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ON OPEN MAPPINGS OF LOCALLY CONNECTED CONTINUA ONTO ARCS

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ABSTRACT. Several structural characterizations of locally connected continua that admit an open mapping onto an arc are obtained.

It is well-known that each arc of an n-cube is its open retract. The Sierpiński universal plane curve, the Menger universal curve and the simplest indecomposable continuum ([7], §48, V, Example 1, p. 204) are also retractable onto arcs under open mappings. Oversteegen [9] has constructed an example of an open and monotone retraction of a smooth dendroid onto an arc. Thus a problem arises to characterize continua which are retractable onto arcs under open mappings. Special attention — with respect to the above problem — is paid to the class of locally connected continua.

Open mappings from a locally connected continuum onto an arc have been investigated by a number of authors, in particular by G. T. Whyburn in his book [11], but under some additional assumptions that concerned either mappings (which were assumed to be light or non-alternating for example, compare [11], Theorems 3.1, p. 189, 2.1, p. 212 and 4.1, p. 218) or the structure of the domain space (dendrites, two-manifolds, etc.). Other particular results were obtained by the authors in [5], where it is shown that each subarc of a graph is its open retract, and in [8], where some conditions are studied that imply the nonexistence of an open retraction of a locally connected continuum onto an arc. Further studies on the problem lead to some characterizations of locally connected continua which admit an open mapping onto an arc. In these characterizations

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both the structure of the set of all separating points and a manner of separation of the continuum by particular points are important, and they play an essential role in the formulation of the considered conditions. In this paper we present the obtained results. The characterizations given in this paper (see the seven equivalent conditions in Theorem 3) as well as the methods by which they are proved are expressed in the same language as those in the Whyburn's book [11]. Thus the research can be understood as a prolongation and a completion of G. T. Whyburn's studies in the area.

All *spaces* considered in this paper are assumed to be metrizable and separable. Given a subset A of a space X, we denote by $\operatorname{cl} A$ the closure, by $\operatorname{bd} A$ the boundary, and by $\operatorname{int} A$ the interior of A in X. A *continuum* means a compact connected space.

We shall use the notion of *order of a point* in the sense of Menger-Urysohn (see e.g. [7], §51, I, p. 274 or [11], p. 48), and we denote by $\operatorname{ord}(p, X)$ order of the space X at a point $p \in X$. Points of order one are called *end points*, and points of order at least three — *ramification points* of X. A *dendrite* means a locally connected continuum containing no simple closed curve. Recall that for a point p of a dendrite X the concept of $\operatorname{ord}(p, X)$ agrees with the number of components of the set $X \setminus \{p\}$ (see [11], Chapter 5, Theorem (1.1) (iv), p. 88). By the *standard universal dendrite of order* 3 we mean a dendrite D_3 such that every ramification point of D_3 is of order 3 and for every arc A contained in D_3 the set of ramification points of D_3 which belong to A is a dense subset of A. It is shown in Theorem 3 of [3], p. 493 that all open images of D_3 are homeomorphic to D_3, so D_3 cannot be openly mapped onto an arc.

We start with recalling some old (and rather forgotten) concepts due to G. T. Whyburn.

A family (a sequence) of subsets of a metric space is called a *null family* (a *null sequence*) provided that for every $\varepsilon > 0$ at most a finite number of its elements are of diameter greater than ε. For a metric space X a subset Y of X is called an *A-set* provided that $X \setminus Y$ is the union of a finite number or of a null sequence of mutually disjoint open sets each having at most one boundary point. The following properties of A-sets in locally connected continua are known (see [11], (3.1) and (3.11), p. 67 and (3.31) and (3.61), p. 69).

Proposition 1 (G. T. Whyburn). *If a continuum X is locally connected and a subset Y of X is an A-set, then Y is a locally connected continuum, and*

(1) *the components of $X \setminus Y$ are open sets, they form a null sequence at most, and the boundary of each component of $X \setminus Y$ is a single point.*

Conversely (see [11], Theorem (3.3), p. 67) the following is true.

Proposition 2 (G. T. Whyburn). *If a continuum X is locally connected and a subcontinuum Y of X is such that the boundary of each component of $X \setminus Y$ is a single point, then Y is an A-set.*

A-sets contained in a locally connected continuum are characterized as follows (see [11], (3.4), p. 69).

Proposition 3 (G. T. Whyburn). *A subcontinuum Y of a locally connected continuum X is an A-set if and only if for each arc ab in X the condition $a, b \in Y$ implies $ab \subset Y$.*

Let two distinct points a and b of a locally connected continuum X be given. We say that a point $p \in X$ *separates the points a and b in X* (or *separates X between a and b*) provided that the points a and b belong to distinct components of $X \setminus \{p\}$. We put

$$E(a, b) = \{ p \in X : p \text{ separates } a \text{ and } b \text{ in } X \}.$$

A point p of a locally connected continuum X is called a *separating point of X* if it separates X between some two distinct points of $X \setminus \{p\}$. Another name used for this concept is a *cut point of X* (see [11], p.41; compare [11], Chapter 3, Section 8, a paragraph preceding (8.1), p. 58). Since for each end point p of X there exist arbitrarily small neighborhoods of p the boundary of each of which consists of a single point, it follows that each end point of X is the limit of a sequence of separating points of X, namely the terms of the sequence are just the points lying in the boundaries of arbitrarily small neighborhoods mentioned in the definition of the end point. Consequently,

(2) the set of end points of a locally connected continuum X is contained in the closure of the set of separating points of X.

Two distinct points a and b of a locally connected continuum X are said to be *conjugate* provided that no point of X separates a and b in X, i.e., if $E(a, b) = \emptyset$. If a point $p \in X$ is neither a separating point nor an end point of X, then the set consisting of p and of all points of X conjugate to p is called a *simple link* of X. By a *cyclic element* of X we mean any separating point of X, or an end point of X, or a simple link of X. The separating points and end point are called *degenerate* cyclic elements of X, while the simple links are called *true* cyclic elements. Denote by $W(a, b)$ the union of all cyclic elements C of X such that

(3) $C \cap (E(a, b) \cup \{a, b\})$ consists of exactly two points.

For every two distinct points a and b of a locally connected continuum X the intersection of all A-sets in X containing a and b is called the *cyclic chain from a to b in X*, and it is denoted by $C(a,b)$. The following result is known (see [11], (5.2), p. 71).

Proposition 4 (G. T. Whyburn). *For every two distinct points a and b of a locally connected continuum X we have*

(4) $C(a,b) = E(a,b) \cup \{a,b\} \cup W(a,b)$.

Besides the above proposition, the structure of cyclic chains in a locally connected continuum X is described in the next known result (due to G. T. Whyburn) which has merely been mentioned without proof in [10], p. 914. For completeness, we present its proof here. To formulate the result, denote by $U(a,b)$ the union of all arcs ab joining the points a and b in X, and note that

(5) $E(a,b) \cup \{a,b\} \subset U(a,b)$.

To prove the result, we recall a concept of cyclic connectedness. A set S is said to be *cyclicly connected* provided that every two points of S lie together in some simple closed curve contained in S. The following theorem, due to W. L. Ayres (see e.g. [11], (9.5), p. 79) will be needed in the sequel.

Proposition 5 (W. L. Ayres). *A locally connected continuum C is cyclicly connected if and only if for each triple of points p_1, p_2, p_3 of C taken in any order p_i, p_j, p_k for $i,j,k \in \{1,2,3\}$ there exists an arc $p_ip_jp_k$ in C.*

Corollary 6. *For every triple of points p_1, p_2, p_3 of a true cyclic element C of a locally connected continuum taken in any order p_i, p_j, p_k for $i,j,k \in \{1,2,3\}$ there exists an arc $p_ip_jp_k$ in C.*

PROOF. Each true cyclic element of a locally connected continuum X is cyclicly connected (see [11], Corollary 1, p. 79). Further, it is an A-set ([11], (3.5) (i), p. 69), and therefore, by Proposition 1, it is a locally connected subcontinuum of X. Now the conclusion follows from the Ayres theorem (Proposition 5 above). □

Proposition 7. *For every two distinct points a and b of a locally connected continuum X we have $C(a,b) = U(a,b)$.*

PROOF. Since the intersection of an arbitrary family of A-sets is an A-set (see [11], (3.5) (ii), p. 69), the cyclic chain $C(a,b)$ is an A-set, whence the inclusion $U(a,b) \subset C(a,b)$ follows from Proposition 3. To prove the opposite inclusion it is enough to show that for each point x in $C(a,b)$ there exists an arc axb from a

to b through x. By (4) and (5) we have only to consider the case $x \in W(a,b)$. Then x is in a cyclic element C of X such that (3) holds. Thus C is a true cyclic element. Denote by y and z points of C such that $C \cap (E(a,b) \cup \{a,b\}) = \{y,z\}$. Since $y,z \in E(a,b) \cup \{a,b\}$, we infer $y,z \in U(a,b)$ by (5). By Corollary 6 above there exists an arc yxz in C. Since $y,z \in U(a,b)$, we have $x \in U(a,b)$. The proof is finished. □

A mapping $f : X \to Y$ of a space X onto Y is said to be *open* if for each open subset of X its image under f is an open subset of Y. Further, f is said to be *interior at a point* $p \in X$ provided that for every open set U in X containing p, the set $f(U)$ contains the point $f(p)$ in its interior. The following statement is obvious.

Statement 8. *A mapping is open if and only if it is interior at each point of its domain.*

Now we are going to discuss some structural conditions concerning a locally connected continuum X under which there exists an open mapping of X onto the closed unit interval $[0,1]$ (or, equivalently, onto an arc). Our starting point is Whyburn's characterization of those locally connected continua which can be mapped onto an arc under a mapping that is simultaneously open and non-alternating. Recall that a mapping $f : X \to Y$ of a space X onto Y is said to be *non-alternating* provided that for each point $y \in Y$ and for any separation of $X \setminus f^{-1}(y)$ into two mutually separated sets X_1 and X_2 we have $f(X_1) \cap f(X_2) = \emptyset$ or, equivalently, $X_i = f^{-1}(f(X_i))$ for $i \in \{1,2\}$ (see [11], (4.2), p. 127 and Chapter 8, Section 2, p. 137-140). The following lemma will be needed in the sequel.

Lemma 9. *Let X be a locally connected continuum, and let $f : X \to [0,1]$ be an open mapping such that $f^{-1}(0)$ and $f^{-1}(1)$ are singletons. Then for every $t \in [0,1] \setminus \{0,1\}$ the set $X \setminus f^{-1}(t)$ consists of two components X_1 and X_2 such that $f(X_1) \cap f(X_2) = \emptyset$, and thus f is non-alternating.*

PROOF. Put $\{a\} = f^{-1}(0)$ and $\{b\} = f^{-1}(1)$. Suppose on the contrary that there exist a number $t \in [0,1] \setminus \{0,1\}$ and a component C of $X \setminus f^{-1}(t)$ which contains neither a nor b. Then C is open as a component of an open subset of the locally connected continuum X, so $f(C)$ is an open subset of $[0,1]$ such that $\operatorname{cl} f(C) = f(C) \cup \{t\}$, and thus either $0 \in f(C)$ or $1 \in f(C)$, a contradiction. □

The above mentioned characterization runs as follows (see [11], (2.1), p. 212 and (4.2), p. 218).

Theorem 10 (G. T. Whyburn). *Let two distinct points a and b of a locally connected continuum X be given. The following conditions are equivalent:*

(6) *there exists a non-alternating open mapping $f : X \to [0,1]$ such that $f(a) = 0$ and $f(b) = 1$;*

(7) *for every arc $ab \subset X$ there exists a non-alternating open retraction $r : X \to ab$;*

(8) $C(a,b) = X$.

Thus, as a consequence of the implication from (8) to (6) we get the following corollary.

Corollary 11. *If a locally connected continuum X contains two distinct points a and b such that* (8) *holds, then there exists an open mapping $f : X \to [0,1]$ such that $f(a) = 0$ and $f(b) = 1$.*

Obviously, the opposite implication to that of Corollary 11 does not hold in general, because according to Theorem 10 we need an additional assumption on the mapping (namely that it is non-alternating) to construct the two needed points $a, b \in X$ satisfying (8). In other words, the existence of points $a, b \in X$ with (8) suffices, but it is not necessary, to the existence of an open mapping from X onto $[0,1]$. To find a necessary condition (which is also sufficient), and which is formulated in the same language, we generalize the concept of a cyclic chain between distinct points of a locally connected continuum X to a cyclic chain between some subsets of X. The authors express their gratitude to Dr. K. Omiljanowski and Dr. J. R. Prajs for fruitful discussions concerning this generalization.

Given a locally connected continuum X, let A and B be disjoint closed subsets of X, both with empty interior. Then by a *cyclic chain $C(A,B)$ from A to B* we mean the union of A and B and of all arcs ab in X such that

(9) $ab \cap A = \{a\}$ and $ab \cap B = \{b\}$,

i.e.,

(10) $C(A,B) = A \cup B \cup \bigcup\{ab \subset X : ab \cap A = \{a\} \text{ and } ab \cap B = \{b\}\}$.

Observe that in case when the sets A and B are singletons $\{a\}$ and $\{b\}$ respectively, the concept of $C(A,B)$ coincides with one of $C(a,b)$ by Proposition 7.

The following characterization of locally connected continua admitting an open mapping onto an arc has been proposed by Dr. J. R. Prajs.

Theorem 12 (J. R. Prajs). *For a locally connected continuum X the following conditions are equivalent:*

(11) *there exists an arc $L \subset X$ and an open retraction $r : X \to L$ of X onto L;*
(12) *there exists an open mapping $f : X \to [0,1]$ of X onto the closed unit interval $[0,1]$;*
(13) *there are two disjoint closed subsets A and B of X, both with empty interior, such that $C(A,B) = X$.*

PROOF. The implication from (11) to (12) is obvious. To show the implication from (12) to (13) assume that there exists an open mapping $f : X \to [0,1]$. Putting $A = f^{-1}(0)$ and $B = f^{-1}(1)$ we see that A and B are disjoint closed subsets of X with $\operatorname{int} A = \operatorname{int} B = \emptyset$. To prove the equality

(14) $C(A,B) = X$

it is enough to verify, according to definition (10) of $C(A,B)$, that for each point x of $X \setminus (A \cup B)$ there is an arc axb from a to b through x such that conditions (9) are satisfied. Shrink A and B to points, and consider the double quotient space $Z = (X/A)/B$. Since A and B are closed, Z is a locally connected continuum. Let $\pi : X \to Z$ be the natural projection. Put $\pi(A) = \{a'\}$ and $\pi(B) = \{b'\}$. Denote by $g : Z \to [0,1]$ the mapping satisfying $g \circ \pi = f$ and note that since f is open, g is open, too ([11], Chapter 8, (3.1), p. 140). By Lemma 9 it is non-alternating, and thus according to Theorem10 we have $Z = C(a', b')$. Denote by L an arc in Z with end points a' and b' and such that $\pi(x) \in L$. Since $\pi|(X \setminus (A \cup B))$ is one-to-one, there is an arc L_0 in X such that $x \in L_0$ and the end points of L_0 lie close to A and to B respectively. Let a and b be points of A and B respectively, close to the end points of L_0. By the local arcwise connectedness of X at a and at b there are small arcs L_a and L_b joining a and b with the end points of L_0 respectively. Then $L_a \cup L_0 \cup L_b$ contains an arc with end points a and b which passes through the point x. Consequently, (14) is shown, and so (13) is satisfied.

To verify the implication from (13) to (11) assume subsets A and B of X do exist having the properties listed in (13). Let L be an irreducible arc between A and B (i.e., such that L intersects both A and B and no proper subarc of L does). As in the previous part of the proof, shrink A and B to points, and consider the double quotient space $Z = (X/A)/B$. Since A and B are closed, Z is a locally connected continuum. Again let $\pi : X \to Z$ be the natural projection. Put $\pi(A) = \{a'\}$ and $\pi(B) = \{b'\}$. Since $A \cap B = \emptyset$, we have $a' \neq b'$. And since $\operatorname{int} A = \operatorname{int} B = \emptyset$, we infer from the definition of the mapping π that for each arc $ab \subset X$ as in (10), i.e., such that conditions (9) hold, the partial mapping $\pi|ab : ab \to a'b'$ is a homeomorphism. Thus by Proposition 7 it follows that $\pi(C(A,B))$ is contained in the cyclic chain $C(a', b')$ from a' to b' in Z. Since π is surjective, condition (14) implies that $Z = \pi(X) = \pi(C(A,B)) \subset C(a',b') \subset Z$, whence it follows $C(a',b') = Z$. Now we infer from Theorem 10 that there exists an open retraction $\sigma : Z \to \pi(L)$ with $\sigma(a') = a'$ and $\sigma(b') = b'$. Put $f = \sigma \circ \pi : X \to \pi(L)$. We claim that f is open. Indeed, since A and B are closed, the set $X \setminus (A \cup B)$ is open; and since the partial mapping $\pi|(X \setminus (A \cup B)) : X \setminus (A \cup B) \to Z \setminus \{a',b'\}$ is a homeomorphism, the mapping $\pi : X \to Z$ is interior at each point of the set $X \setminus (A \cup B)$. Therefore, since σ is open, the mapping f is interior at each point of $X \setminus (A \cup B)$. Further, since each point of $A \cup B$ is sent under π into $\{a',b'\}$, and since $\operatorname{int} A = \operatorname{int} B = \emptyset$ by assumption, we infer from the openness of σ that f is interior at each point of $A \cup B$. Therefore f is open according to Statement 8. Since $\pi|L$ is a homeomorphism, the composition $(\pi|L)^{-1} \circ f : X \to L$ is the needed open retraction. This finishes the proof. □

Corollary 13. *If a locally connected continuum X contains two disjoint closed subsets A and B of X, both with empty interior, such that $C(A,B) = X$, then there exists an open mapping $f : X \to [0,1]$ of X onto the closed unit interval $[0,1]$ such that*

(15) $f^{-1}(0) = A$ *and* $f^{-1}(1) = B$.

Proof. The retraction $(\pi|L)^{-1} \circ f : X \to L$ as constructed in the proof of the implication from (13) to (11) of Theorem 12 composed with a homeomorphism of L onto $[0,1]$ satisfies the required conditions. □

Remark 1. Observe that in the proof of implication from (12) to (11) of Theorem 12 we used intermediate condition (13), and therefore in the construction of the needed retraction we did not exploit the mapping f of condition (12). One could expect that to construct the retraction it would be enough to compose f and a homeomorphism. We will show that this is not the case. To this aim denote by D the unit disc in the plane, i.e., put $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

Example 14. *There is an open mapping $f : D \to [0,1]$ and an arc $L \subset D$ such that for no homeomorphism $h : [0,1] \to L$ the composition $h \circ f$ is a retraction.*

Proof. Consider the pseudo-circle P (see [1] and [6] for the definition and basic properties) located in the disc D so that it is disjoint with the boundary $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ of D. Thus $D \setminus P$ is the union of two disjoint open subsets U_1 and U_2 of D such that U_1 is homeomorphic to the plane $\mathbb{R}^2$, and U_2 is homeomorphic to $D \setminus \{(0,0)\}$. Let $h_1 : U_1 \to \mathbb{R}^2$ and $h_2 : U_2 \to D \setminus \{(0,0)\}$ be the homeomorphisms. Denote by $\|(x,y)\| = \sqrt{x^2 + y^2}$ the standard norm in the plane $\mathbb{R}^2$. Consider a binary relation ρ on D defined for points $p, q \in D$ by the following conditions.

$$p \rho q \Leftrightarrow \quad \text{either } p, q \in P \text{ or there is an index } i \in \{1, 2\} \text{ such that } p, q \in U_i \text{ and } \|h_i(p)\| = \|h_i(q)\|.$$

Then the decomposition space D/ρ is homeomorphic to $[0, 1]$. Denote by $f : D \to [0, 1]$ a mapping which is the composition of the quotient mapping from D onto D/ρ and a homeomorphism from D/ρ onto $[0, 1]$ and such that $f^{-1}(0)$ is a point of U_1, $f^{-1}(1/2) = P$, and $f^{-1}(1) = S$, and let L be an arbitrary arc in D whose one end point is the point $f^{-1}(0)$ and the other one is in the boundary $f^{-1}(1) = S$ of D as the only point of $L \cap S$. Suppose on the contrary that there is a homeomorphism $h : [0, 1] \to L$ such that $h \circ f$ is a retraction of D onto L. Then $P \cap L$ is a singleton $\{h(1/2)\}$ which locally separates P. This contradicts hereditary indecomposability of P. □

Now we are going to show that one cannot omit the sets A and B as uniands in the definition of a cyclic chain $C(A, B)$ (see the union in the right member of the equality (10)). In other words, if we put

$$C'(A, B) = \bigcap \{\, ab \subset X : ab \cap A = \{a\} \text{ and } ab \cap B = \{b\}\,\},$$

where the sets A and B are disjoint, closed and having empty interiors, then Theorem 12 with $C'(A, B)$ in place of $C(A, B)$ is not true. This can be seen by the following example, the main idea of which is due to Dr. J. R. Prajs.

Example 15 (J. R. Prajs). *There exists a locally connected continuum X admitting an open mapping onto the closed unit interval* $[0, 1]$ *such that for every two disjoint closed sets A and B both with empty interior we have $C'(A, B) \neq X$.*

PROOF. The construction is made in the Euclidean 3-space $\mathbb{R}^3$. Denote by Q the closed unit square $[0, 1] \times [0, 1] \times \{0\}$. In its relative interior, i.e., in the set $(0, 1) \times (0, 1) \times \{0\}$ locate a continuum K being the union of a closed segment J and of a spiral line L which is a one-to-one continuous image of the non-negative reals $[0, +\infty)$ and which approximates J going around it close and close to J in such a way that $J = \operatorname{cl} L \setminus L$ and that no point of J is accessible from the complement of $Q \setminus K$, i.e.,

(16) there is no arc $ab \subset Q$ such that $a \in J, b \in Q \setminus K$ and $ab \cap K = \{a\}$.

Further, for each $n \in \mathbb{N}$, let $p_n \in Q \setminus K$ with $K = \operatorname{cl}\{\, p_n : n \in \mathbb{N}\,\} \setminus \{\, p_n : n \in \mathbb{N}\,\}$, and let $p_n q_n$ be a straight line segment perpendicular to the plane $\mathbb{R}^2 \times \{0\}$, with

$\lim \operatorname{diam} p_n q_n = 0$. Put $X = Q \cup \bigcup \{ p_n q_n : n \in \mathbb{N} \}$. Then X is a locally connected continuum. We define an open mapping $f : X \to [0,1]$ as the composition of four intermediate mappings. The first step is shrinking the continuum $K \subset Q \subset X$ to a point. Let $f_1 : X \to X/K = X_1$ be the quotient mapping. Then for each $n \in \mathbb{N}$ the partial mapping $f_1 | p_n q_n$ is a homeomorphism, $f_1(Q)$ is homeomorphic to Q, and the sequence of points $f_1(p_n)$ converges to the point $f_1(K)$. So, without loss of generality, we can assume that $f_1(Q) = Q$, $f_1(K) = (1/3, 1/2, 0)$, for each $n \in \mathbb{N}$, $f_1(p_n) \in [1/3, 2/3] \times \{1/2\} \times \{0\}$, and that the arcs $f_1(p_n q_n)$ are straight line segments perpendicular to the plane $\mathbb{R}^2 \times \{0\}$. Therefore, under these assumptions, we have $X_1 = Q \cup \bigcup \{ f_1(p_n q_n) : n \in \mathbb{N} \}$.

The second step is projecting the square Q onto its middle segment $M = [0,1] \times \{1/2\} \times \{0\}$. More precisely, we define a continuum $X_2 = M \cup \bigcup \{ f_1(p_n q_n) : n \in \mathbb{N} \}$ and a mapping $f_2 : X_1 \to X_2$ such that $f_2 | Q : Q \to M$ is defined by $f_2((x, y, 0)) = (x, 1/2, 0)$ and $f_2 | (\bigcup \{ f_1(p_n q_n) : n \in \mathbb{N} \})$ is the identity.

The third step is retracting X_2 onto the middle third part T_1 of M. Precisely, putting

$$T_i = [i/3, (i+1)/3] \times \{1/2\} \times \{0\} \quad \text{for} \quad i \in \{0, 1, 2\}$$

we have $M = T_0 \cup T_1 \cup T_2$ and we define $f_3 : X_2 \to T_1$ as follows. The partial mapping $f_3 | T_1 : T_1 \to T_1$ is the identity, and $f_3 | T_0 : T_0 \to T_1$ and $f_3 | T_2 : T_2 \to T_1$ are homeomorphisms. So, $f_3 | M : M \to T_1$ is defined. For each $n \in \mathbb{N}$ we put $f_3(f_1(q_n)) = (1/3, 1/2, 0)$ and we take $f_3 | f_1(p_n q_n)$ as a homeomorphism of the straight line segment $f_1(p_n q_n)$ onto the straight line segment in T_1 joining $(1/3, 1/2, 0)$ with $f_1(p_n)$. The reader can verify that the composition $f_3 \circ f_2 \circ f_1 : X \to T_1$ is an open mapping (note that neither f_1 nor f_2 is open).

The fourth mapping $f_4 : T_1 \to [0,1]$ is a homeomorphism. Putting $f = f_4 \circ f_3 \circ f_2 \circ f_1$ we see that f is open.

To show the final part of the conclusion consider arbitrary subsets A and B of X such that $A = \operatorname{cl} A$, $B = \operatorname{cl} B$, $A \cap B = \emptyset$, and $\operatorname{int} A = \operatorname{int} B = \emptyset$. Then either A or B contains almost all end points q_n of X. Let these end points be in A. Since $K = \operatorname{Lim} \{ p_n : n \in \mathbb{N} \} = \operatorname{Lim} \{ q_n : n \in \mathbb{N} \}$, we have $J \subset K \subset \operatorname{cl} \{ q_n : n \in \mathbb{N} \} \subset \operatorname{cl} A = A$. Consider an arc ab with $a \in J \subset A$ and $b \in B$. If $b \in p_n q_n$ for some $n \in \mathbb{N}$, then the arc $a p_n \subset ab$ lies in Q; if $b \in Q$, then $ab \subset Q$. In any case each such arc has to intersect K out of the point a, according to (16). Thus the segment J is not in $C'(A, B)$, whence $C'(A, B) \neq X$. □

Remark 2. As another argument for the existence of a mapping $f : X \to [0,1]$ of Example 15 one can apply Theorem 12 putting $A = \{ q_n : n \in \mathbb{N} \} \cup K$, $B = \{p\}$ for some point $p \in Q \setminus K$ and verifying that the equality (14) holds true.

Remark 3. It follows from the construction of the continuum X of Example 15 (in particular from the final statement of the formulation of the example), that there are no points $a, b \in X$ such that $C(a,b) = X$. Therefore, according to Theorem 10, the open mapping $f : X \to [0,1]$ defined in Example 15, as well as any other open mapping from X onto $[0,1]$ such that $f(a) \neq f(b)$ is not non-alternating.

Now we intend to characterize those locally connected continua X which are cyclic chains, i.e., such that $X = C(A,B)$ for some two disjoint closed subsets A and B of X, both with empty interior. To this aim let us come back to Proposition 2 and observe that just the same situation is described in Theorem 10 of [8], p. 451, where some conditions are considered which concern the structure of the boundary $\operatorname{bd} Y$ of Y as a subcontinuum of X, and which prevent the existence of an open retraction from X onto an arc. Note that, since an arc is an absolute retract, a retraction from X onto an arc $A \subset X$ always does exist. So, the obstructions refer rather to openness than to retractness. Thus a more general situation can be considered of a mapping that maps X onto an arbitrary arc, not necessary being a subset of X, or even onto an arbitrary (linear) graph. By a *graph* we mean any linear graph, that is, a continuum being the union of a finite number of arcs disjoint except for their end points. Since every arc contained in a graph G is an open retract of G (see [5], Theorem 4, p. 345), and since the closed interval $[0,1]$ is a graph, we get the following corollary to Theorem 12.

Corollary 16. *A locally connected continuum X can be mapped onto a graph G under an open mapping $f : X \to G$ if and only if there are two disjoint closed subsets A and B of X, both with empty interior, such that $C(A,B) = X$.*

Proposition 17. *Let a continuum X be locally connected. If there is an open mapping of X onto a graph, then*

(17) *for each point $p \in X$ its complement $X \setminus \{p\}$ has finitely many components.*

PROOF. Assume (17) is not satisfied. Then there is a point $p \in X$ whose complement $X \setminus \{p\}$ has infinitely many components. The boundary of each component of $X \setminus \{p\}$ is the singleton $\{p\}$, thus Proposition 2 implies that $\{p\}$ is an A-set, and therefore the components form a null sequence according to condition (1) of

Proposition 1. For each $n \in \mathbb{N}$ let C_n denote the closure of a component of the complement $X \setminus \{p\}$. Thus C_n is a continuum with the nonempty interior. We can assume that $C_m \cap C_n = \{p\}$ for $m \neq n$ and that $\lim \operatorname{diam} C_n = 0$. Suppose on the contrary that there are a graph G and an open surjective mapping $f : X \to G$. Therefore, for each $n \in \mathbb{N}$ the set $f(C_n)$ is a subcontinuum of G with the nonempty interior, and $\lim \operatorname{diam} f(C_n) = 0$. Thus there is an $n_0 \in \mathbb{N}$ such that for each $n > n_0$ we have $\operatorname{bd} f(C_n) \setminus \{f(p)\} \neq \emptyset$. For these n take a point $p_n \in C_n$ such that $f(p_n) \in \operatorname{bd} f(C_n) \setminus \{f(p)\}$ and choose in X an open connected neighbourhood U_n about p_n so that $p \notin U_n$ and $U_n \cap C_m = \emptyset$ for $m, n > n_0$ and $m \neq n$. Then $U_n \subset C_n$, whence $f(U_n) \subset f(C_n)$. Since f is open, we infer that $f(U_n) \subset \operatorname{int} f(C_n)$, and so $f(p_n) \in f(U_n)$ is an interior point of $f(C_n)$, a contradiction. The proof is complete. □

Remark 4. The opposite implication to that of Proposition 17 is not true. Indeed, the standard universal dendrite D_3 of order 3 contains no point the complement of which has infinitely many component, and it cannot be openly mapped onto any graph because all open images of D_3 are homeomorphic to D_3 (Theorem 3 of [3], p. 493).

As a consequence of Theorem 12 and Proposition 17 we get the following corollary.

Corollary 18. *Let a locally connected continuum X contain two disjoint closed subsets A and B, both with empty interior, such that $X = C(A, B)$. Then* (17) *holds.*

To show the next result we need a lemma. Given a metric space X, let $E(X)$ denote the set of end points of X, i.e., the set of points of order 1 in X. Recall that if B is a subset of a metric space X, then B^d stands for the set of all accumulation points of B.

Lemma 19. *Let a locally connected continuum X be mapped onto a graph G under an open mapping $f : X \to G$, and let an A-set Y in X be given. Then $f((\operatorname{bd} Y)^d) \subset E(G)$.*

PROOF. Suppose on the contrary that there is a point $x \in (\operatorname{bd} Y)^d$ such that $f(x) \in G \setminus E(G)$. Take a sequence of distinct points $x_n \in \operatorname{bd} Y$ with $x = \lim x_n$. For each $n \in \mathbb{N}$ let C_n denote the closure of the component of the complement $X \setminus Y$ such that $x_n \in C_n$, i.e., $C_n \cap Y = \{x_n\}$. Thus each C_n is a subcontinuum of X with the nonempty interior. According to the definition of an A-set we see that C_n's form a null sequence and we have $\operatorname{Lim} C_n = \{x\}$. Therefore for each

$n \in \mathbb{N}$ the set $f(C_n)$ is a subcontinuum of G with the nonempty interior, and $\operatorname{Lim} f(C_n) = \{f(x)\}$. Thus there is an $n_0 \in \mathbb{N}$ such that for each $n > n_0$ we have $\emptyset \neq \operatorname{bd} f(C_n) \setminus \{f(x)\} \subset G \setminus E(G)$. For these n take a point $p_n \in C_n$ such that $f(p_n) \in \operatorname{bd} f(C_n) \setminus \{f(x)\}$ and choose in X an open connected neighbourhood U_n about p_n so that $x_n \notin U_n$ and $U_n \cap C_m = \emptyset$ for $m, n > n_0$ and $m \neq n$. Then $U_n \subset C_n$, whence $f(U_n) \subset f(C_n)$. Since f is open, we infer that $f(U_n) \subset \operatorname{int} f(C_n)$, and so $f(p_n) \in f(U_n)$ is an interior point of $f(C_n)$, a contradiction. The proof is complete. □

Remark 5. Observe that if in condition (11) of Theorem 12 we substitute the phrase "for every arc $L \subset X$" in place of "there exists an arc $L \subset X$" (analogously as it is stated in condition (7) of Whyburn's Theorem 10 above), then the obtained statement is no longer equivalent to (12) and (13). In fact, if $Y \subset X$ is an A-set such that $(\operatorname{bd} Y)^d \neq \emptyset$ and L is an arc which contains a point of $(\operatorname{bd} Y)^d$ as a non-end point of itself, then by Lemma 19 there is no open retraction of X onto L.

Proposition 20. *Let a locally connected continuum X contain two disjoint closed subsets A and B, both with empty interior, such that $X = C(A, B)$, and let an A-set Y in X be given. Then*

(18) *the complement $X \setminus (\operatorname{bd} Y)^d$ has finitely many components.*

PROOF. By Theorem 12 there is an open mapping $f : X \to [0,1]$ such that (15) is satisfied. Taking $[0,1]$ for G in Lemma 19 we infer that $f((\operatorname{bd} Y)^d) \subset \{0,1\}$, so $(\operatorname{bd} Y)^d \subset A \cup B$. Let $F = (\operatorname{bd} Y)^d$ and note that F is closed. Shrink the intersections $A \cap F$ and $B \cap F$ to points and observe that the double quotient space $K = (X/(A \cap F))/(B \cap F)$ is a locally connected continuum. Let $\pi : X \to K$ be the natural projection. Put $a' = \pi(A \cap F)$ and $b' = \pi(B \cap F)$. Define a mapping $\sigma : K \to [0,1]$ such that $f = \sigma \circ \pi$. Openness of f implies that σ is open, too (see [11], (3.1), p. 140). Assume on the contrary that $X \setminus F$ has infinitely many components. Consequently, the set $K \setminus \{a', b'\}$ also has infinitely many components. Since the sets $K \setminus \{a'\}$ and $K \setminus \{b'\}$ have finitely many components by Proposition 17, it follows that there are infinitely many components of $K \setminus \{a', b'\}$ whose boundary is $\{a', b'\}$. Let $K_1, K_2, \ldots$ be a convergent sequence of the closures of these components. Then the limit continuum is not locally connected at any point distinct from a' and b', a contradiction. The proof is finished. □

The following lemma is needed to prove the converse to Proposition 20.

Denote by $S(X)$ the set of all separating points of a locally connected continuum X.

Lemma 21. *Let a locally connected continuum X be given such that for each A-set Y in X condition* (18) *is satisfied. Then*

(19) $\operatorname{int}\operatorname{cl} E(X) = \emptyset$.

PROOF. Suppose on the contrary that there is a nonempty open set U of the continuum X such that

(20) $U \subset \operatorname{cl} E(X)$.

We intend to construct an A-set Y in X such that the complement $X \setminus (\operatorname{bd} Y)^d$ has infinitely many components. Observe that by (20) and (2) we have $U \subset \operatorname{cl} S(X)$, whence it follows that $U \cap S(X)$ is a dense subset of U. Take a sequence of mutually disjoint nonempty open sets $\{U_i : i \in \mathbb{N}\}$ contained in U, and for each $i \in \mathbb{N}$ choose a point $y_i \in U_i \cap S(X)$. Next, again for each $i \in \mathbb{N}$, consider a null sequence of mutually disjoint nonempty open sets $\{U_{i,j} : j \in \mathbb{N}\}$ having the singleton $\{y_i\}$ as its limit. In each set $U_{i,j}$ choose a point $x_{i,j}$ and arrange the double sequence $\{x_{i,j} : i, j \in \mathbb{N}\}$ in an ordinary one: $\{x_n : n \in \mathbb{N}\}$. We see, by the definition of the points x_n, that

(21) each point y_i is an accumulation point of the set $\{x_n : n \in \mathbb{N}\}$.

We shall define now, by induction, a sequence of subcontinua $\{X_n : n \in \mathbb{N}\}$. The needed A-set Y will be the intersection of this sequence. Let B_1 be an open ball of radius 1 centered at x_1. By (20) we can choose $e_1 \in B_1 \cap E(X)$ and $s_1 \in B_1 \cap S(X)$ so that s_1 separates e_1 from x_1. Let C_1 be the component of $X \setminus \{s_1\}$ containing e_1, and define $X_1 = X \setminus C_1$. Note that X_1 is a subcontinuum of X. Assume that for some positive integer n a subcontinuum X_n of X has been defined. Let B_{n+1} be an open ball of radius $1/(n+1)$ centered at x_{n+1}. Since $x_{n+1} \in U$, again by (20) we can choose a point $e_{n+1} \in B_{n+1} \cap E(X)$ and $s_{n+1} \in B_{n+1} \cap S(X)$ so that s_{n+1} separates e_{n+1} from x_{n+1}. Let C_{n+1} be the component of $X \setminus \{s_{n+1}\}$ containing e_{n+1}, and define $X_{n+1} = X_n \setminus C_{n+1}$. Note that X_{n+1} is a subcontinuum of X_n. The inductive procedure is finished. Putting $Y = \bigcap\{X_n : n \in \mathbb{N}\}$ we see that Y is a subcontinuum of X. Moreover, it follows from the construction that the components of $X \setminus Y$ are just the sets C_n and $\{s_n\} = \operatorname{bd} C_n$ for each n. Thus Y is an A-set by Proposition 2. Again by construction we have $\operatorname{bd} Y = \{s_n : n \in \mathbb{N}\}$, whence by (21) it follows that $\{y_i : i \in \mathbb{N}\} \subset (\operatorname{bd} Y)^d$. Since each y_i is a separating point of X, we conclude that $X \setminus (\operatorname{bd} Y)^d$ has infinitely many components. This contradiction completes the proof. □

Observation 22. *For every subset Z of a locally connected continuum X there exists in X an A-set $M(Z)$ which is minimal with respect to containing the set Z.*

PROOF. Indeed, it is enough to recall that the intersection of any family of A-sets is an A-set (see [11], (3.5), (ii), p. 69), and take as $M(Z)$ the intersection of all A-sets containing the set Z. □

Since a continuum X is not disconnected by any subset of the set $E(X)$ of end points of X, we have the following observation.

Observation 23. *Given a continuum X, let H be a subset of the set $E(X)$ of end points of X. Then the sets $X \setminus Z$ and $X \setminus (Z \setminus H)$, where Z is any subset of X, have the same number of components.*

Proposition 24. *Let a locally connected continuum X be given such that for each A-set Y in X condition* (18) *holds. Then also*

(22) *the set $X \setminus [E(X) \cup \operatorname{bd} M(S(X))]^d$ has finitely many components.*

PROOF. Put $F = \operatorname{cl}([E(X) \cup \operatorname{bd} M(S(X))]^d \setminus E(X))$ and observe that, by Lemma 21, we have $\operatorname{int} F = \emptyset$. We claim that there exists an A-set Y in X such that

(23) $(\operatorname{bd} Y)^d = F$.

To this aim we take in the set F a dense countable (or, maybe, finite) subset of points $\{ p_i : i \in \mathbb{N} \}$. For every point p_i we choose a sequence of points $x_{i,j} \in [E(X) \cup \operatorname{bd} M(S(X))] \setminus F$ tending to p_i when j tends to infinity, and such that $\operatorname{dist}(x_{i,j}, F) < 1/(i+j)$. Therefore every $x_{i,j}$ is an isolated point of the set $\{ x_{i,j} : i, j \in \mathbb{N} \}$. Arrange all points of this set in a sequence $\{ x_n : n \in \mathbb{N} \} \subset E(X) \cup \operatorname{bd} M(S(X))$. We will define, by induction, a decreasing sequence of A-sets Y_n. Put $Y_0 = X$ and assume that for some $n \in \mathbb{N}$ an A-set Y_n is defined such that $\operatorname{bd} Y_n = \{ q_1, \dots, q_n \}$, $\{x_{n+1}, x_{n+2}, \dots \} \subset Y_n$ and $Y_0 \supset \cdots \supset Y_n$. To define Y_{n+1} consider the point x_{n+1}. If $x_{n+1} \in E(X)$, let U_{n+1} denote an open set containing x_{n+1}, with a one-point boundary, and disjoint with the set $F \cup \operatorname{bd} Y_n \cup \{ x_m : n+1 \neq m \in \mathbb{N} \}$. If $x_{n+1} \in \operatorname{bd} M(S(X))$, let U_{n+1} denote the component of $X \setminus M(S(X))$ such that $\operatorname{bd} U_{n+1} = \{x_{n+1}\}$. In this case put $q_{n+1} = x_{n+1}$. Then we define $Y_{n+1} = Y_n \setminus U_{n+1}$. Note that $\operatorname{bd} Y_{n+1} = \operatorname{bd} Y_n \cup \{q_{n+1}\} = \{q_1, \dots, q_n, q_{n+1} \}$. It can be observed that Y_{n+1} is an A-set and meets all the needed conditions. Put $Y = \bigcap \{ Y_n : n \in \mathbb{N} \}$. Then, since the intersection of any family of A-sets is an A-set (see [11], (3.5), (ii), p. 69), Y is an A-set. By construction we have $F \subset Y$ and $\operatorname{bd} Y = \{ q_n : n \in \mathbb{N} \}$.

To verify (23) take a point $y \in (\operatorname{bd} Y)^d$. Then there is a sequence $\{q_{n_i}\}$ of points of $\operatorname{bd} Y$ having y as its limit. Note that the distances $d(q_{n_i}, x_{n_i})$ tend to zero as the subindex i tends to infinity, whence also the sequence $\{x_{n_i}\}$ has the point y as its limit, and thus $y \in F$. To show the other inclusion, take a point y in F and choose a sequence of points $x_{n_i} \in E(X) \cup \operatorname{bd} M(S(X))$ tending to y. As previously we see that also the sequence $\{q_{n_i}\}$ has y as its limit. Since $q_{n_i} \in \operatorname{bd} Y$, we infer that $y \in (\operatorname{bd} Y)^d$. Thus the equality (23) is shown.

According to assumption (18), the complement $X \setminus (\operatorname{bd} Y)^d$ has finitely many components, so by (23) the set $X \setminus F$ has finitely many components. Since

$$[E(X) \cup \operatorname{bd} M(S(X))]^d \setminus E(X) \subset F \subset [E(X) \cup \operatorname{bd} M(S(X))]^d,$$

we see that F differs from $[E(X) \cup \operatorname{bd} M(S(X))]^d$ by a subset of the set $E(X)$, and therefore Observation 23 implies that the number of components of $X \setminus [E(X) \cup \operatorname{bd} M(S(X))]^d$ is finite, too. Hence the conclusion (22) follows. □

To formulate the next proposition we need an auxiliary notation. In a locally connected continuum X take the A-set $M(S(X))$ and in every component C of the complement $X \setminus M(S(X))$ choose exactly one point c. Denote by J the set of all chosen points. In other words, we denote by J any subset of $X \setminus M(S(X))$ such that

(24) for every component C of $X \setminus M(S(X))$ the intersection $J \cap C$ is a singleton.

Proposition 25. *Let a locally connected continuum X be given, and let a set $J \subset X$ satisfy* (24). *If conditions* (17) *and* (22) *are satisfied, then*

(25) *the set $X \setminus [E(X) \cup J]^d$ has finitely many components.*

PROOF. Since J consists of isolated points, we infer from (17) that $J^d = [\operatorname{bd} M(S(X))]^d$, and therefore (25) is satisfied. □

Proposition 26. *Let a locally connected continuum X be given, and let a set $J \subset X$ satisfy* (24). *Then conditions* (19) *and* (25) *imply* (13).

PROOF. Put $A = \operatorname{cl}(E(X) \cup J)$, and observe that, by construction,

(26) if $s \in S(X)$, then each component of $X \setminus \{s\}$ intersects A.

Note further that, since J consists of isolated points only, it follows from condition (19) that $\operatorname{int} A = \emptyset$. So, by (25), the set $X \setminus A^d$ has a finite number of components. In every component of the set $X \setminus A^d$ choose exactly one point not belonging to A, and let B denote the (finite) set of the chosen points. Note that A and B are disjoint. Thus both A and B satisfy the needed conditions. We have to show that $C(A, B) = X$.

So, take a point x of $X \setminus (A \cup B)$ and let D be the component of $X \setminus A^d$ to which the point x belongs. By the definition of the set B there is exactly one point $b \in B \cap D$. We claim that

(27) there exists an arc L from x to b in D with $L \cap A = \emptyset$.

Indeed, take an arbitrary arc L' from x to b in D. Then

(28) $L' \cap A^d = \emptyset = L' \cap E(X)$.

Since every component of $X \setminus A$ is contained in a corresponding component of $X \setminus A^d$, by (28) we have $L' \cap A \subset J$. Modifying the arc L' in the set $X \setminus M(S(X))$ one can get an arc L with end points x and b such that, still keeping (28) with L in place of L', the condition $L \cap J = \emptyset$ holds true. Since $A = E(X) \cup J \cup A^d$ by the definition of A, we see that L is the needed arc.

Consider three cases. First, if $x \in X \setminus M(S(X))$, let C be the component of $X \setminus M(S(X))$ containing x, let c be the only point of J in C and let s be the only point of the boundary of C in X. Then $\operatorname{cl} C = C \cup \{s\}$ and therefore $S(X) \cap \operatorname{cl} C = \{s\}$. Thus $\operatorname{cl} C$ is an A-set ([11], (3.32), p. 69) which contains no separating point of itself. Since a locally connected continuum is cyclicly connected if and only if it contains no separating point (the cyclic connectedness theorem, [11], (9.3), p. 79), we infer that $\operatorname{cl} C$ is cyclicly connected. If $b \in \operatorname{cl} C$, it follows from Proposition 5 that there is an arc $cxb \subset \operatorname{cl} C$. Note that $cxs \cap A = \{c\}$. Thus the arc cxb is the needed one, i.e., it satisfies (9) with $a = c$, whence it follows that $x \in C(A, B)$. If b is not in $\operatorname{cl} C$, then applying Proposition 5 again, we get an arc $cxs \subset \operatorname{cl} C$. Note that $cxs \cap A = \{c\}$. Taking an arc $sb \subset D$ disjoint with A, according to (27), we see that $sb \cap (A \cup B) = \{b\}$, and thus the union $cxs \cup sb$ is the needed arc, i.e., the arc satisfying (9) with $a = c$, whence it follows that $x \in C(A, B)$.

Second, if $x \in S(X) \subset M(S(X))$, let K stand for the component of $X \setminus \{x\}$ containing the point b. By (27) there is an arc $xb \subset \operatorname{cl} K = K \cup \{x\}$ which is disjoint with A. By (26) each other component of $X \setminus \{x\}$ intersects A. Let K^* be one of these components, and let $a \in A \cap K^*$. Choosing an arc $ax \subset \operatorname{cl} K^* = K^* \cup \{x\}$ we see that $ax \cap B = \emptyset$, so the union $ax \cup xb$ is an arc containing x and satisfying (9). Thus again $x \in C(A, B)$.

Third, if $x \in M(S(X)) \setminus S(X)$, we claim that there are points a_1, a_2 in A and an arc $a_1 x a_2 \subset X$ such that $a_1 x a_2 \cap A = \{a_1, a_2\}$. Indeed, since $x \in M(S(X)) \setminus S(X)$, there are points $s_1, s_2 \in S(X)$ and an arc $s_1 s_2 \subset X$ such that $x \in s_1 s_2$. For $i \in \{1, 2\}$ and for a component of $X \setminus \{s_i\}$ not containing x choose a point $a_i^* \in A$. In the arc $a_1^* s_1 x s_2 a_2^*$ naturally ordered from a_1^* to a_2^* let a_1 be the last point in $a_1^* x \cap A$, and let a_2 be the first point in $xa_2^* \cap A$. Then $a_1xa_2 \subset a_1^*s_1xs_2a_2^*$ is the needed arc.

By (27), there is an arc L from x to b disjoint from A. Let c be the last point of the intersection $L \cap a_1xa_2$ in the natural ordering of L from x to b and take the arc $cb \subset L$. If $c \in a_1x$, put $L_0 = a_2xc \subset a_1xa_2$; if $c \in xa_2$, put $L_0 = a_1xc \subset a_1xa_2$. In both these subcases the union $L_0 \cup cb$ is the needed arc, i.e., the arc satisfying (9) and containing the point x. So, all cases have been considered, and thus the proof is complete. □

Theorem 12, Corollary 18 and Propositions 20, 24, 25 and 26 can be summarized as follows.

Theorem 27. *For every locally connected continuum X the following conditions are equivalent:*

(29) *there is an arc $L \subset X$ which is an open retract of X;*

(30) *the continuum X can be mapped onto the closed unit interval $[0,1]$ under an open mapping;*

(31) *there exists a graph G such that the continuum X can be mapped onto G under an open mapping;*

(32) *for each an A-set Y in X and for each point $p \in X$ the complements $X \setminus (\operatorname{bd} Y)^d$ and $X \setminus p$ have finitely many components;*

(33) $\operatorname{int}\operatorname{cl} E(X) = \emptyset$, *the complement $X \setminus [E(X) \cup \operatorname{bd} M(S(X))]^d$ has finitely many components, and for each point $p \in X$ its complement $X \setminus \{p\}$ has finitely many components;*

(34) $\operatorname{int}\operatorname{cl} E(X) = \emptyset$, *and for every set $J \subset X$ satisfying (24) the set $X \setminus [E(X) \cup J]^d$ has finitely many components;*

(35) *there are two disjoint closed subsets A and B of X, both with empty interior, such that $C(A, B) = X$.*

Remark 6. (a) The assumption (17) cannot be omitted in Proposition 25 (and in conditions (32) and (33) of Theorem 27). In fact, if X is the one-point union of countably many circles with diameters tending to zero, then the point p which is the only point of the intersection of every two and of all these circles does not satisfy (17), and we have $E(X) = \emptyset$ and $S(X) = \{p\}$, whence (22) is satisfied, while the conclusion of Proposition 25 does not hold according to Corollary 18.

(b) The operator M cannot be neglected in assumption (22) of Proposition 25, i.e., we cannot take the complement $X \setminus [E(X) \cup \operatorname{bd} S(X)]^d$ in place of one considered in (22) by the following example. In the closed unit segment $[0,1] \times \{0\}$

in the plane take the standard Cantor ternary set C and replace each component of $[0,1] \times \{0\} \setminus C$ by a circle having the closure of the component as the diameter. Then the union of all (countably many and mutually disjoint) circles together with C is a locally connected continuum X for which we have $S(X) = C$ and $[E(X) \cup \operatorname{bd} S(X)]^d = C$, thus the complement $X \setminus [E(X) \cup \operatorname{bd} S(X)]^d$ has infinitely many components, while the natural projection of X onto $[0,1] \times \{0\}$ is open (and $M(S(X)) = X$).

(c) The assumption (19) cannot be omitted in Proposition 26 (and in conditions (33) and (34) of Theorem 27). Indeed, let X be the standard universal dendrite D_3 of order 3. Since all points of D_3 are of order 3 at most, and since for dendrites order of a point equals the number of components of the complement of the point ([11], (1.1), (iv), p. 88), we see that condition (17) of Proposition 25 is satisfied. Further, the set $E(X)$ is dense in X whence the condition $M(S(X)) = X$ follows, and so assumption (22) of Proposition 25 is satisfied, too. So, assumption (25) of Proposition 26 holds by Proposition 25, while there is no open mapping of $X = D_3$ onto $[0,1]$ because all open images of X are homeomorphic to X ([3], Theorem 3, p. 493).

Studying open mappings between dendrites, the first and the second named authors together with J. R. Prajs obtained a characterization of these dendrites which can be openly mapped onto an arc ([4], Theorem 6.61, p. 33). However, an even better result was shown by the third and the fourth named authors as Theorem 2 of [8], p. 455, namely the characterization is valid not only for dendrites but for local dendrites. We derive that and one more characterizations from Theorem 27. Recall that a locally connected continuum is called a *local dendrite* provided that each its point has a closed neighbourhood which is a dendrite, or — equivalently — if the continuum contains a finite number of simple closed curves at most ([7], §51, VII, Theorem 4, p. 303). To prove the characterization we need a statement.

Statement 28. *Let a continuum X be given. The numbers of components of the following four sets are equal:*

(36) $$X \setminus (E(X))^d,$$

(37) $$X \setminus ((E(X))^d \setminus E(X)),$$

(38) $$X \setminus (\operatorname{cl} E(X) \setminus E(X)),$$

(39) $$X \setminus \operatorname{cl} E(X).$$

PROOF. The numbers of components of the sets (36) and (37) as well as of the sets (38) and (39) are equal by Observation 23. The sets (37) and (38) are the same since for each subset A of a space X we have $\operatorname{cl} A \setminus A = A^d \setminus A$. □

Theorem 29. *If the locally connected continuum X is a local dendrite, then each of conditions (29)–(35) of Theorem 27 is equivalent to any the following two:*

(40) $\operatorname{int}\operatorname{cl} E(X) = \emptyset$ *and the set* $X \setminus \operatorname{cl} E(X)$ *has finitely many components;*

(41) *the set* $\operatorname{cl} E(X) \setminus E(X)$ *is finite, and* $\operatorname{ord}(p, X)$ *is finite for each point p of X.*

PROOF. Since X contains finitely many simple closed curves only, the set $\operatorname{bd} M(S(X))$ is finite, whence

(42) $(\operatorname{bd} M(S(X)))^d = \emptyset$,

and by condition (33) of Theorem 27 we infer, in addition to the first part of (40), that the set $X \setminus (E(X))^d$ has finitely many components. Now the second part of (40) is a consequence of Statement 28.

To show that (40) implies (41) we first claim that

(43) the set $\operatorname{cl} E(X)$ contains no nondegenerate continuum.

Assume on the contrary that a nondegenerate continuum V is contained in $\operatorname{cl} E(X)$. Since $\operatorname{int}\operatorname{cl} E(X) = \emptyset$, there is a sequence of points $a_1, a_2, \ldots, a_n, \ldots$ in $X \setminus \operatorname{cl} E(X)$ such that $\operatorname{cl}\{a_n : n \in \mathbb{N}\} = V$. Consider irreducible arcs between points a_n and the continuum V. Because of local connectedness of X the arcs form a null sequence. Since X contains finitely many simple closed curves only, the points a_n are contained in infinitely many components of $X \setminus \operatorname{cl} E(X)$, a contradiction to (40). So (43) is shown.

Since every subset of $X \setminus E(X)$ that has infinitely many components separates the local dendrite X into infinitely many components, the first part of (41) is established.

To prove the second part, suppose there is a point p in X such that $\operatorname{ord}(p, X)$ is infinite. Since X contains no small simple closed curves, $p \in \operatorname{cl} E(X)$. By (43) the number of components of $X \setminus \operatorname{cl} E(X)$ is greater than or equal to the number of components of $X \setminus \{p\}$ which is infinite, contrary to (40). This finishes the proof of the implication from (40) to (41).

To show that (41) implies (33) note that $\operatorname{int} E(X) = \emptyset$ and thus, by (41), we get $\operatorname{int}\operatorname{cl} E(X) = \emptyset$. Further, it follows from (41) that no point of the finite set $\operatorname{cl} E(X) \setminus E(X)$ disconnects X into infinitely many components, i.e., that the number of components of $X \setminus (\operatorname{cl} E(X) \setminus E(X))$ is finite. Therefore (33) follows from (42). The proof is complete. □

Remark 7. Equivalence of conditions (30) and (31) holds for any space X (not necessarily being a locally connected continuum). Conditions (32) and (33) have no meaning for a continuum X which fails to be locally connected; and (35) is not equivalent to (30) if the continuum X is not locally connected. In fact, the circle of pseudo-arcs (see [2] for the definition) contains no arcs, so (35) does not hold, while (30) is satisfied.

Problem. *What continua can be openly mapped onto an arc?*

Remark 8. For dendrites condition (30) is preserved under open mappings in the sense that if a dendrite X can be openly mapped onto $[0,1]$, then every open image of X can be also ([4], Corollary 6.66, p. 34). However, it is not true for arbitrary locally connected continua because putting $X = D_3 \times [0,1]$, where D_3 is the standard universal dendrite of order 3, we see that X can be openly mapped onto $[0,1]$ (under the projection onto the second factor), while its open image D_3 (under the projection onto the first factor) cannot be.

Proposition 17 and Theorem 27 can be considered as an extension and a rectification of Theorem 1 of [8], p. 451. Three conditions are considered in that theorem which suffice to nonexistence of an open retraction of a locally connected continuum onto an arc. The first of them is identical with the negation of ours (17), and therefore Proposition 17 generalizes part (i) of Theorem 1 of [8]. Recall that this condition is sufficient but not necessary (see Remark 4). The second condition discussed in Theorem 1 of [8], which states that there is an A-set Y in X whose boundary contains a nondegenerate subcontinuum, has been erroneously formulated. Below we present an example (viz. Example 30) showing incorrectness of that formulation. Finally, the third condition which says that there is an A-set Y in X such that the complement of $(\operatorname{bd} Y)^d$ has infinitely many components, and which coincides with the negation of ours (18), is not only sufficient, but also necessary to demonstrate the nonexistence of an open mapping of a locally connected continuum X onto an arc. Thus Theorem 27 complements part (iii) of Theorem 1 of [8].

Example 30. *The locally connected continuum X of Example 15 contains a subcontinuum Y such that:*

(44) *Y is an A-set;*

(45) *the boundary of Y contains a nondegenerate subcontinuum;*

(46) *there exists an open retraction of X onto an arc.*

PROOF. We will use the notation of Example 15. For each $n \in \mathbb{N}$ let q_n^* be any interior point of the arc $p_n q_n$, and put $Y = Q \cup \bigcup\{p_n q_n^* : n \in \mathbb{N}\}$. Then Y is an A-set in X and $\operatorname{bd} Y = \{q_n^* : n \in \mathbb{N}\} \cup K$. Thus X satisfies (44) and (45). It also satisfies (46) by Theorem 12. □

Remark 9. Let X be the locally connected continuum of Example 15. Note that $\operatorname{cl} E(X)$ contains the subcontinuum K in X (we are still using the notations of Example 15). More precisely, we have $\operatorname{cl} E(X) = K \cup E(X)$. Therefore, the same Example 30 shows that part (II) of Corollary 1 of [8], p. 453, is also incorrect, i.e., $\operatorname{cl} E(X)$ contains a nondegenerate continuum, while there is an open retraction of X onto an arc.

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