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SEPARABLE ZERO-DIMENSIONAL SPACES WHICH ARE CONTINUOUS IMAGES OF ORDERED COMPACTA

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ABSTRACT. A structure theorem is proved about separable zero-dimensional spaces which are continuous images of ordered compacta and it is shown that not all spaces in this class are orderable themselves.

1. INTRODUCTION

All spaces considered in this paper are Hausdorff.

The paper is devoted to study the class $\mathcal{S}$ of all separable zero-dimensional spaces which are continuous images of ordered compacta. Theorem 3.1 shows that members of $\mathcal{S}$ can be obtained by resolving a metrizable zero-dimensional compactum into at most two-point sets in a rather canonical fashion. The general method of resolutions is presented in the survey paper [**18**], while the more specific approach adapted here follows the construction in [**12**, p. 388].

The other goal of the paper is to show that not all members of $\mathcal{S}$ are orderable. The argument is presented in Theorem 4.2 which was initially planned to be put in a joint paper of the second-named author with S. W. Williams and H. Zhou. That paper was never written though it was quoted in other publications, e.g. in [**2**], [**9**], [**12**] or [**18**]. The result of Theorem 4.2 shows that the results published in [**8**, (3.1) and (3.2)] and [**13**] are false. The corrected version of [**8**, (3.1)] is quoted in Theorem 4.3, below.

Another motivation for the results obtained here is their relation to studies of monotonically normal compacta. The first intensive study of the concept of monotone normality can be found in [**3**] where among many other results it was shown that each continuous image of an ordered compactum is a monotonically

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normal space. The considerations in [**3**] motivated a 1973 conjecture in [**15**] that monotone normality coincides with orderability in the class of compact, separable and zero-dimensional spaces. The conjecture reappeared in [**17**], [**1**], [**16**], [**7**] and [= **10**]. It led to the following more general question which was asked in 1986 in [**7**] and then was repeated in [**8**] and [**10**]: Is each monotonically normal compactum the continuous image of an ordered one? Theorem 4.2 shows that the conjecture of [**15**] is not true. However, it does not settle the more general question because the space X used in the proof of Theorem 4.2 belongs to the class $\mathcal{S}$.

The structure of monotonically normal compacta is almost unknown so far. The only two results which are available are concerned with the class $\mathcal{M}$ of separable monotonically normal compacta. Each member of $\mathcal{M}$ is perfectly normal, [**14**]. Moreover, if a member of $\mathcal{M}$ is obtained by means of a resolution, then all but countably many fibers of the resolution consist of at most two points, [**11**]. The latter result is somewhat similar to what is proved in Theorem 3.1 about the structure of members of $\mathcal{S}$.

Since this paper was written the authors have learned about the elegant result of Mary Ellen Rudin that "Every separable monotonically normal compactum is the continuous image of an ordered compactum".

The starting point to proving Theorems 3.1 and 4.3 is the following result of [**8**, (2.1)]: a zero-dimensional space which is the continuous image of an ordered compactum can be embedded into a dendron. In order to keep the present paper reasonably self-contained it is necessary to summarize basic results about dendrons. This is done in the preliminary Section 2, while the main results of the paper are put in the Sections 3 and 4.

2. About dendrons

A space Z is said to be a *dendron* if Z is compact and connected and for every two distinct points x and x' of Z there exists $z \in Z$ such that x and x' belong to distinct components of $Z - \{z\}$. Metrizable dendrons are also called *dendrites*. A rather comprehensive survey of basic results on dendrons can be found in [**4**]. We would like to recall the following facts:

2.1. [**4**] Each dendron is locally connected. Hence, if Z is a dendron and A is a closed subset of Z, then each component of $Z - A$ is an open subset of Z.

2.2. [**4**] If Z is a dendron, then the collection of all sets which are components of $Z - \{z\}$ for $z \in Z$ is a subbasis of open sets in Z.

2.3. [4] Dendrons are rim-finite continua, i.e., each dendron admits a basis of open sets which have finite boundaries.

2.4. [4] Dendrons are hereditarily unicoherent continua, i.e., the intersection of any two subcontinua of a dendron is a connected set.

2.5. [4] Dendrons are uniquely arc-connected spaces, i.e., if Z is a dendron and x and x' are distinct points of Z, then there exists the unique orderable subcontinuum $[x, x']$ of Z whose end-points are x and x'. Furthermore, $[x, x'] = \{x, x'\} \cup \{z \in Z : x$ and x' belong to distinct components of $Z - \{z\}\}$.

2.6. (see e.g. [5] or [6]) Dendrons are nested continua, i.e., if Z is a dendron and $\mathcal{L}$ is a collection of orderable subcontinua of Z such that $\mathcal{L}$ is linearly ordered by inclusion, then $\bigcup \mathcal{L}$ is contained in an orderable subcontinuum $[x, x']$ of Z.

2.7. (see e.g. [**19**, (6.1) on p. 134 and (2.21) on p. 138]) If Z is a dendron and $\mathcal{J}$ is a family of pairwise disjoint subcontinua of Z, then the decomposition $\mathcal{G}$ of Z into members of $\mathcal{J}$ and points is upper semi-continuous, and the quotient space $Z/\mathcal{G}$ is a dendron again.

2.8. [4] If Z is a dendron then density(Z) = weight(Z). Hence, separable dendrons are metrizable.

2.9. [4] Each dendron is an image of an orderable continuum under a continuous map. Hence, each closed subset of a dendron is a continuous image of an orderable compactum.

Let Z be a dendron and $z \in Z$. We define the *order of ramification*, $r(z)$, of z in Z to be the number of components of $Z - \{z\}$. We shall say that z is an *end-point* of Z if $r(z) = 1$, and z is a *ramification point* of Z if $r(z) \geq 3$. We shall use the following notation: $E_Z = \{x \in Z : x$ is an end-point of $Z\}$ and $R_Z = \{x \in Z : x$ is a ramification point of $Z\}$.

Let Z be a dendron and $A \subset Z$. We shall say that A is a *strong T-set* in Z if A is non-empty and closed, and each component of $Z - A$ is homeomorphic to the real line.

Let Z be a dendron and A be a strong T-set in Z. By local connectedness of Z, each component of $Z - A$ is open in Z. Let J be a component of $Z - A$. By the properties (2.5) and (2.6), there exist points $a_J, b_J \in Z$ such that $\mathrm{cl}(J) = [a_J, b_J]$ and $J = [a_J, b_J] - \{a_J, b_J\}$. Therefore, if $z \in J$ then $Z - \{z\}$ has exactly two components. One of the components contains a_J while the other one contains b_J. Thus, $r(z) = 2$ for each $z \in J$. This proves the following lemma.

Lemma 2.10. If A is a strong T-set in a dendron Z, then $E_Z \subset A$ and $R_Z \subset A$.

Lemma 2.11. Let A be a separable strong T-set in a dendron Z, and $\{z_0, z_1, \dots\}$ be a countable dense subset of A. Then $Z - E_Z \subset \bigcup_{n=1}^{\infty}[z_0, z_n]$.

PROOF. Suppose that $x \in Z - E_Z$. Since $r(x) > 1$, there is a component J of $Z - \{x\}$ which does not contain z_0. By (2.6), $J\cap = E_Z \neq \emptyset$. Hence, by Lemma 2.10, $J \cap A \neq \emptyset$. Since J is an open subset of Z and $\{z_0, z_1, \dots\}$ is a countable dense subset of A, there exists a positive integer n such that $z_n \in J$. Therefore, either $x = z_0$ or z_0 and z_n belong to distinct components of $Z - \{x\}$. It follows that $x \in [z_0, z_n]$. □

Lemma 2.12. Suppose that Z is a dendron which contains a separable strong T-set. Then the set R_Z is at most countable.

PROOF. Let A denote a separable strong T-subset of Z and let $\{z_0, z_1, \dots\}$ be a countable dense subset of A. Then $R_Z \subset Z - E_Z \subset \bigcup_{n=1}^{\infty}[z_0, z_n]$. Hence, it suffices to show that $R_Z \cap [z_0, z_n]$ is a countable set for each n.

Let n be a positive integer. For each $x \in R_Z \cap [z_0, z_n]$, there exists a component L_x of $Z - \{x\}$ such that $L_x \cap [z_0, z_n] = \emptyset$. By (2.4), $L_x \cap L_{x'} = \emptyset$ if x and x' are distinct points of $R_Z \cap [z_0, z_n]$. Thus, sets of the form $L_x \cap A$, $x \in R_Z \cap [z_0, z_n]$, are open and pairwise disjoint subsets of A. As in the proof of Lemma 2.11, all these sets $L_x \cap A$ are non-empty. Since A is separable, $R_Z \cap [z_0, z_n]$ is at most countable. □

3. THE STRUCTURE THEOREM.

The *double arrow space* is $([0,1] \times \{0,1\}) - \{(0,0),(1,1)\}$ with its lexicographic ordering and the order topology.

Let X be a separable and zero-dimensional space which is a continuous image of an orderable compactum. The following properties of X are well-known and quite easy to prove:

3.1. X is a continuous image of a separable and orderable compactum.

3.2. Since each separable and orderable compactum is a continuous image of the double arrow space, X is a continuous image of the double arrow space.

3.3. Since the double arrow space is hereditarily separable, X is hereditarily separable, too.

We shall also need the following well-known fact:

3.4. If Y is a separable and orderable compactum, then there exist $A, A^* \subset X$ such that A is closed, A has no isolated points, A^* is at most countable, and $Y = A \cup A^*$.

The following partial converse of (2.9) is available:

3.5. [8, Theorem 2.1] A zero-dimensional space which is a continuous image of an orderable compactum is homeomorphic to a strong T-subset of a dendron.

Construction 1. Let Z be a dendron, A a non-empty closed subset of Z, and C a subset of A such that $r(x) = 2$ for each $x \in C$. Thus, $Z - \{x\}$ has exactly two components $K_{x,0}$ and $K_{x,1}$ for each $x \in C$.

Let $\mathrm{s}(A, C, Z) = (A - C) \cup (C \times \{0, 1\})$ and define $\pi : \mathrm{s}(A, C, Z) \to A$ by

$$\pi(s) = \begin{cases} s & \text{if } s \in A - C \\ x & \text{if } s = (x, i) \in C \times \{0, 1\}. \end{cases}$$

Let $\mathcal{S}$ be the collection which consists of all sets $\pi^{-1}(U)$, where U is an open set in A, and of all sets $\pi^{-1}(K_{x,i}) \cup \{(x, i)\}$, where $(x, i) \in C \times \{0, 1\}$. Topologize $\mathrm{s}(A, C, Z)$ by letting $\mathcal{S}$ be a subbasis of open sets. The resulting space $\mathrm{s}(A, C, Z)$ is called the *dendritic resolution* of A by means of C with respect to Z. Rougly speaking, it is formed from A by splitting each point of C into two points in the directions dictated by Z. For instance, the double arrow space coincides with $\mathrm{s}\big([0, 1],]0, 1[, [0, 1]\big)$.

The following lemma is easily proved.

Lemma 3.6. $\mathrm{s}(A, C, Z)$ is a compact space and $\pi : \mathrm{s}(A, C, Z) \to A$ is a continuous mapping.

Construction 2. Dendritic resolutions can be viewed as subsets of dendrons. In fact, let Z, A and C be as in Construction 1. Let $\mathbf{t}(C, Z) = (Z - C) \cup \big(C \times [0, 1]\big)$ and define $p : \mathbf{t}(C, Z) \to Z$ by

$$p(t) = \begin{cases} t & \text{if } t \in Z - C \\ x & \text{if } t = (x, i) \in C \times [0, 1]. \end{cases}$$

It can be easily shown that $\mathbf{t}(C, Z)$ becomes a dendron when topologized by using the open subbasis

$$\Big\{p^{-1}(U) : U \text{ is open in } Z\Big\} \cup \Big\{p^{-1}(K_{x,0}) \cup \big(\{x\} \times [0, i[\big) : x \in C,\ i \in]0, 1[\Big\} \cup$$
$$\Big\{p^{-1}(K_{x,1}) \cup \big(\{x\} \times]i, 1]\big) : x \in C,\ i \in]0, 1[\Big\}.$$

Clearly, $\mathbf{s}(A, C, Z)$ is a subspace of $\mathbf{t}(C, Z)$ and the restriction of p to $\mathbf{s}(A, C, Z)$ coincides with π. We remark that if A is a strong T-set in Z, then $\mathbf{s}(A, C, Z)$ is a strong T-set in $\mathbf{t}(C, Z)$.

Theorem 3.1. *Let X be a zero-dimensional space which is a continuous image of a separable and orderable compactum. Then there exist a dendrite Z and a strong T-set A in Z such that X is homeomorphic to a dendritic resolution of A with respect to Z.*

PROOF. By (3.5), there exists a dendron Y such that $X \subset Y$ and X is a strong T-set in Y. Let $\{x_0, x_1, \dots\}$ be a countable dense subset of X. By Lemma 2.11, $Y - E_Y \subset \bigcup_{n=1}^{\infty}[x_0, x_n]$. Moreover, each component of $Y - X$ is a component of $[x_0, x_n] - X$ for some n. Indeed, by Lemma 2.10, $E_Y \subset X$. Since $x_0 \in X$, each component of $Y - X$ is contained in some $[x_0, x_n]$.

Let n be a positive integer. Observe that each component of $[x_0, x_n] - X$ is a component of $Y - X$. Hence, $X \cap [x_0, x_n]$ is a strong T-set in $[x_0, x_n]$. Moreover, by (3.3), $X \cap [x_0, x_n]$ is separable.

Let $\mathcal{P}_n = \{\mathrm{cl}(J) : J \text{ is a component of } [x_0, x_n] - X\}$. Then each $I \in \mathcal{P}_n$ is of the form $I = [x, x']$ for some $x, x' \in X \cap [x_0, x_n]$, and $I \cap X = \{x, x'\}$. Of course, if $y \in X \cap [x_0, x_n]$, y can belong to at most two members of $\mathcal{P}_n$. Since $X \cap [x_0, x_n]$ is separable, there exist X_n and X_n^* such that X_n is closed, X_n has no isolated points, X_n^* is at most countable, and $X \cap [x_0, x_n] = X_n \cup X_n^*$. By Lemmas 2.10 and 2.12, we may assume that $R_Y \cap [x_0, x_n] \subset X_n^*$.

Let $\mathcal{R}_n^* = \{I \in \mathcal{P}_n : I \cap X_n^* \neq \emptyset\}$ and $\mathcal{R}_n = \mathcal{P}_n - \mathcal{R}_n^*$. Then $\mathcal{R}_n^*$ is at most countable. Suppose that $I, I' \in \mathcal{R}_n$ and $I \neq I'$. Then I and I' are disjoint. In fact, if $I \cap I' \neq \emptyset$, then $I \cap I' = \{y\}$ for some isolated point y of $X \cap [x_0, x_n]$. But then $y \in X_n^*$, and so $I, I' \in \mathcal{R}_n^*$.

Let $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$. We shall prove that $\mathcal{R}$ consists of pairwise disjoint sets. Suppose that $I, I' \in \mathcal{R}$, $I \neq I'$ and $I \cap I' \neq \emptyset$. Since the members of each collection $\mathcal{R}_n$ are pairwise disjoint, there exist distinct positive integers m and m' such that $I \in \mathcal{R}_m$ and $I' \in \mathcal{R}_{m'}$. By (2.4), $[x_0, x_m] \cap [x_0, x_{m'}] = [x_0, x]$ for some $x \in R_Y$. It follows that $I = [x, y]$ and $I' = [x, y']$ for some $y \in [x, x_m]$ and $y' \in [x, x_{m'}]$. Since $x \in R_Y$, $x \in X_n^*$ for each n such that $x \in [x_0, x_n]$. Hence, $I \notin \mathcal{R}$ and $I' \notin \mathcal{R}$; a contradiction which proves that $\mathcal{R}$ consists of pairwise disjoint sets.

Let $\mathcal{G}$ denote the decomposition of Y into members of $\mathcal{R}$ and singletons. By (2.7), $\mathcal{G}$ is upper semi-continuous. Therefore, the quotient space $Y/\mathcal{G}$ is Hausdorff. Let $Z = Y/\mathcal{G}$ and $g : Y \to Z$ denote the quotient space and the quotient map, respectively. Since g is a monotone map (i.e., its fibers are continua), Z is a dendron. Observe that $Z = g(X) \cup g\big(\bigcup\bigcup_{n=1}^{\infty} \mathcal{R}_n^*\big)$. Since X is separable and each $\mathcal{R}_n^*$ is a countable family of separable sets, Z is separable. By (2.8), Z is metrizable, i.e., it is a dendrite.

Let $A = g(X)$. Then A is a closed subset of Z. Let $C = g(\bigcup \mathcal{R})$. If $I \in \mathcal{R}$ then $I \cap X \neq \emptyset$ and I is collapsed to a point by g. Hence $C \subset A$.

Let K be a component of $Z - A$. Since $K \cap A = \emptyset$, it follows that there exists the unique component J of $Y - X$ such that $K \cap g(J) \neq \emptyset$. Furthermore, $\mathrm{cl}(J) \in \mathcal{R}_n^*$ for some n and $g|J$ is a homeomorphism of J onto K. This proves that A is a strong T-set in Z.

Let $z \in C$. Then there exists the unique $I_z \in \mathcal{R}$ such that $z = g(I_z)$. Since $I_z \cap R_Y = \emptyset$, $Y - I_z$ has exactly two components. Denote them by $J_{I_z,0}$ and $J_{I_z,1}$. Let $K_{z,0} = g\big(J_{I_z,i}\big)$ for $i = 0, 1$. Then $K_{z,0}$ and $K_{z,1}$ are distinct components of $Z - \{z\}$ and $K_{z,0} \cup K_{z,1} = Z - \{z\}$.

Therefore, $r(z) = 2$ for each $z \in C$.

It remains to prove that X is homeomorphic to $\mathrm{s}(A, C, Z)$. Let $z \in C$ and $I_z \in \mathcal{R}$ be as above. Then $I_z = [x, x']$ for some $x, x' \in X - R_Y$. Furthermore, if $i \in \{0, 1\}$, the boundary of $J_{I_z,i}$ consists of a single point which is one of x and x', and $\mathrm{bd}(J_{I_z,0}) \cup \mathrm{bd}(J_{I_z,1}) = \{x, x'\}$. Let $x_{z,i}$ denote the unique boundary point of $J_{I_z,i}$, $i = 0, 1$.

Define $h : X \to \mathrm{s}(A, C, Z) = (A - C) \cup \big(C \times \{0, 1\}\big)$ by

$$h(x) = \begin{cases} x & \text{if } g(x) \in A - C \\ (x, i) & \text{if } z = g(x) \in C,\ i \in \{0,1\} \text{ and } x = x_{z,i}. \end{cases}$$

An easy straightforward proof shows that h is continuous, one-to-one and onto. Hence, h is a homeomorphism of X onto $\mathrm{s}(A, C, Z)$. □

Remark. It is quite easy to see that the dendron Z, its strong T-subset A, and the set $C \subset A$ as given in Theorem 3.1 can be chosen in a manner such that A has the additional property of being zero-dimensional.

4. The example

In this section we let Q denote the subset of all rational numbers and P denote the subset of all irrational numbers of $]0, 1[$.

Construction 3. Each rational number $\frac{p}{q} \in]0, 1[$ is assumed here to be in its irreducible form, i.e., p and q are to be relatively prime positive integers.

Let $Z = (\{[0,1]\times\{0\}) \cup \bigcup \left\{ \{\frac{p}{q}\} \times [0, \frac{1}{q}] : \frac{p}{q} \in Q \right\}$ and let Z carry the topology induced from the plane $\mathbb{R}^2$. Then Z is a dendrite, $E_Z = \{(0,0),(1,0)\} \cup \left\{ (\frac{p}{q}, \frac{1}{q}) : \frac{p}{q} \in Q \right\}$, and $R_Z = \left\{ (\frac{p}{q}, 0) : \frac{p}{q} \in Q \right\}$. Let $A = ([0,1]\times\{0\}) \cup E_Z$ and $C = P \times \{0\}$. Then $C \subset A \subset Z$, A is a strong T-set in Z, and $r(z) = 2$ for each $z \in C$.

Let $X = \mathbf{s}(A, C, Z)$.

Construction 4. Let $Y = ([0,1] \times \{0,1\}) \cup (Q \times \{2\})$. Consider Y with its lexicographic ordering and the order topology. Let $\mathcal{G}$ denote the decomposition of Y into singletons and the sets $\{(q,0),(q,2)\}$, $q \in Q$.

Let $X = Y/\mathcal{G}$.

Construction 5. Let $Y = ([0,1] \times \{0\}) \cup (P \times \{1\})$, and let $\leq$ denote the lexicographic ordering on Y, and take Y with its order topology. Then Y is a linearly ordered compact space and $Q \times \{0\}$ is a dense subset.

Let $X = Y \cup Q$ with the following topology: The points of Q are isolated, and basic neighbourhoods of each $(t,i) \in Y$ are of the form $U \cup \{s \in Q : s \neq t \text{ and } (s,0) \in U\}$, where U is an open neighbourhood of (t,i) in Y.

A straightforward proof of the following Proposition 4.1 is left for the reader.

Proposition 4.1. *The three spaces X of Constructions 3-5 are homeomorphic.*

Theorem 4.2. *There exists a space which is compact, separable, zero-dimensional, the continuous image of an ordered compactum, monotonically normal, and yet not orderable.*

PROOF. It is enough to prove that the space X obtained in Construction 5 is not orderable. We are going to follow the notation introduced in Construction 5. Let $\leq$ denote the lexicographic ordering of Y.

Suppose that X is orderable and let $\sqsubseteq$ be a linear ordering on X which induces the original topology of X. Since we are going to consider two different linear orderings $\leq$ and $\sqsubseteq$, the need arises to deal with the two kinds of closed intervals. Let $[a,b] = \{y \in Y : a \leq y \leq b\}$ when $a, b \in Y$ and $a \leq b$, and let $I(c,d) = \{x \in X : c \sqsubseteq x \sqsubseteq d\}$ when $c, d \in X$ and $c \sqsubseteq d$. In addition, intervals of real numbers are denoted as usual.

Since X is compact, $(X, \sqsubseteq)$ is Dedekind complete. Let x_0 and x_1 denote the smallest and the biggest points of $(X, \sqsubseteq)$, respectively. Since Y is a closed subset of X, there exist the smallest point y_0 of Y and the biggest point y_1 of Y in $(X, \sqsubseteq)$.

Let (a, b) be a pair of adjacent jump points in $(Y, \sqsubseteq)$, i.e., $a, b \in Y$, $a \sqsubseteq b$, $a \neq b$ and $I(a,b) \cap Y = \{a, b\}$. Since Y is compact and Q is discrete, $I(a', b')$ is finite for all $a', b' \in Q$ such that $a \sqsubseteq a' \sqsubseteq b' \sqsubseteq b$. If $I(a,b) \cap Q \neq \emptyset$, let $c_{(a,b)} \in I(a,b) \cap Q$.

Observe also that if $q \in Q$ and $q \sqsubseteq y_0$ then $I(x_0, q)$ is a finite set, and if $q \in Q$ and $y_1 \sqsubseteq q$ then $I(q, x_1)$ is a finite set.

Let $q \in Q$. Then $q \sqsubseteq y_0$, $y_1 \sqsubseteq q$, or there exists a pair (a, b) of adjacent jump points in $(Y, \sqsubseteq)$ such that $q \in I(a, b)$.

If $q \sqsubseteq y_0$, let

$$f\big((q,0)\big) = \begin{cases} y_0 & \text{if } y_0 \neq (q,0) \\ y_1 & \text{otherwise.} \end{cases}$$

If $y_1 \sqsubseteq q$, let

$$f\big((q,0)\big) = \begin{cases} y_1 & \text{if } y_1 \neq (q,0) \\ y_0 & \text{otherwise.} \end{cases}$$

If $q \in I(a, b)$ for some pair of adjacent jump points in $(Y, \sqsubseteq)$, let

$$f\big((q,0)\big) = \begin{cases} a & \text{if } b = (q,0) \text{ or if both } q \in I(a, c_{(a,b)}) \text{ and } a \neq (q,0) \\ b & \text{otherwise.} \end{cases}$$

Then f is a function, $f : Q \times \{0\} \to Y$. Obviously, f has no fixed point. □

Claim 1. If $A \subset Q$, then the set of cluster points of $A \times \{0\}$ in X coincides with the union of the set of cluster points of $f\big(A \times \{0\}\big)$ in X and the set $\Big\{x \in f\big(A \times \{0\}\big) : f^{-1}(x) \cap \big(A \times \{0\}\big) \text{ is infinite}\Big\}$.

Proof of Claim 1. Let x be a cluster point of $A \times \{0\}$ in X. Then x is also a cluster point of A in X. Hence, there exists a $\sqsubseteq$-monotone, say $\sqsubseteq$-increasing, sequence $(q_n)_{n=1}^{\infty}$ in A which converges to x. If $x \in \mathrm{cl}\big(I(x_0, x) \cap Y - \{x\}\big)$, then there is a subsequence $(q_{n_i})_{i=1}^{\infty}$ such that $q_{n_i} \sqsubseteq f\big((q_{n_{i+1}}, 0)\big) \sqsubseteq q_{n_{i+2}}$ and $q_{n_i} \neq f\big((q_{n_{i+1}}, 0)\big) \neq q_{n_{i+2}}$ for $i = 1, 2, \ldots$ Hence, the sequence $\Big(f\big((q_{n_i}, 0)\big)\Big)_{i=1}^{\infty}$ converges to x, i.e., x is a cluster point of $f\big(A \times \{0\}\big)$. If $x \notin \mathrm{cl}\big(I(x_0, x) \cap Y - \{x\}\big)$, then there exists a positive integer n_0 such that $f\big((q_n, 0)\big) = x$ for all $n > n_0$, and so $f^{-1}(x) \cap \big(A \times \{0\}\big)$ is infinite.

Now, suppose that x is a cluster point of $f\big(A \times \{0\}\big)$ in X. Then there exists a sequence $(q_n)_{n=1}^{\infty}$ in A such that $\Big(f\big((q_n, 0)\big)\Big)_{n=1}^{\infty}$ is a $\sqsubseteq$-monotone, say $\sqsubseteq$-increasing, sequence of distinct points converging to x. Clearly, there exists a subsequence $(q_{n_i})_{i=1}^{\infty}$ such that $q_{n_i} \sqsubseteq f\big((q_{n_{i+1}}, 0)\big) \sqsubseteq q_{n_{i+2}}$ and $q_{n_i} \neq f\big((q_{n_{i+1}}, 0)\big) \neq$

$q_{n_{i+2}}$ for all i. Hence, $(q_{n_i})_{i=1}^{\infty}$ converges to x, i.e., x is a cluster point of A, and so a cluster point of $A \times \{0\}$.

Finally, suppose that $x \in f(A \times \{0\})$ and the set $f^{-1}(x) \cap (A \times \{0\})$ is infinite. Let $\delta : Q \times \{0\} \to Q$ denote the projection, $\delta((q,0)) = q$. The following four cases are possible concerning the set $A_x = \delta(f^{-1}(x)) \cap A$:
(1) $A_x \subset I(x_0, y_0)$ and $x = y_0$, (2) $A_x \subset I(y_1, x_1)$ and $x = y_1$, (3) there exists a pair (a,b) of adjacent jump points in $(Y, \sqsubseteq)$ such that $A_x \subset I(a, c_{(a,b)})$ and $x = a$, or (4) there exists a pair (a,b) of adjacent jump points in $(Y, \sqsubseteq)$ such that $A_x \subset I(c_{(a,b)}, b)$ and $x = b$. In either of these cases A_x is a $\sqsubseteq$-monotone sequence $(q_n)_{n=1}^{\infty}$ of distinct points which converges to x. Then $\big((q_n, 0)\big)_{n=1}^{\infty}$ converges to x as well, and so x is a cluster point of $A \times \{0\}$. This completes the proof of Claim 1.

For each $n \in \{0, 1, \ldots, \infty\}$ let P_n denote the set of all $p \in P$ such that the set

$$\Big\{q \in Q : \big[(0,0),(p,0)\big] \text{ contains exactly one of the points } (q,0) \text{ and } f\big((q,0)\big)\Big\}$$

consists of n elements.

Note that P_∞ is empty because otherwise there would exist $p \in P_\infty$ and an infinite set A contained in Q such that either $A \times \{0\} \subset \big[(0,0),(p,0)\big]$ and $f(A \times \{0\}) \subset \big[(p,1),(1,0)\big]$ or $f(A \times \{0\}) \subset \big[(0,0),(p,0)\big]$ and $A \times \{0\} \subset \big[(p,1),(1,0)\big]$. This would imply that $A \times \{0\}$ has a cluster point not in the closure of $f(A \times \{0\})$, contradicting Claim 1.

<u>Claim 2</u>. For each non-negative integer n, P_n is the union of a countable set and a nowhere dense subset of P.

Proof of Claim 2. As before, let $\delta((q,0)) = q$ for each $q \in Q$.

Suppose that n is a non-negative integer such that P_n is not the union of a countable set and a nowhere dense set. Hence, there exists an open interval J contained in $[0,1]$ such that $J \subset \mathrm{cl}(P_n - C)$ for each countable set C. Let $p_0 \in (J \cap P_n) - \delta(f(Q \times \{0\}))$. Let $F_0 = \Big\{q \in Q : \big[(0,0),(p_0,0)\big]$ contains exactly one of the points $(q,0)$ and $f\big((q,0)\big)\Big\}$. Then F_0 has exactly n elements.

Let q_0 denote the biggest element of the finite set
$\Big((F_0 \times \{0\}) \cup f(F_0 \times \{0\})\Big) \cap [(0,0),(p_0,0)] - \{(p_0,0)\}$ in $(Y, \leq)$. Let $q_1 \in Q \cap J$ be such that $q_0 < (q_1, 0) < (p_0, 0)$. Then there exists $p_1 \in J \cap P_n$ such that $(p_1, 0)$ is between $(q_1, 0)$ and $f\big((q_1,0)\big)$ as well as between q_0 and $(p_0, 0)$. It follows that the set $F_1 = \Big\{q \in Q : \big[(0,0),(p_1,0)\big]$ contains exactly one of the points

$(q,0)$ and $f\big((q,0)\big)\big\}$ contains the $(n+1)$-point set $F_0 \cup \{q_1\}$. This contradicts the fact that $p_1 \in P_n$. The proof of Claim 2 is complete.

Since $P_\infty = \emptyset$ and $P = \bigcup_{n=0}^{\infty} P_n$, Claim 2 implies that the set P of all irrational numbers between 0 and 1 is the union of countably many nowhere dense sets. This contradiction which has originated from the hypothesis that X is an orderable space concludes the proof of Theorem 4.2.

Theorem 4.3. *Let X be a compact, separable and zero-dimensional space. Then the following conditions are equivalent:*

(i) *X is orderable;*

(ii) *there exists a dendron Z such that $X \subset Z$, X is a strong T-set in Z, and the set $\mathrm{cl}(I \cap R_Z)$ is metrizable for each orderable subcontinuum I of Z.*

PROOF. If X is orderable, then it can be embedded into an orderable continuum Z as a strong T-set. Then $R_Z = \emptyset$. The alleged proof of [**8**, (3.1)] shows that (ii) implies (i) as it was already remarked in [**9**]. □

Remark. Let R be a subset of P. Let $Y_R = \big([0,1] \times \{0\}\big) \cup \big(R \times \{1\}\big)$ and consider the lexicographic ordering and the order topology on Y_R. Let $X_R = Y_R \cup Q$ with the topology defined as in Construction 5.

(a) The argument of Theorem 4.2 shows that X_R is not orderable if R is not the union of countably many nowhere dense subsets of $[0,1]$.

(b) It is not hard to prove that X_R is orderable when $R = A \cup \bigcup_{n=1}^{\infty} R_n$, where A is a countable dense subset of P, and each R_n is a nowhere dense subset of $[0,1]$ such that no component of $[0,1] - R_n$ is of the form $\{q\}$ with $q \in Q$.

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