{-1}(xy) \in H$; If $xy \in G$, then $x = (xy)y^{-1} \in G$; Either leads to contradiction. □

Remark: Recall there is a similar property for vector spaces.

Problem3.

(a)

Proof. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in H$, A$B = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$ then

$$(ae + bg) + (ce + dg) = (a + c)e + (b + d)g = e + g = 1$$

$$(af + bh) + (cf + dh) = (a + c)f + (b + d)h = f + h = 1$$

and $\det(AB) = \det(A)\det(B) \neq 0$, thus $AB \in H$. $A^{-1} = \begin{pmatrix} \frac{d}{\det(A)} & -\frac{b}{\det(A)} \\ -\frac{c}{\det(A)} & \frac{a}{\det(A)} \end{pmatrix}$ then

$$\frac{d-c}{ad-bc} = \frac{d-c}{(1-c)d-(1-d)c} = \frac{d-c}{d-c} = 1.$$

$$\frac{a-b}{ad-bc} = \frac{a-b}{a(1-b)-b(1-a)} = \frac{a-b}{a-b} = 1.$$

Hence $A^{-1} \in H$. We have proved H is closed under multiplication and inverse thus H is a subgroup of G. □

(b)Similar. Don't forget to check $a^2 + b^2 = 1$ when checking closedness.
Problem4. Not true. Can take $G = (\mathbb{Z}, +)$ and $H = \{+1, -1\}$ for example.
Problem5.

Proof. For every $x \in H$ Take $x = y$ in the condition we have $xx^{-1} = e \in H$. So H has identity. Take e, x in the condition we have $ex^{-1} = x^{-1} \in H$. So H is closed under inverse. For $x, y \in H$, take x, y^{-1} in the condition we have

$$x(y^{-1})^{-1} = xy \in H.$$

Then H is closed under operation of G. Hence H is a subgroup of G.

□