UH - Math 6353 - Dr. Heier - Take Home Final Exam - Spring 2017
Due: Monday, May 8, 2017, at 5pm, as a (scanned) pdf to heier@math.uh.edu

1. (20 points) Let (M, g) be a hermitian manifold with associated $(1,1)$-form ω. Prove that g is Kähler (i.e., $d\omega = 0$) if and only if for every $p \in M$, there exists an open neighborhood $U \ni p$ and a smooth real function ϕ defined on U such that $\omega = \sqrt{-1}\partial\bar{\partial}\phi$ on U. Hint: This is a local statement and the $\partial\bar{\partial}$-Lemma does **not** apply. Use the Poincaré-Lemma instead, which you may cite freely.

2. (20 points) Prove that the set of upper triangular invertible $(n \times n)$-matrices is a solvable subgroup of $GL_n(\mathbb{C})$. Is it normal? Prove your answer.

3. (20 points) Using Matsushima's criterion, prove that there does not exist a Kähler-Einstein metric on $\mathbb{P}^2$ blown up at two distinct points.

4. (20 points) On the unit two ball $\mathbb{B}^2$ in $\mathbb{C}^2$ consider the Bergman metric, i.e., the hermitian metric g whose associated $(1,1)$-form is given by

$$\omega = -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(1 - z_1\bar{z}_1 - z_2\bar{z}_2).$$

Compute the Ricci curvature form Ric. Compare Ric and ω. Also, using the trace formula, compute the scalar curvature.

5. (10 points) Let M be a non-singular cubic surface in $\mathbb{P}^3$. Let C be an irreducible curve in M with $C^2 < 0$. Prove that C must be satisfy $C^2 = -1$ and is a line in the ambient projective space $\mathbb{P}^3$.

6. (10 points) Let X be a K3 surface. Prove that X is not the blow-up of any smooth compact complex surface.