University of Houston – Spring 2020 – Dr. G. Heier
Advanced Linear Algebra I (Math 4377/6308)

Contents

These lecture notes are based on the textbook Linear Algebra, 4th edition, by Friedberg, Insel, and Spence, ISBN 0-13-008451-4. They are provided "as is" and as a courtesy only. They do not replace use of the textbook or attending class.

0 Foundational material: The appendices

0.1 Appendix A: Sets

Definition 0.1. A *set* is a collection of objects, called elements.

Example 0.2.

- $\{1,2,3\} = \{2,1,1,1,2,3\}$ (no notion of "multiplicity", no notion of order)
- $[1,2]$ = the interval of reals between 1 and 2, including 1 and 2.
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ (later)
- $\left\{ \begin{pmatrix} 8 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix} \right\}$ = set of two vectors
- $\emptyset$: the empty set

Given two sets A, B, there are several operations that yield new sets from these. Most important are the following:

- $A \cup B$ (union of A and B)
- $A \cap B$ (intersection of A and B)
- $A \times B = \{(a,b) : a \in A, b \in B\}$ (product of A and B)

Definition 0.3. Let A be a set. A *relation* on A is a subset S of $A \times A$. Write $x \sim y$ if and only if $(x,y) \in S$.

Example 0.4.

- $A = \{1,2,3\}$, $S = \{(1,2),(1,3),(2,3)\}$. This relation is "<".

- $A = \{1,2,3\}$, $S = \{(1,2),(1,3),(2,3),(2,1),(2,3),(3,1),(3,2)\}$. This relation is "$\neq$".
- $A = \{1,2,3\}$, $S = \{(1,1),(2,2),(3,3)\}$. This relation is "$=$".

Recall the following symbols. $\forall$: "for all", $\exists$: "there exists".

Definition 0.5. Let A be a set with a relation S. Then S is called an *equivalence relation* if and only if

i. $\forall x \in A : x \sim x$ (reflexive)

ii. $\forall x, y \in A : x \sim y \Leftrightarrow y \sim x$ (symmetric)

iii. $\forall x, y, z \in A : (x \sim y \text{ and } y \sim z) \Rightarrow x \sim z$ (transitive)

Example 0.6. Let $A = \mathbb{Z}$. Let $x \sim y \Leftrightarrow \exists k \in \mathbb{Z} : x - y = 5k$. This defines an equivalence relation.

i. reflexive: Let $x \in \mathbb{Z}$. Then $x - x = 0 = 5 \cdot 0$. Done.

ii. symmetric: Let $x, y \in \mathbb{Z}$ with $x \sim y$. Then $x - y = 5k$ implies $y - x = 5 \cdot (-k)$. Done.

iii. transitive: Let $x, y, z \in \mathbb{Z}$ with $x \sim y$ and $y \sim z$. Then $x - y = 5k_1$ and $y - z = 5k_2$ implies (by adding the two equalities) $x - z = 5 \cdot (k_1 + k_2)$. Done.

0.2 Appendix B: Functions

Definition 0.7. Let A, B be sets. A *function* $f : A \to B$ is a rule that associates to each element $x \in A$ a unique element of B, denoted $f(x)$. The set A is called the *domain*, the set B is called the *codomain*.

Definition 0.8.
- For $S \subseteq A$, $f(S) = \{f(x) : x \in S\}$ (*image of S under f*). $f(A)$ is called the *range*.
- For $T \subseteq B$, $f^{-1}(T) = \{x \in A : f(x) \in T\}$ (*pre-image of T under f*)
- $f : A \to B = g : A \to B \Leftrightarrow \forall x \in A : f(x) = g(x)$

Definition 0.9.
- $f : A \to B$ is *injective* if and only if $f(x) = f(y) \Rightarrow x = y$.
- $f : A \to B$ is *surjective* if and only if $\forall b \in B \exists a \in A : f(a) = b$.
- For $S \in A$, the restriction of f to S is $f|_S : S \to B, x \mapsto f(x)$.

0.3 Appendix C: Fields

Definition 0.10. Let A be a set. A binary operation is any map $A \times A \to A$. We are very familiar with $\mathbb{Q}$ and $\mathbb{R}$ and the properties that the two binary operations $+$ and $\cdot$ have.

Definition 0.11. A field F is a set with two binary operations labelled $+$ and $\cdot$ such that

i. $a+b=b+a, \quad a \cdot b = b \cdot a$ (commutativity)

ii. $(a+b)+c=a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity)

iii. $\exists 0 \in F : a+0=a \ \forall a$
   $\exists 1 \in F : 1 \cdot a = a \ \forall a$ (neutral elements)

iv. $\forall a \in A : \exists b \in A : a+c=0$
   $\forall a \in A \backslash \{0\} : \exists b \in A : a \cdot b = 1$ (inverse elements)

v. $a \cdot (b+c) = a \cdot b + a \cdot c$ (distributive law)

Theorem 0.12 (Cancellation Laws)**.** *Let F be a field and $a, b, c \in F$.*

i. $a+b=c+b \Rightarrow a=c$

ii. $a \cdot b = c \cdot b$ *and* $b \neq 0 \Rightarrow a=c$

Proof. Part i. Let d be an additive inverse of b. Now, observe that $(a+b)+d=a$ and $(c+b)+d=c$. Done.

Part ii is done in detail in the textbook. □

Proposition 0.13. *The neutral element of addition is unique.*

Proof. Let 0 and $0'$ be two neutral elements of addition. Then

$$0 = 0 + 0' = 0'.$$

□

Fields enjoy several other important properties (not listed here).

Example 0.14. Some examples of fields.

- $\mathbb{Q}, \mathbb{R}, \mathbb{C}$
- $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ (see also Homework 1).

0.4 Appendix D: Complex Numbers

Motivation: In $\mathbb{R}$, $x^2 - 1 = 0$ has two solutions, namely $-1, 1$. However, the almost identical equation $x^2 + 1 = 0$ has no solutions. This means that the reals "leave something to be desired." In response, we introduce the imaginary unit i, which has the property $i^2 = -1$.

Definition 0.15. A *complex number* is an expression of the form $z = a + bi$ with $a, b \in \mathbb{R}$. Sum and product are defined by

$$z + w = (a + bi) + (c + di) = a + c + (b + d)i$$

and

$$zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Remark 0.16. Memorize the multiplication by multiplying out as one would do naively, and then use $i^2 = -1$. Do some examples!

Theorem 0.17. *The complex numbers with sum and multiplication as above form a field.*

Proof. This just involves tedious checking of all the properties–you should try a few yourself at home. $\square$

Remark 0.18. The multiplicative inverse of $z = a + bi$ is

$$\frac{1}{a + bi} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} + i\frac{-b}{a^2 + b^2}.$$

Definition 0.19. The *complex conjugate* of $z = a + bi$ is $\bar{z} = a - bi$.

Proposition 0.20.

i. $\bar{\bar{z}} = z$

ii. $\overline{z + w} = \bar{z} + \bar{w}$

iii. $\overline{zw} = \bar{z} \cdot \bar{w}$

iv. $\overline{\frac{z}{w}} = \frac{\bar{z}}{\bar{w}}$

Remark 0.21. It is now clear that there is a bijection $\mathbb{C} \to \mathbb{R}^2$ via $a+bi \mapsto (a,b)$. By Pythagoras' Theorem, the length of a straight line from the origin to the point (a,b) is $\sqrt{a^2+b^2}$.

Definition 0.22. The *absolute value* (or *modulus*) of $z = a+bi$ is $|z| = \sqrt{a^2+b^2}$.

Remark 0.23. We have

$$z\bar{z} = (a+bi)(a-bi) = a^2+b^2.$$

Thus,

$$|z| = \sqrt{z\bar{z}}.$$

Properties 0.24.

i. $|zw| = |z||w|$

ii. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

iii. $|z+w| \leq |z| + |w|$

Theorem 0.25 (Fundamental Theorem of Algebra)**.** *Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a complex polynomial (i.e., $a_i \in \mathbb{C}$). Then $\exists z_0 \in \mathbb{C} : p(z_0) = 0$.*

Proof. No proof is given here. This is a comparatively hard theorem to prove. □

Corollary 0.26. *For p as above, $\exists r_1, \ldots, r_n \in \mathbb{C}$ such that*

$$p(z) = a_n(z-r_1)\ldots(z-r_n).$$

Proof. Long division! □

Remark 0.27. The formula often taught in high school to solve quadratic equations still works. For example, to solve $x^2 - 2x + 5 = 0$, write $x = -\frac{-2}{2} \pm \sqrt{1-5} = 1 \pm \sqrt{-4} = 1 \pm \sqrt{4(-1)} = 1 \pm 2\sqrt{-1} = 1 \pm 2i$.

1 Vector spaces

1.1 Introduction

Geometrically, a vector in, say, $\mathbb{R}^2$, is the datum of a direction and a magnitude. Thus, it can be represented by an arrow which points in the given direction and has the given length. Two vector can be added using the parallelogram rule (see the textbook for some nice pictures explaining this).

Physically, the vectors may, e.g., represent forces that are exerted on an object. The result of the addition is the resulting net force that the object experiences when the original two forces are applied.

Algebraically, when $v = (a_1, a_2)$ and $w = (b_1, b_2)$, then $v + w = (a_1 + b_1, a_2 + b_2)$. Scalar multiplication is defined via $t(a_1, a_2) = (ta_1, ta_2)$.

Definition 1.1. The (non-zero) vectors v and w are parallel if and only if $\exists t \in \mathbb{R} : tv = w$.

A vector can be interpreted as the *displacement vector* between its start and end point. If the start point is (x_1, x_2) and the end point is (y_1, y_2), then the displacement vector is $(y_1 - x_1, y_2 - x_2)$.

Definition 1.2. The *line* through the points $A = (x_1, x_2)$ and $B = (y_1, y_2)$ is

$$\{(x_1, x_2) + t(y_1 - x_1, y_2 - x_2) : t \in \mathbb{R}\}.$$

Definition 1.3. The *line* through the points $A = (x_1, x_2, x_3)$ and $B = (y_1, y_2, y_3)$ is

$$\{(x_1, x_2, x_3) + t(y_1 - x_1, y_2 - x_2, y_3 - x_3) : t \in \mathbb{R}\}.$$

Definition 1.4. The *plane* through the points $A = (x_1, x_2, x_3)$, $B = (y_1, y_2, y_3)$ and $C = (z_1, z_2, z_3)$ (not all three on a line) is

$$\{(x_1, x_2, x_3) + s(y_1 - x_1, y_2 - x_2, y_3 - x_3) + t(z_1 - x_1, z_2 - x_2, z_3 - x_3) : s, t \in \mathbb{R}\}.$$

Example 1.5.

i. The line through $(1, 1, 2)$ and $(0, 3, -1)$ is

$$\{(1, 1, 2) + t(-1, 2 - 3) : t \in \mathbb{R}\}.$$

ii. The plane through the points $A = (1, 0, -1)$, $B = (0, 1, 2)$ and $C = (1, 1, 0)$ is

$$\{(1, 0, -1) + s(-1, 1, 3) + t(0, 1, 1) : s, t \in \mathbb{R}\}.$$

Now, observe that vector addition and scalar multiplication satisfy certain laws, e.g., $v + w = w + v$, $1 \cdot v = v$, $(ab)v = a(bv)$. In Section 1.2, we will distill these obvious properties into an abstract definition.

Quiz 1.
1. (5 points) Let $\mathbb{R}$ be the set of real numbers. Define a relation R on $\mathbb{R}$ by $(x, y) \in R$ if and only if $x - y$ is an integer. Prove that R is an equivalence relation.

2. Let the function $f : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is even} \\ 3x+1 & \text{if } x \text{ is odd} \end{cases}.$$

i. (2.5 points) Is f injective? Prove your answer.

ii. (2.5 points) Is f surjective? Prove your answer.

Answer to 1.

i. Reflexivity. Let $x \in \mathbb{R}$. Then $x - x = 0 \in \mathbb{Z}$.

ii. Symmetry. Let $(x, y) \in R$. Then there exists an integer k such that $x - y = k$. After multiplying both sides by -1, we obtain $y - x = -k \in \mathbb{Z}$, which implies $(y, x) \in R$.

iii. Transitivity. Let $(x, y), (y, z) \in R$. Then there exist integer k, ℓ such that $x - y = k$ and $y - z = \ell$. Adding the two equations yields $x - z = k + \ell \in \mathbb{Z}$, which implies $(x, z) \in R$.

Answer to 2.

i. The function f is not injective, because $f(2) = 4 = f(1)$.

ii. The function f is not surjective, because there are no odd numbers in its Range.

1.2 Vector Spaces

Definition 1.6. A *vector space* (or *linear space*) V over a field F (think $F = \mathbb{R}$, or $\mathbb{C}$) is a set with a binary operation denoted "+" and a second map $\cdot : F \times V \to V$ such that

i. $\forall x, y \in V : x + y = y + x$

ii. $\forall x, y, z \in V : (x + y) + z = x + (y + z)$

iii. $\exists 0 \in V : \forall x \in V : x + 0 = x$

iv. $\forall x \in V \exists y \in V : x + y = 0$

v. $\forall x \in V : 1x = x$, where 1 is the neutral element of multiplication in F

vi. $\forall a, b \in F \forall x \in V : (ab)x = a(bx)$

vii. $\forall a \in F \forall x, y \in V : a(x + y) = ax + ay$

viii. $\forall a, b \in F \forall x \in V : (a + b)x = ax + bx$

Definition 1.7. The elements of F are called *scalars*. The elements of V are called *vectors*. Because of item ii above, sums like $x + y + z + w$ are well-defined.

Remark 1.8. To simplify typing, we will usually not adorn vectors with an arrow, i.e., we will write x instead of $\vec{x}$ and 0 instead of $\vec{0}$. Note that the neutral element of addition in the field is also denoted with 0, but it should always be clear from the context what is meant.

Example 1.9. i. (THE example, see later section on isomorphisms) Take a field F. (We will mostly just take $\mathbb{R}$, or perhaps $\mathbb{C}$.) An n-tuple is $(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n \in F$. Note that $\{n\text{-tuples}\} \cong F^n$ naturally. Define $(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$. Also, $c(a_1, \ldots, a_n) = (ca_1, \ldots, ca_n)$ for $c \in F$.

ii. $\mathrm{Mat}_{m,n}(F)$ is a vector space with componentwise addition and scalar multiplication

The following examples of vector spaces are substantially different from the examples above. They are "infinite dimensional," more about that later.

Example 1.10. i. Let S be a set of real numbers Let $\mathcal{F}$ be the set of all real-valued functions on S. Then $\mathcal{F}$ is a vector space (over $\mathbb{R}$) with the usual addition and scalar multiplication of real-valued functions.

ii. Let S now be an interval of reals. Consider in $\mathcal{F}$ only those functions that are continuous. This is also a vector space (use the summation theorem for continuous functions from calculus)

iii. Consider in $\mathcal{F}$ only those functions that are differentiable. This is also a vector space (use the summation theorem for differentiable functions from calculus)

iv. Assume that $x_0 \in S$. Then $\{f : S \to \mathbb{R} \mid f(x_0) = 0\}$ is a vector space. (check it!). $\{f : S \to \mathbb{R} \mid f(x_0) = 1\}$ is not!

Again, we would like to infer more properties of vector spaces from the original list of 8 properties. To start, we observe that the zero vector is unique, with the same proof as in the case of fields in the Introduction. Moreover, we also have a cancellation law:

Theorem 1.11 (Cancellation law for vector spaces)**.** *Let V be a vector space and $x, y, z \in V$. If $x + z = y + z$, then $x = y$.*

Proof. Let v be such that $z + v = 0$ (condition iv). Then

$$\begin{aligned} x &= x + 0 = x + (z + v) = (x + z) + v \\ &= (y + z) + v = y + (z + v) = y + 0 = y \end{aligned}$$

due to conditions ii and iii. □

Corollary 1.12. *The additive inverse is unique.*

Theorem 1.13. *Let V be a vector space. Then the following statements are true.*

i. $\forall x \in V : 0x = \vec{0}$

ii. $\forall x \in V \ \forall a \in F : (-a)x = -(ax) = a(-x)$

iii. $\forall a \in F : a\vec{0} = \vec{0}$

Proof. The textbook has detailed proofs of i and ii. The item iii is left to the reader. □

1.3 Subspaces

Definition 1.14. A subset W of a vector space V over the field F is called a *subspace of* V if W is a vector space with $+$ and scalar multiplication from V.

Example 1.15.

- $\{\vec{0}\}$, V
- $\mathbb{R}^2 \cong \{(a, b, 0) | a, b \in \mathbb{R}\} \subset \mathbb{R}^3$

- The above examples 1.10ii and 1.10iii in 1.10i.

In order to verify that W is a subspace of V, it is not necessary to check all the vector space axioms in the definition of a vector space. For example, the restricted addition is clearly commutative since it was already commutative before the restriction.

Theorem 1.16. *A nonempty subset W of the vector space V is a subspace of V if and only if*

i. $\forall x, y \in W : x + y \in W$ *(closedness under +)*

ii. $\forall c \in F \forall x \in W : c \cdot x \in W$ *(closedness under scalar multiplication)*

Proof. First, observe that the implication $\Rightarrow$ is trivial. The proof of the other direction consists of some easy verifications. For example, let's see why $\vec{0} \in W$: Take an arbitrary element x of W. Since W is nonempty, such an element exists. Now, simply observe that $0 \cdot x = \vec{0}$, which is an element of W by ii. The remaining details are left to the reader. $\square$

More examples:

Example 1.17.
- Let $W = \{(a, b) | a + b = 0\} \subset \mathbb{R}^2$. Closedness under $+$ is checked as follows. Let $(a, b), (c, d) \in W$. Then the result of the addition is $(a+c, b+d)$, which satisfies $(a+c)+(b+d) = (a+b)+(c+d) = 0+0 = 0$. Closedness under scalar multiplication is seen as follows. Let $(a, b) \in W$ and c a scalar. Then the result of the scalar multiplication is (ca, cb), which satisfies $ac + cb = c(a + b) = c0 = 0$.

- Let $W = \{(a, b, c) | 3a - b + 2c = 0\} \subset \mathbb{R}^3$. Check it as in i.

- What about $W = \{(a, b, c) | 3a - b + 2c = 1\} \subset \mathbb{R}^3$? Let $(a, b, c), (d, e, f) \in W$. Then the result of their addition is $(a+d, b+e, c+f)$, which satisfies $3(a+d) - b - e + 2(c+f) = 3a - b + 2c + 3d - e + 2f = 1 + 1 = 2 \neq 1$. Thus, W is not closed under addition and not a subspace.

- Let $V = \{f : S \to \mathbb{R}\}$. Let $p \in S$. Let $W_p = \{f \in V : f(p) = 0\}$. Then W_p is a subspace of V: for $f, g \in W_p$, $(f + g)(p) = f(p) + g(p) = 0 + 0 = 0$. Furthermore, for $c \in \mathbb{R}$, $(cf)(p) = cf(p) = c0 = 0$.

- Any intersection of subspaces in a vector space is itself a subspace.

A major class of examples is given by sums and direct sums of subspaces.

Definition 1.18. Let V be a vector space. Let S, T be nonempty subsets of V. Then let $S + T = \{x + y | x \in S, y \in T\}$. We call $S + T$ the *sum* of S and T.

Definition 1.19. Let V be a vector space. Let W, U be subspaces of V. Then we call V the *direct sum* of W, U if $W + U = V$ and $W \cap U = \{0\}$. Write $V = W \oplus U$.

Proposition 1.20. *Let V be a vector space. Let W, U be subspaces of V. Then the sum $W + U$ is a subspace of V (containing both W and U).*

Proof. $(w_1 + u_1) + (w_2 + u_2) = (w_1 + w_2) + (u_1 + u_2)$, which is the sum of a vector in W, namely $w_1 + w_2$, and a vector in U, namely $u_1 + u_2$. Thus, $W + U$ is closed under addition. The closedness under scalar multiplication is completely analogous. $\square$

Example 1.21.

- $\{(a, b, 0, c)|a, b, c \in \mathbb{R}\} + \{(d, 0, e, f)|d, e, f \in \mathbb{R}\} = \mathbb{R}^4$. But this is not a direct sum.
- $\{(a, 0, 0, b)|a, b \in \mathbb{R}\} \oplus \{(0, c, d, 0)|c, d \in \mathbb{R}\} = \mathbb{R}^4$. This is a direct sum.
- $\{(a, 0, 0)|a \in \mathbb{R}\} \oplus \{(0, b, 0)|b \in \mathbb{R}\} = \{(a, b, 0)|a, b \in \mathbb{R}\}$

1.4 Linear Combinations and systems of linear equations

Definition 1.22. Let V be a vector space and S a nonempty subset of V. We call $v \in V$ a *linear combination* of vectors in S if there exist vectors $u_1, \ldots, u_n \in S$ and scalars $a_1, \ldots, a_n \in F$ such that $v = a_1 u_1 + \ldots + a_n u_n$.

Example 1.23.

- $(3, 4, 1) = 3(1, 0, 0) + 4(0, 1, 0) + 1(0, 0, 1)$.
- If we want to write $(3, 1, 2)$ as a linear combination of $(1, 0, 1), (0, 1, 1), (1, 2, 1)$, how do we find the coefficients a_1, a_2, a_3? Answer: Make the Ansatz

  $$(3, 1, 2) = a_1(1, 0, 1) + a_2(0, 1, 1) + a_3(1, 2, 1)$$

  and solve the system of linear equations

  $$a_1 + a_3 = 3, \quad a_2 + 2a_3 = 1, \quad a_1 + a_2 + a_3 = 2.$$

  Solution: $a_1 = 2, a_2 = -1, a_3 = 1$.

Quiz 2.
1. (5 points) Let $V = \mathbb{R}^2$. Let $W = \{(a, b) \in V \ : \ a^2 - b^2 = 0\}$. Is W a subspace of V? Prove your answer.

2. (5 points) Let $V = \mathbb{R}^3$. Let $W = \{(s, s - t, t) \in V \ : \ s, t \in \mathbb{R}\}$. Is W a subspace of V? Prove your answer.

Answer to 1. No. We have $(1, 1)$ and $(1, -1)$ in W, but their sum $(1, 1) + (1, -1) = (2, 0)$ is not in W.

Answer to 2. Yes. W is clearly non-empty. Moreover, if $(s_1, s_1 - t_1, t_1)$ and $(s_2, s_2 - t_2, t_2)$ are in W, then so is their sum $(s_1 + s_2, s_1 + s_2 - t_1 - t_2, t_1 + t_2)$ (take $s = s_1 + s_2$ and $t = t_1 + t_2$). Also, if $c \in \mathbb{R}$, then $c(s_1, s_1 - t_1, t_1) = (cs_1, cs_1 - ct_1, ct_1)$ is in W (take $s = cs_1$).

Example 1.24.

- Finding the a_i in a given situation may or may not be possible. E.g., writing

$$(3, 1, 2) = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(1, 2, 0)$$

is clearly impossible.

- There may be many choices for the a_i:

$$(2, 6, 8) = a_1(1, 2, 1)+a_2(-2, -4, -2)+a_3(0, 2, 3)+a_4(2, 0, -3)+a_5(-3, 8, 16)$$

is equivalent to

$$(a_1, a_2, a_3, a_4, a_5) \in \{(-4 + 2s - t, s, 7 - 3t, 3 + 2t, t) | s, t \in \mathbb{R}\}.$$

(There are two "free variables".)

There are three types of operations that we used to solve the above systems of linear equations:

i. Interchange the order of any two equations.

ii. Multiply an equation by a nonzero scalar.

iii. Add one equation to another.

Key point: These operations do not change the set of solutions.

Definition 1.25. Let V be a vector space. Let S be a nonempty subset of V. We call $\operatorname{span}(S)$ the set of all vectors in V that can be written as a linear combination of vectors in S.

Example 1.26. Let $S = \{(1,0,0),(0,1,0),(2,1,0)\}$. Then $\text{span}(S) = \{(s,t,0)|s,t \in \mathbb{R}\}$.

Theorem 1.27. *The span of any subset S of a vector space V is a subspace of V.*

Proof. Let $v = a_1u_1 + \ldots + a_nu_n \in \text{span}(S)$. Then $cv = (ca_1)u_1 + \ldots + (ca_n)u_n \in \text{span}(S)$. Thus, closedness under scalar multiplication is ok.

Let $v = a_1v_1 + \ldots + a_nv_n \in \text{span}(S)$ and let $w = b_1w_1 + \ldots + b_mw_m \in \text{span}(S)$. Then

$$v + w = a_1v_1 + \ldots + a_nv_n + b_1w_1 + \ldots + b_mw_m.$$

Thus, closedness under addition is ok. $\square$

Example 1.28.

- $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ spans $\mathbb{R}^3$.
- $S = \{(1,2),(2,1)\}$ spans $\mathbb{R}^2$.
- $S = \{(1,1),(2,2)\}$. $\text{span}(S) = \{(s,s)|s \in \mathbb{R}\} \neq \mathbb{R}^2$.
- $S = \{(1,2)\}$ does not span $\mathbb{R}^2$.
- Which (a,b,c) are in $\text{span}(\{(1,1,2),(0,1,1),(2,1,3)\})$? Answer: Those that satisfy $a + b = c$.

1.5 Linear dependence and linear independence

Motivation: Let W be a subspace of V. We are interested in a set $S \subset W$ such that $\text{span}(S) = W$ and S is "as small as possible".

Definition 1.29. A subset S of a vector space V is called *linearly dependent* if there exist a finite number of vectors $u_1, \ldots, u_n \in S$ and scalars $a_1, \ldots, a_n$, not all equal to zero, such that

$$a_1u_1 + \ldots + a_nu_n = 0.$$

We also say that the vectors in S are linearly dependent.

Example 1.30. Let $S = \{(1,3,-4,2),(2,2,-4,0),(1,-3,2-4),(-1,0,1,0)\}$. Then

$$4(1,3,-4,2) - 3(2,2,-4,0) + 2(1,-3,2-4) + 0(-1,0,1,0) = (0,0,0,0).$$

Thus, S is linearly dependent.

Definition 1.31. If S is not linearly dependent, we say S is *linearly independent.*

Remark 1.32. Linear independence is equivalent to: "$\sum a_i v_i = \vec{0} \Rightarrow$ all $a_i = 0$".

Remark 1.33. The empty set $\emptyset$ is linearly independent. The singleton set $\{v\}$ is linearly independent if and only if $v \neq \vec{0}$.

Theorem 1.34. *Let V be a vector space. If $S_1 \subseteq S_2$ and S_1 is linearly dependent, then S_2 is linearly dependent.*

Proof. This is immediate from the definition. □

Theorem 1.35. *Let S be a linearly independent subset of V. Let $v \in V \setminus S$. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \operatorname{span}(S)$.*

Proof. "$\Rightarrow$". Write $a_1 u_1 + \ldots + a_n u_n + a_{n+1} v = 0$ with not all a_i equal to zero and $u_i \in S$.

Claim: $a_{n+1} \neq 0$.

Proof of claim: If $a_{n+1} = 0$, then at least one of $a_1, \ldots, a_n$ is not equal to zero and $a_1 u_1 + \ldots + a_n u_n = 0$. Contradiction to linear independence of S.

So, $a_{n+1} \neq 0$, and we can write $v = \frac{-a_1}{a_{n+1}} u_1 + \ldots + \frac{-a_n}{a_{n+1}} u_n$, qed.

"$\Leftarrow$". Write $v = a_1 u_1 + \ldots + a_n u_n$. Then $a_1 u_1 + \ldots + a_n u_n + (-1)v = 0$. Thus, $S \cup \{v\}$ is linearly dependent, qed. □

Quiz 3.
(10 points) Let V be a vector space over $\mathbb{R}$. Let $v, w \in V$. Prove that if $\{v, w\}$ is linearly independent then $\{v - w, v + w\}$ is linearly independent.

Answer: Make the Ansatz $a(v - w) + b(v + w) = 0$. We have to prove that $a = b = 0$. The Ansatz implies $(a + b)v + (b - a)w = 0$. By the linear independence of $\{v, w\}$, we can infer that $a + b = 0$ and $b - a = 0$, which is a linear system of two homogeneous equations in 2 variables a, b. The only solution of this system is $a = b = 0$, q.e.d.

1.6 Bases and dimension

Definition 1.36. Let V be a vector space. A *basis* β is a linearly independent subset of V which satisfies $\text{span}(\beta) = V$.

Theorem 1.37. *Let V be a vector space. Let $\beta = \{u_1, \ldots, u_n\}$ be a subset of V. Then*

$$\beta \text{ is a basis } \Leftrightarrow \forall v \in V : \exists!\, a_1, \ldots, a_n \in F : v = a_1u_1 + \ldots + a_nu_n.$$

(Recall that $\exists!$ means **unique** *existence.)*

Proof. "$\Rightarrow$". Spanning property is already known. We just have to prove uniqueness.

Let

$$a_1u_1 + \ldots + a_nu_n = v = b_1u_1 + \ldots + b_nu_n.$$

This implies

$$(a_1 - b_1)u_1 + \ldots + (a_n - b_n)u_n = 0.$$

Linear independence implies $a_1 - b_1 = 0, \ldots, a_n - b_n = 0$. Done.

"$\Leftarrow$". Spanning property is already known. To show lin. indep., just observe that

$$a_1u_1 + \ldots + a_nu_n = 0$$

is solved by the trivial solution $a_1 = \ldots = a_n = 0$. However, by assumption, this is the only solution. Thus, we have established linear independence. Done. □

Theorem 1.38. *Let V be a vector space. Let S be a finite subset of V with $\text{span}(S) = V$. Then there exists a subset of S which is a basis for V. In particular, V has a finite basis.*

Proof. We conduct this proof by induction over the cardinality of S.

If $\#S = 1$, then $S = \{v\}$, and S is clearly linearly independent (unless we are in a trivial cases).

Now, assume that we know the theorem for $\#S = n$. We have to prove it for $\#S = n + 1$.

If S is not a basis, then S is lin. dep. Claim: $\exists v \in S : V = \text{span}(S) = \text{span}(S \setminus \{v\})$. Proof of Claim: lin. dep. means that there is a linear combination

$$a_1 u_1 + \ldots + a_n u_n = 0$$

with some $a_{i_0} \neq 0$. We can solve the above equation for u_{i_0}. It is now clear that a linear combination of the vectors $u_1, \ldots, u_n$ can be expressed as a linear combination of the vectors $u_1, \ldots, u_{i_0-1}, u_{i_0+1}, \ldots u_n$. Thus, letting $v = u_{i_0}$ establishes the Claim.

If we let $\tilde{S} = S \setminus \{u_{i_0}\}$, then we can apply the induction hypothesis to obtain that $\tilde{S}$ contains a basis. Since $\tilde{S} \subset S$, we can conclude that S contains a basis. Done. $\square$

Example 1.39.

- Let $S = \{(1,0),(1,1),(2,3)\}$. Observe that S spans $\mathbb{R}^2$, but S is lin. dep. After removing any one of the three vectors from S, we obtain a basis.
- Let $S = \{(1,0),(0,1),(0,2)\}$. Observe that S spans $\mathbb{R}^2$, but S is lin. dep. After removing the second or third vector, we obtain a basis. However, removing the first vector does not yield a basis.
- Let $S = \{(2,-3,5),(8,-12,20),(1,0,-2),(0,2,-1),(7,2,0)\}$. Observe that S spans $\mathbb{R}^3$, but S is lin. dep.. Consider the span of the first vector. Obviously, the span remains unchanged after adding the second vector (which is 4 times the first), so the second vector should be removed. The third vector is not a multiple of the first, so we keep it. A direct computation shows that the first, third and fourth vector are lin. indep. and span $\mathbb{R}^3$. The fifth can be disregarded.

Theorem 1.40 (Replacement Theorem)**.** *Let V be a vector space. Let $V = \text{span}(G)$, where G is a subset of V of cardinality n. Let L be a linearly independent subset of V of cardinality m. Then the following holds.*

- $m \leq n$
- *there exists a subset $H \subseteq G$ of cardinality $n - m$ such that* $\text{span}(L \cup H) = V$

Remark 1.41. A typical situation is for example $m = 2$ and $n = 5$, i.e., $L = \{v_1, v_2\}$ and $G = \{w_1, w_2, w_3, w_4, w_5\}$. The replacement theorem now says that there are two vectors in G that can be **replaced** with the two vectors from L such that the set obtained by the replacement still spans V. In other words, L can be injected into G and the result still spans V.

Corollary 1.42. *Let V be a vector space with a finite basis. Then all bases contain the same number of elements.*

Proof. Let β basis of cardinality m and γ basis of cardinality n. Since β lin. indep. and γ spans, we have $m \leq n$. By symmetry, we have $m = n$. $\square$

Definition 1.43. A vector space is called *finite dimensional* if there exists a basis consisting of finitely many vectors. The unique cardinality of a basis of a finite dimensional vector space is called the *dimension* of V, denoted $\dim(V)$.

Example 1.44. $\dim(\mathbb{R}^n) = n, \dim(Mat_{m\times n}) = mn$. (Consider the standard bases.)

Here are some more Corollaries.

Corollary 1.45. *Let V be a vector space of dimension n. Then any generating set S of V contains at least n elements.*

Proof. By Theorem 1.38, S contains a basis. By Corollary 1.42, that basis has n elements. So S contains at least n elements. $\square$

Corollary 1.46. *Let V be a vector space and $S \subset V$ a subset. If $V = \operatorname{span}(S)$ and $\#S = \dim(V)$, then S is a basis.*

Proof. By Theorem 1.38, S contains a basis. This basis must have $\dim V = \#S$ elements. Thus, this basis is S itself. $\square$

Corollary 1.47. *Let V be a vector space and $S \subset V$ a subset. If S is lin. indep. and $\#S = \dim(V)$, then S is a basis.*

Proof. Take any basis G. Apply the Replacement Theorem with G and $L = S$. Since $\#G = \#S = \dim V$, we have $H = \emptyset$ and $V = \operatorname{span} G = \operatorname{span} S$. $\square$

Corollary 1.48. *Let V be a vector space. Every lin. indep. subset S of V can be extended to a basis.*

Proof. Take any basis G of V. Apply the Replacement Theorem with $S = L$ and G. $\square$

Finally, let us prove the Replacement Theorem.

Proof of Replacement Theorem. For a fixed $n = \#G$, we do induction over $\#L = m$.

For $m = 0$, we have $L = \emptyset$. Take $H = G$. Done.

Induction step: "$m \to m + 1$".

Let $L = \{v_1, \ldots, v_{m+1}\}$, let $\tilde{L} = \{v_1, \ldots, v_m\}$.

Induction hypothesis $\Rightarrow \exists \tilde{H} = \{u_1, \ldots, u_{n-m}\}$ such that $V = \operatorname{span}(\tilde{L} \cup \tilde{H})$.

Write

$$v_{m+1} = a_1 v_1 + \ldots + a_m v_m + b_1 u_1 + \ldots + b_{n-m} u_{n-m}. \qquad (1)$$

Since L is lin. indep. we know that there exists i such that $b_i \neq 0$. Thus $n - m > 0$, i.e., $n \geq m + 1$. This proves the first part of the claim for $m + 1$.

It remains to show that if, w.l.o.g., $b_1 \neq 0$, then $H = \{u_2, \ldots, u_{n-m}\}$ works, i.e., $V = \operatorname{span}(L \cup H)$. Let $v \in V$ be arbitrary. We know we can write

$$v = \alpha_1 v_1 + \ldots + \alpha_m v_m + \gamma_1 u_1 + \ldots + \gamma_{n-m} u_{n-m}. \qquad (2)$$

If we solve (1) for u_1 and substitute into (2), we see that v can be written as a linear combination of $v_1, \ldots, v_{m+1}, u_2, \ldots, u_{n-m}$. Done. □

Now, let us discuss the dimension of subspaces.

Theorem 1.49. *Let V be a vector space. Let W be a subspace of V. Assume $\dim V$ is finite. Then $\dim W \leq \dim V$ and equality holds if and only if $V = W$.*

Proof. This is immediate from the Replacement Theorem. □

In the following examples, the task is to find a basis for (and the dimension of) the subspace W.

Example 1.50.
- Let $V = \mathbb{R}^3$. Let $W = \{(a_1, a_2, a_3) \mid a_1 + a_3 = 0 \text{ and } a_1 + a_2 - a_3 = 0\}$. Solving the system

  $$a_1 + a_3 = 0 \text{ and } a_1 + a_2 - a_3 = 0$$

  yields $W = \{(-t, 2t, t) \mid t \in \mathbb{R}\}$. Thus $\{(-1, 2, 1)\}$ is a basis, and the dimension of W is one.

Quiz 4:

(10 points) Let $G = \{(1,0,1,0),(1,1,0,1),(1,-2,1,2),(0,2,1,2)\}$. Let $L = \{(2,-1,1,3)\}$. You may assume without proof that G spans $\mathbb{R}^4$. Find a subset $H \subset G$ of cardinality 3 such that $H \cup L$ spans $\mathbb{R}^4$. Prove the spanning property with an explicit computation.

Answer. Note that $(2,-1,1,3) = (1,1,0,1)+(1,-2,1,2)$, so $(2,-1,1,3)$ may replace either $(1,1,0,1)$ or $(1,-2,1,2)$. To show that, e.g.,

$$\{(1,0,1,0),(2,-1,1,3),(1,-2,1,2),(0,2,1,2)\}$$

spans $\mathbb{R}^4$, start solving a system as follows:

$$\begin{aligned}
1a_1 + 2a_2 + 1a_3 + 0a_4 &= a\\
0a_1 - 1a_2 - 2a_3 + 2a_4 &= b\\
1a_1 + 1a_2 + 1a_3 + 1a_4 &= c\\
0a_1 + 3a_2 + 2a_3 + 2a_4 &= d
\end{aligned}$$

This system can be transformed to echelon form (do it!) and is thus solvable. □

Example 1.51.
- Let $V = \mathbb{R}^5$. Let $W = \{(a_1,a_2,a_3,a_4,a_5) \mid a_1+a_3+a_5 = 0 \text{ and } a_2 = a_4\}$. Solving the system

  $$a_1 + a_3 + a_5 = 0 \text{ and } a_2 = a_4$$

  yields

  $$\begin{aligned}
  W &= \{(-a_3-a_5, a_4, a_3, a_4, a_5) \mid a_3,a_4,a_5 \in \mathbb{R}\}\\
  &= \{a_3(-1,0,1,0,0)+a_4(0,1,0,1,0)+a_5(-1,0,0,0,1) \mid a_3,a_4,a_5 \in \mathbb{R}\}
  \end{aligned}$$

  Thus $\{(-1,0,1,0,0),(0,1,0,1,0),(-1,0,0,0,1)\}$ is a basis for W, and the dimension of W is three.

2 Linear transformations and matrices

2.1 Linear transformations, null spaces, and ranges

Definition 2.1. Let V, W be vector spaces over the same field F. We call a function $T : V \to W$ a *linear transformation* from V to W if

i. $\forall x, y \in V : T(x+y) = T(x) + T(y)$

ii. $\forall c \in F \forall x \in V : T(cx) = cT(x)$

Remark 2.2. We say T is *linear* for short.

Properties 2.3.

- $T(\vec{0}) = \vec{0}$
- $T(x-y) = T(x) - T(y)$
- $T(a_1v_1 + \ldots + a_nv_n) = a_1T(v_1) + \ldots + a_nT(v_n)$

Example 2.4.

- $T(a_1, a_2) = (2a_1 + a_2, a_1)$ (Check it!)
- $T : \mathbb{R}^5 \to \mathbb{R}^7, T(a_1, \ldots, a_5) = (a_1, a_2, 0, a_5, 0, 0, a_1)$
- $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), T(f) = \frac{df}{dx}$

Definition 2.5. LetV, W be vector spaces. Let $T : V \to W$ linear. We define the *null space* (aka *kernel*) of T to be

$$N(T) = \{x \in V \ : \ T(x) = \vec{0}\}.$$

Remark 2.6. Recall that the range of T is

$$R(T) = \{T(x) \ : \ x \in V\}.$$

Example 2.7. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$. To find the null space, set

$$T(a_1, a_2, a_3) = (0, 0) \Leftrightarrow a_1 - a_2 = 0 \text{ and } 2a_3 = 0.$$

The solution of the above system of two equations in three variables is

$$N(T) = \{(t, t, 0) | t \in \mathbb{R}\}.$$

Moreover, it is clear that T is onto, so $R(T) = \mathbb{R}^2$.

Theorem 2.8. *Let V, W be vector spaces and $T : V \to W$ linear. Then*

i. $N(T)$ is a subspace of V

ii. $R(T)$ is a subspace of W

Proof. i. $N(T)$ is non-empty because $\vec{0} \in N(T)$. Now, we just have to check closedness. If $T(x) = 0$ and $T(y) = 0$, then $T(ax + by) = aT(x) + bT(y) = 0 + 0 = 0$. Done.

ii. Again, just closedness. If $T(x) = v_1$ and $T(y) = v_2$, then $av_1 + bv_2 = aT(x) + bT(y) = T(ax + by)$. Done. □

Theorem 2.9. *Let V, W be vector spaces and $T : V \to W$ linear. Let $\{v_1, \ldots, v_n\}$ be a basis for V. Then $R(T) = \operatorname{span}\{T(v_1), \ldots, T(v_n)\}$.*

Proof. Let $v = a_1v_1 + \ldots + a_nv_n$. Then $T(v) = T(a_1v_1 + \ldots + a_nv_n) = a_1T(v_1) + \ldots + a_nT(v_n) \in \operatorname{span}\{T(v_1), \ldots, T(v_n)\}$. Done. □

Example 2.10. Problem: Find (a basis for) $R(T)$ when $T : \mathbb{R}^3 \to \mathbb{R}^3, T(a_1, a_2, a_3) = (a_1 - 2a_2, a_2 + a_3, 2a_1 + a_2 + 5a_3)$.

First, we note $T(1, 0, 0) = (1, 0, 2)$, $T(0, 1, 0) = (-2, 1, 1)$, $T(0, 0, 1) = (0, 1, 5)$. Thus, according to Theorem 2.9, $R(T) = \operatorname{span}\{(1, 0, 2), (-2, 1, 1), (0, 1, 5)\}$. Now, note that twice the first vector plus the second equals the third, so $R(T) = \operatorname{span}\{(1, 0, 2), (-2, 1, 1)\}$. The set $\{(1, 0, 2), (-2, 1, 1)\}$ is clearly a basis for $R(T)$, and $\dim R(T) = 2$.

Note that $N(T)$ is easily computed to be one-dimensional, and $\dim N(T) + \dim R(T) = 3$.

Definition 2.11. Let V, W be vector spaces and $T : V \to W$ linear. If $N(T), R(T)$ are finite dimensional, then let

$$\operatorname{nullity}(T) = \dim N(T), \quad \operatorname{rank}(T) = \dim R(T).$$

Theorem 2.12 (Dimension Theorem)**.** *Let V, W be vector spaces and $T : V \to W$ linear. If V is finite-dimensional, then*

$$\operatorname{nullity}(T) + \operatorname{rank}(T) = \dim V.$$

Proof. Let $\{v_1, \ldots, v_k\}$ be a basis for $N(T)$. In particular, $k = \operatorname{nullity}(T)$. Let $n = \dim V$. The Replacement Theorem implies that there are vectors $v_{k+1}, \ldots, v_n \in V$ such that $\{v_1, \ldots, v_n\}$ is a basis for V.

Claim: $\{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis for $R(T)$.

Spanning: Let $v = a_1v_1 + \ldots + a_nv_n \in V$ arbitrary. Then

$$T(v) = a_1T(v_1) + \ldots + a_kT(v_k) + a_{k+1}T(v_{k+1}) + \ldots + a_nT(v_n).$$

However, the first k summands are zero due to $v_1, \ldots, v_k \in N(T)$.

Lin. indep.: Write

$$b_{k+1}T(v_{k+1}) + \ldots + b_n T(v_n) = 0.$$

We have to conclude $b_{k+1} = \ldots = b_n = 0$. To do this, note that $b_{k+1}T(v_{k+1}) + \ldots + b_n T(v_n) = T(b_{k+1}v_{k+1} + \ldots + b_n v_n)$, i.e., $b_{k+1}v_{k+1} + \ldots + b_n v_n \in N(T)$. Thus, there exist $a_1, \ldots, a_k$:

$$a_1 v_1 + \ldots + a_k v_k = b_{k+1}v_{k+1} + \ldots + b_n v_n.$$

Since $\{v_1, \ldots, v_n\}$ is a basis for V and thus lin. indep., this is only possible if

$$a_1 = \ldots = a_k = b_{k+1} = \ldots = b_n = 0.$$

□

Theorem 2.13. *Let V, W vector spaces. Let $T : V \to W$ linear. Then T is one-to-one if and only if $N(T) = \{\vec{0}\}$.*

Proof. $\Rightarrow$. Saw: $T(\vec{0}) = \vec{0}$. Since T is one-to-one, this implies $N(T) = \{\vec{0}\}$.

$\Leftarrow$. Assume $T(x) = T(y)$. Then $T(x) - T(y) = 0$. By linearity of T, $T(x - y) = \vec{0}$. By assumption, $x - y = \vec{0}$. Done. □

Theorem 2.14. *Let V, W vector spaces. Let $\{v_1, \ldots, v_n\}$ be a basis for V. Let $w_1, \ldots, w_n$ be a list of arbitrary vectors in W. Then there exists a unique $T : V \to W$ linear such that $T(v_i) = w_i$ for all $i = 1, \ldots, n$.*

Proof. Recall that an arbitrary $v \in V$ can be written as $v = \sum_i a_i v_i$ with unique coefficients a_i. Then set $T(v) = \sum_i a_i w_i$. It is easy to check that this defines a well-defined linear map as required in the Theorem. Uniqueness is also clear. □

Corollary 2.15. *Let V, W vector spaces. Let $U, T : V \to W$ linear with $U(v_i) = T(v_i)$ on a basis $\{v_1, \ldots, v_n\}$ for V. Then $U = T$.*

Quiz 5:

(10 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$(a_1, a_2, a_3, a_4) \mapsto (a_1 + a_2 - a_3, a_2 + a_4, a_1 - a_2 - a_3 - 2a_4, 2a_1 + 3a_2 - 2a_3 + a_4).$$

Find bases for the null space and range of T. Justify your reasoning carefully.

Answer. To find the null space $N(T)$, solve $T(a_1, a_2, a_3, a_4) = (0, 0, 0, 0)$. This turns out to be equivalent to the system

$$a_1 + a_2 - a_3 = 0, a_2 + a_4 = 0.$$

Thus, $N(T) = \{(s + t, -t, s, t) \mid s, t \in \mathbb{R}\}$.

By the Dimension Theorem, the rank of T is $4 - 2 = 2$, so any two linearly independent vectors in the range of T will form a basis. Simply take $T(1, 0, 0, 0) = (1, 0, 1, 2)$ and $T(0, 1, 0, 0) = (1, 1, -1, 3)$ (which are clearly linearly independent).

2.2 The matrix representation of a linear transformation

Definition 2.16. Let V be a finite dimensional vector space. An *ordered basis* for V is a basis endowed with a specific order.

Example 2.17. As ordered bases,

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \neq \{(0, 1, 0), (1, 0, 0), (0, 0, 1)\}.$$

Definition 2.18. Let $\beta = \{u_1, \ldots, u_n\}$ ordered basis for V. We saw earlier:

$$\forall x \in V \ \exists! a_1, \ldots, a_n : x = a_1 u_1 + \ldots + a_n u_n.$$

Write

$$[x]_\beta = (a_1, \ldots, a_n)$$

for the *coordinate vector of* x relative to β. In particular, $[u_i]_\beta = e_i$.

Definition 2.19. Take V with $\beta = \{v_1, \ldots, v_n\}$, W with $\gamma = \{w_1, \ldots, w_m\}$. Let $T : V \to W$ linear. Write

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

for $j = 1, \ldots, n$. Call the matrix (a_{ij}) the *matrix representation of* T with respect to β and γ. When $V = W$ and $\beta = \gamma$, write $A = [T]_\beta$.

Remark 2.20. The key fact to remember is that the j-th **column** of the matrix representation is $[T(v_j)]_\gamma$.

Example 2.21. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$. Let β and γ be the respective standard bases. Then

$$T(1,0) = (1,0,2), \quad T(0,1) = (3,0,-4).$$

Thus,

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}.$$

(b) Same map, but with $\gamma' = \{e_2, e_1, e_3\}$:

$$[T]_\beta^{\gamma'} = \begin{pmatrix} 0 & 0 \\ 1 & 3 \\ 2 & -4 \end{pmatrix}.$$

(c) Same map as in (a), but with $\beta' = \{e_2, e_1\}$:

$$[T]_{\beta'}^\gamma = \begin{pmatrix} 0 & 0 \\ 3 & 1 \\ -4 & 2 \end{pmatrix}.$$

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard basis and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$. By solving a system of linear equations, we find

$$T(1,0) = (1,1,2) = -\frac{1}{3}(1,1,0) + \frac{2}{3}(2,2,3),$$

and

$$T(0,1) = (-1,0,1) = -(1,1,0) + (0,1,1).$$

Thus,

$$[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}.$$

Definition 2.22. Let $U, T : V \to W$ be linear. Then

$$(U + T)(x) = U(x) + T(x)$$

and

$$(cT)(x) = cT(x).$$

Theorem 2.23. *Let V, W be given vector spaces. The set of all linear transformations $V \to W$ is a vector space with $+$ and $\cdot$ defined as above. Write $\mathcal{L}(V, W)$ for this vector space. Write $\mathcal{L}(V)$ for $\mathcal{L}(V, V)$. (Write V^* for $\mathcal{L}(V, F)$.)*

Proof. Check the axioms! □

Definition 2.24. Let $U, T : V \to W$ linear. Then

i. $[U + T]_\beta^\gamma = [U]_\beta^\gamma + [T]_\beta^\gamma$

ii. $[aT]_\beta^\gamma = a[T]_\beta^\gamma$

Proof. Write $U(v_j) = \sum a_{ij} w_i$, $T(v_j) = \sum b_{ij} w_i$. Then

$$(U + T)(v_j) = U(v_j) + T(v_j) = \sum a_{ij} w_i + \sum b_{ij} w_i = \sum (a_{ij} + b_{ij}) w_i.$$

Thus, the ij entry of $[U + T]_\beta^\gamma$ is $a_{ij} + b_{ij}$. □

2.3 Composition of linear transformations and matrix multiplication

Theorem 2.25. *Let V, W, Z be vector spaces over the same field. Let $T : V \to W$ and $U : W \to Z$ be linear. Then $U \circ T : V \to Z$ is linear.*

Proof.

$$(U \circ T)(ax + by) = U(T(ax + by)) = U(aT(x) + bT(y)) = aU(T(x)) + bU(T(y))$$

$$= a(U \circ T)(x) + b(U \circ T)(y).$$

□

Theorem 2.26. *Let $U, S, T : V \to V$ linear. Then*

- $U \circ (S + T) = U \circ S + U \circ T$
- $(U + S) \circ T = U \circ T + S \circ T$
- $U \circ (S \circ T) = (U \circ S) \circ T$

- $\mathrm{id} \circ U = U \circ \mathrm{id} = U$
- $a(U \circ S) = (aU) \circ S = U \circ (aS)$

Now, let us investigate the matrix of a composition of linear transformations. Let $T : V \to W$, $U : W \to Z$. Let $\alpha = \{v_j | j = 1, \ldots, n\}, \beta = \{w_k | k = 1, \ldots, m\}, \gamma = \{z_i | i = 1, \ldots, p\}$ be the corresponding ordered basis, in alphabetical order. Let $[T]_\alpha^\beta = B$, $[U]_\beta^\gamma = A$. Then

$$(U \circ T)(v_j) = U(T(v_j)) = U(\sum_k b_{kj} w_k) = \sum_k b_{kj} U(w_k) =$$

$$\sum_k b_{kj} (\sum_i a_{ik} z_i) = \sum_i (\sum_k a_{ik} b_{kj}) z_i.$$

Consequently, if $C = [U \circ T]_\alpha^\gamma$, $c_{ij} = \sum_k a_{ik} b_{kj}$.

Quiz 6:

1. (5 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(a_1, a_2) = (a_1 + 2a_2, a_1 - a_2)$. Let $\beta = \{(1, 0), (0, 1)\}$ and $\gamma = \{(-1, 2), (1, -1)\}$. Compute $[T]_\beta^\gamma$.

2. (5 points) Let V be an n-dimensional vector space. Let $T : V \to V$ be a linear transformation. Let W be a subspace of V with

$$T(W) := \{T(w) \mid w \in W\} \subset W.$$

Assume that $\dim W = k$. Prove that there is an ordered basis for V such that $[T]_\beta$ is of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where A is a $k \times k$ matrix and 0 is the $(n - k) \times k$ zero matrix.

Answer. 1. Solve $T(1, 0) = (1, 1) = a(-1, 2) + b(1, -1)$ for a and b. This gives $a = 2$, $b = 3$.

Solve $T(0, 1) = (2, -1) = a(-1, 2) + b(1, -1)$ for a and b. This gives $a = 1$, $b = 3$. Thus,

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix}.$$

2. Pick an ordered basis $\{v_1, \ldots, v_k\}$ for W and complete it to an ordered basis $\{v_1, \ldots, v_n\}$ for V, which can be done due to the Replacement Theorem. Now, note that for $j = 1, \ldots, k$:

$$T(v_j) = a_{1,j} v_1 + \ldots + a_{k,j} v_k + 0 v_{k+1} + \ldots + 0 v_n$$

due to $T(W) \subset W$ and the uniqueness of coordinates. Recall that the n-tuples $(a_{1,j}, \ldots, a_{k,j}, 0, \ldots, 0)$ $(j = 1, \ldots, k)$ form the first k columns of the matrix $[T]_\beta$. Done.

Definition 2.27. Let A be a $p \times m$ matrix and B an $m \times n$ matrix. Define the *matrix product* of A and B to be the $p \times n$ matrix given by

$$(AB)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj},$$

where $i = 1, \ldots, p$, $j = 1, \ldots, n$.

We have just established the following theorem.

Theorem 2.28. *Let $T : V \to W$, $U : W \to Z$. Then $[U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$.*