UH Math 3330-01 Dr.Heier-Spring 2017
HW5 Answer Key

Yan He

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Problem1. (a) The group $\mathbb{Z}_2 \times \mathbb{Z}_4$ is of order 8. Thus every subgroup the order must be divisor of 8. For every order list cyclic group first and then non cyclic.

order 1: $\{(0,0)\}$

order 2: $<(1,0)>, <(0,2)>, <(1,2)>$

order 4: $<(0,1)>, <1,1>, \{(0,0),(0,2),(1,0),(1,2)\}$

order 8: $\mathbb{Z}_2 \times \mathbb{Z}_4$

(b) Similarly, the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is also order 8. Then the subgroups:

order 1: $\{(0,0,0)\}$

order 2: $<(1,0,0)>, <(0,1,0)>, <(0,0,1)>, <(0,1,1)>, <(1,0,1)>$ $, <(1,1,0)>, <(1,1,1)>$ 7 choices.

order 4: cannot be cyclic because $\mathbb{Z}_2$ only has at most order 2 element.

$\{(1,0,0),(0,1,0),(1,1,0),(0,0,0)\}$

$\{(1,0,0),(0,0,1),(1,0,1),(0,0,0)\}$

$\{(1,0,0),(0,1,1),(1,1,1),(0,0,0)\}$

$\{(0,1,0),(0,0,1),(0,1,1),(0,0,0)\}$

$\{(0,1,0),(1,0,1),(1,1,1),(0,0,0)\}$

5 choices.

order 8: itself.

Problem2.

(a)omitted.

(b)

Proof. From (a) we know both G and H are cyclic. By Theorem 6.1 we know $|G|$ and $|H|$ are co-prime. So if $G \times H$ is cyclic, so its subgroups are all cyclic. Let $K = <(g,h)> \leq G$ for $g \in G, h \in H$ then $ord(g)||G|$, $ord(h)||H|$, $<g>$ $\times <h> \leq K$, thus $gcd(ord(g), ord(h)) = 1$ because $gcd(|G|,|H|) = 1$. So

$$|K| = ord(g,h) = lcm(ord(g), ord(h)) = ord(g)ord(h).$$

So $|K| = |<g>||h| = |<g> \times <h>|$. So $K = <g> \times <h>$. □

Problem3.

(a) omitted. (b) omitted. (c) $A = \{0,1\}$, $B = 0$. $f : A \to B$ s.t $f(0) = f(1) = 0$. $S = \{0\}, T = \{1\}$ Then $f(S \cap T) = f(\emptyset) = \emptyset$ where $f(S) \cap f(T) = \{1\}$.

Problem4. (a)

Proof. First suppose $f : A \to B$ is injective. Then for every $y \in Im(f)$ corresponds to one and only one $x_y \in A$ s.t. $f(x) = y$, in other words we have $x_{f(x)} = x$. Fix $x_0 \in A$, define $g : B \to A, y \mapsto x_y$for $y \in Imf$ and $y \mapsto x_0$for $y \notin Imf$. Then for every $x \in A$, $g \circ f(x) = g(f(x)) = x_{f(x)} = x$ so $g \circ f = id_A$. On the other hand, suppose there exists function $g : B \to A$ s.t. $g \circ f = id_A$. Then for every $x, y \in A$ s.t. $f(x) = f(y)$ then

$$g(f(x)) = g(f(y)).$$

So $g(f(x)) = g \circ f(x) = id_A(x) = x$, $g(f(y)) = y$ for the same reason, so $x = y$. We have $f(x) = f(y)$ implies $x = y$ so f is injective.

□

(b)

Proof. If f is surjective, then for every $y \in B$ there exists $x \in A$ s.t. $f(x) = y$. Define $g : B \to A$, let $g(y)$ be arbitrary $x \in A$ such that $f(x) = y$. Then for every $y \in B$, $f \circ g(y) = f(g(y)) = y$. So $f \circ g = id_B$. On the other hand, if there exists a function $g : B \to A$ s.t. $f \circ g = id_B$, then for every $y \in B$ corresponds to $g(y) \in A$ s.t. $f(g(y)) = y$. So f surjective. □

Problem5. (a) not true. For example

(b)not true. For example