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FOLIATIONS ON CONSTANT CURVATURE SURFACES AND NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

In [2], Chern and Tenenblat found a relation between foliations on a surface of constant curvature and the MKdV equation. They gave a class of such foliations such that the geodesic curvature of the leaves of a foliation in this class satisfies the MKdV equation, if one chooses a coordinate system on the surface accordingly. Similar relations between foliations on a surface of constant curvature $K \neq 0$ and the sine-Gordon equation, and between foliations on a surface of constant negative curvature and the KdV equation, were given by Tian in [9]. Naturally, one would like to know if any other (*real*) nonlinear partial differential equations, especially any other soliton equations such as the sinh-Gordon equation, the Calogero-Degasperis-Fokas equation, the Sawada-Kotera equation, and the Kaup-Kupershmidt equation, are related to classes of foliations on a surface of constant curvature. If there are more, then one would like to have constructive procedures to find a class of foliations for a given nonlinear partial differential equation (PDE).

In this paper, we first observe that in order for a nonlinear PDE to describe a class of foliations on a surface of constant negative curvature (pseudo-spherical surface) or a surface of constant positive curvature (spherical surface), the equation must be the compatibility condition for a special type of $\mathrm{sl}(2,\mathbb{R})$- or $\mathrm{su}(2)$-linear system. Then, we look for necessary and sufficient conditions for the existence of a class of foliations on a pseudo-spherical or spherical surface described by a given nonlinear PDE of the form $u_t = \mathcal{F}(u, u_x, \ldots, u_{x\cdots x})$ or $u_{xt} = \mathcal{F}(u, u_x, \ldots, u_{x\cdots x})$ or $u_{xt} = \mathcal{F}(u, u_x, u_t)$. Under two technical assumptions (one of them will be given in §2 and the other in the theorems in §3–5), we show that the existence of such a linear system for an equation of one of the above forms together with some minor restriction(s) on the entrices in the coefficient matrices of the linear system form necessary and sufficient conditions. Moreover, our proof in each case gives a general way for computing the class of foliations. These general ways are unified treatments of the known examples and yield many new examples. In particular, we obtain four classes of foliations on a pseudo-spherical surface described by the sinh-Gordon equation, the Calogero-Degasperis-Fokas equation, the Sawada-Kotera equation, and the Kaup-Kupershmidt equation, respectively.

We note that Pinkall recently gave in [7] a very interesting relation between evolutions of an affine curve and the KdV equation, and that relations between soliton equations and classes of surfaces in $\mathbb{R}^3$ are discussed in [10].

The organization of this paper is as follows. In §2, we review an example and give a general definition. We treat nonlinear PDE's of the form $u_t = \mathcal{F}(u, u_x, \ldots, u_{x\cdots x})$ in §3, while §4 is devoted to nonlinear PDE's of the form $u_{xt} = \mathcal{F}(u, u_x, \ldots, u_{x\cdots x})$. Finally, nonlinear PDE's of the form $u_{xt} = \mathcal{F}(u, u_x, u_t)$ are discussed in §5.

We are indebted to Josef Dorfmeister for his interest in this work, and would like to thank Qiming Liu for bringing the references [1], [4] and [6] to our attention and for suggesting that the Kaup-Kupershmidt equation be included as a concrete example. We are also grateful to the referee, whose suggestions led improvements in our presentation. The author was partially supported by NSF Grant DMS–9205293.

2. An Example and a General Definition

In this section, we first review the example of Chern and Tenenblat in [2], then give the general definition to be used in the sequel, and finally present a necessary condition for a nonlinear PDE to describe a class of foliations.

Let M be a surface endowed with a Riemannian metric of constant Gaussian curvature K. Locally, let $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be an orthonormal frame field and $\{\omega_1, \omega_2\}$ its dual coframe field. The structure equations of M can then be written as

$$(2.1) \qquad \mathrm{d}\omega_1 = \omega_{12} \wedge \omega_2, \qquad \mathrm{d}\omega_2 = \omega_1 \wedge \omega_{12}, \qquad \mathrm{d}\omega_{12} = -K\omega_1 \wedge \omega_2,$$

where ω_{12} is the corresponding connection form on M.

Given a foliation of M by curves, suppose that at each point $y \in M$, $\boldsymbol{e}_1$ is tangent to the leaf through y of the foliation; namely, the foliation is defined by

$\omega_2 = 0$. Let

$$\omega_{12} = p\omega_1 + q\omega_2, \tag{2.2}$$

then p and q are the local invariants of the foliation (under a fixed local orientation).

Assume that p and q satisfy

$$q = \frac{\partial_1^2 p}{\partial_1 p}, \tag{2.3}$$

then there is a (local) coordinate system (x, t) on M satisfying

$$\frac{\partial}{\partial x} = \eta e_1, \qquad \frac{\partial}{\partial t} = \eta^3\big(-K + \frac{1}{2}p^2\big)e_1 + \eta^3\partial_1 p \cdot e_2, \tag{2.4}$$

where $\eta \neq 0$ is a free (real) parameter and one needs the assumption

$$\partial_1 p \neq 0. \tag{2.5}$$

Moreover, $u = \eta p$ is a solution to the MKdV equation

$$u_t = \frac{3}{2}u^2 u_x + u_{xxx}. \tag{2.6}$$

Proofs of these claims can be found in [1].

Conversely, given a domain $D \subseteq \mathbb{R}^2$ and a solution $u : D \to \mathbb{R}$ to the MKdV equation satisfying

$$u_x \neq 0 \tag{2.7}$$

at each point of D, if we define a Riemannian metric on D by specifying an orthonormal frame field $\{e_1, e_2\}$ via

$$\frac{\partial}{\partial x} = \eta e_1, \qquad \frac{\partial}{\partial t} = \big(-\eta^3 K + \frac{1}{2}\eta u^2\big)e_1 + (\eta\partial_x u)e_2, \tag{2.8}$$

then one can verify that D has constant curvature K, the geodesic curvature p of the x-lines is equal to u/η, and the foliation of D by the x-lines satisfies (2.3).

Altogether, we see that the class of foliations on M satisfying (2.3) is *described* by the MKdV equation.

With the above example in mind, we are ready to give a general definition, in which we will not write down any obviously necessary assumptions like (2.5) or (2.7).

Definition 1. The class of foliations on a surface M of constant curvature K satisfying

$$F(p, q, \partial_1 p, \partial_2 p, ..., \partial_2^\ell \cdots \partial_1^k q) = 0 \tag{2.9}$$

is said *to be described by the nonlinear partial differential equation*

$$\mathcal{F}(u, \partial_x u, \partial_t u, ..., \partial_t^n \partial_x^m u) = 0 \tag{2.10}$$

if there exist (real) smooth functions α, $\tilde{\alpha}$, γ, $\tilde{\gamma}$, δ, $\tilde{\delta}$ and f such that for every foliation satisfying (2.9) there holds

$$[\alpha(p,q)\,\boldsymbol{e}_1,\ \gamma(p,q,...,\partial_2^l\cdots\partial_1^k q)\,\boldsymbol{e}_1 + \delta(p,q,...,\partial_2^l\cdots\partial_1^k q)\,\boldsymbol{e}_2] = 0, \tag{2.11}$$

and if we define

$$\frac{\partial}{\partial x} = \alpha(p,q)\,\boldsymbol{e}_1, \tag{2.12}$$

$$\frac{\partial}{\partial t} = \gamma(p,q,...,\partial_2^l\cdots\partial_1^k q)\,\boldsymbol{e}_1 + \delta(p,q,...,\partial_2^l\cdots\partial_1^k q)\,\boldsymbol{e}_2, \tag{2.13}$$

$$u = f(p,q,\partial_1 p,\partial_2 p,...,\partial_2^l\cdots\partial_1^k q), \tag{2.14}$$

then u satisfies (2.10), and that for every solution $u : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ to (2.10) if we define a Riemannian metric on D by giving an orthonormal frame field $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ via

$$\begin{aligned} \frac{\partial}{\partial x} &= \tilde{\alpha}(u, \partial_x u, \partial_t u, ..., \partial_t^n \partial_x^m u)\boldsymbol{e}_1, \\ \frac{\partial}{\partial t} &= \tilde{\gamma}(u, \partial_x u, \partial_t u, ..., \partial_t^n \partial_x^m u)\boldsymbol{e}_1 + \tilde{\delta}(u, \partial_x u, \partial_t u, ..., \partial_t^n \partial_x^m u)\boldsymbol{e}_2, \end{aligned} \tag{2.15}$$

then D has constant Gaussian curvature K, the foliation of D by the x-lines satisfies (2.9), and u can be expressed in terms of the corresponding p and q by (2.14).

Remark. We avoid considering foliations consisting of t-lines by interchanging the variables x and t in (2.10) when necessary.

Every nonlinear PDE considered in this paper is assumed to have solutions to all its smooth initial value problems in a certain range, no matter how small the range is. This is our first technical assumption mentioned in the introduction. Moreover, from now on, we *only look at the cases* $K \neq 0$. By rescaling, we can then assume $K = -1$ or 1. If $K = -1$, the structure equations of M form the compatibility condition for the linear system

$$\mathrm{d}V = \Omega V \tag{2.16}$$

on $V : M \to \mathrm{SL}(2, \mathbb{R})$, where

$$\Omega = \frac{1}{2}\begin{pmatrix} \omega_2 & \omega_1 - \omega_{12} \\ \omega_1 + \omega_{12} & -\omega_2 \end{pmatrix}; \tag{2.17}$$

if $K = 1$, the structure equations of M form the compatibility condition for the linear system

$$\mathrm{d}W = \Theta W \tag{2.18}$$

on $W : M \to \mathrm{SU}(2)$, where

$$\Theta = \frac{1}{2}\begin{pmatrix} \mathrm{i}\omega_2 & \omega_1 + \mathrm{i}\omega_{12} \\ -\omega_1 + \mathrm{i}\omega_{12} & -\mathrm{i}\omega_2 \end{pmatrix}. \tag{2.19}$$

In the soliton theory, this kind of linear systems with a free parameter are called scattering systems and used intensively. As a consequence of this observation, we have the following necessary condition for a nonlinear PDE to describe a class of foliations on a pseudo-spherical or spherical surface.

Proposition 2.1. *If a nonlinear partial differential equation describes a class of foliations on a pseudo-spherical surface, then it is the compatibility condition for an* $\mathrm{sl}(2,\mathbb{R})$*-linear system*

$$\begin{aligned} V_x &= \frac{1}{2}\begin{pmatrix} 0 & \sigma \\ \tau & 0 \end{pmatrix} V, \\ V_t &= \frac{1}{2}\begin{pmatrix} A & B \\ C & -A \end{pmatrix} V; \end{aligned} \tag{2.20}$$

if a nonlinear partial differential equation describes a class of foliations on a spherical surface, then it is the compatibility condition for an $\mathrm{su}(2)$*-linear system*

$$\begin{aligned} W_x &= \frac{1}{2}\begin{pmatrix} 0 & \sigma + \mathrm{i}\tau \\ -\sigma + \mathrm{i}\tau & 0 \end{pmatrix} W, \\ W_t &= \frac{1}{2}\begin{pmatrix} \mathrm{i}A & B + \mathrm{i}C \\ -B + \mathrm{i}C & -\mathrm{i}A \end{pmatrix} W. \end{aligned} \tag{2.21}$$

PROOF. (2.15) can be rewritten as

$$\begin{aligned} \omega_1 &= \tilde{\alpha}(u, \partial_x u, \partial_t u, \dots, \partial_t^n \partial_x^m u)\,\mathrm{d}x + \tilde{\gamma}(u, \partial_x u, \partial_t u, \dots, \partial_t^n \partial_x^m u)\,\mathrm{d}t, \\ \omega_2 &= \tilde{\delta}(u, \partial_x u, \partial_t u, \dots, \partial_t^n \partial_x^m u)\,\mathrm{d}t. \end{aligned} \tag{2.22}$$

So, the corresponding linear system (2.16) or (2.18) has the form (2.20) or (2.21) after these substitutions. □

Remark. Some existing $\mathrm{sl}(2,\mathbb{R})$-linear systems not of the form (2.20) or (2.21) are equivalent to ones of the form (2.20) or (2.21). For example, the $\mathrm{sl}(2,\mathbb{R})$-linear system

$$\begin{aligned} \tilde{V}_x &= \frac{1}{2}\begin{pmatrix} \phi & \psi \\ -\psi & -\phi \end{pmatrix}\tilde{V}, \\ \tilde{V}_t &= \frac{1}{2}\begin{pmatrix} P & Q \\ R & -P \end{pmatrix}\tilde{V} \end{aligned} \tag{2.23}$$

is equivalent to

$$\begin{aligned} V_x &= \frac{1}{2}\begin{pmatrix} 0 & \phi-\psi \\ \phi+\psi & 0 \end{pmatrix}V, \\ V_t &= \frac{1}{4}\begin{pmatrix} Q+R & 2P-Q+R \\ 2P+Q-R & -Q-R \end{pmatrix}V \end{aligned} \tag{2.24}$$

via the transformation

$$V = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\tilde{V}. \tag{2.25}$$

In this way, one obtains sl$(2,\mathbb{R})$-linear systems of the form (2.20) for the MKdV equation and the sine-Gordon equation from their usual scattering systems (see Example 3 of §3 Example 7 of §4). A similar transformation (involving a free parameter) works for the KdV equation.

Remark. In general, the existence of this kind of linear systems for a given nonlinear PDE can be discussed via the method effectively used by Chern and Tenenblat in [3]. For example, detailed calculations along their lines yield, among others, the sl$(2,\mathbb{R})$-linear system

$$\begin{aligned} V_x &= \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2}\,\eta\cos u \\ -\frac{\sqrt{2}}{\eta}\sin u & 0 \end{pmatrix}V, \\ V_t &= \frac{1}{2}\begin{pmatrix} -u_x & B(u,u_x,u_{xx}) \\ C(u,u_x,u_{xx}) & u_x \end{pmatrix}V, \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} B(u,u_x,u_{xx}) &= \eta\Big(-\sqrt{2}\sin^3 u + \sqrt{2}\sin u + \frac{1}{\sqrt{2}}\cos u\, u_x^2 - \sqrt{2}\sin u\, u_{xx}\Big), \\ C(u,u_x,u_{xx}) &= \frac{1}{\eta}\Big(\sqrt{2}\cos^3 u - \sqrt{2}\cos u - \frac{1}{\sqrt{2}}\sin u\, u_x^2 - \sqrt{2}\cos u\, u_{xx}\Big), \end{aligned} \tag{2.27}$$

for the Calogero-Degasperis-Fokas equation [1] & [4]

$$u_t = \frac{3}{2}\sin(2u)\,u_x + \frac{1}{2}u_x^3 + u_{xxx}, \tag{2.28}$$

for which an explicit Bäcklund transformation is given in [11].

In the next three sections, we want to give some necessary and sufficient conditions that yield classes of foliations described by nonlinear PDE's from such linear systems for these equations.

3. Some Equations $u_t = \mathcal{F}(u, u_x, ..., u_{x\cdots x})$ Describing Foliations

Consider a nonlinear PDE

$$u_t = \mathcal{F}(u, u_x, ..., u_{x\cdots x}). \tag{3.1}$$

By Proposition 2.1, we can assume that it is the compatibility condition for an sl$(2,\mathbb{R})$-linear system of the form (2.20) or an su(2)-linear system of the form (2.21). First, for the case of an sl$(2,\mathbb{R})$-linear system, we have the following result.

Theorem 3.1. *A nonlinear partial differential equation $u_t = \mathcal{F}(u, u_x, ..., u_{x\cdots x})$ describes a class of foliations on a pseudo-spherical surface via an* sl$(2,\mathbb{R})$*-linear system*

$$\begin{aligned} V_x &= \frac{1}{2}\begin{pmatrix} 0 & \beta(u) \\ \gamma(u) & 0 \end{pmatrix} V, \\ V_t &= \frac{1}{2}\begin{pmatrix} A(u, u_x, ..., u_{x\cdots x}) & B(u, u_x, ..., u_{x\cdots x}) \\ C(u, u_x, ..., u_{x\cdots x}) & -A(u, u_x, ..., u_{x\cdots x}) \end{pmatrix} V \end{aligned} \tag{3.2}$$

for it if and only if

$$\frac{-\beta(u) + \gamma(u)}{\beta(u) + \gamma(u)} \neq \textit{constant}. \tag{3.3}$$

Note that here we only consider an sl$(2,\mathbb{R})$-linear system whose first equation coefficient matrix involves u only. This is our second technical assumption mentioned in the introduction. In each of the other theorems of this paper, there will be a similar assumption.

Proof. Assume that the nonlinear PDE in question is the compatibility condition of an sl$(2,\mathbb{R})$-linear system (3.2) satisfying (3.3). For a general solution u, define a metric on its domain by giving an orthonormal frame field $\{e_1, e_2\}$ via

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\beta(u) + \gamma(u)}{2} e_1, \\ \frac{\partial}{\partial t} &= \frac{B(u, \partial_x u, ..., \partial_x^n u) + C(u, \partial_x u, ..., \partial_x^n u)}{2} e_1 + A(u, \partial_x u, ..., \partial_x^n u) e_2, \end{aligned} \tag{3.4}$$

then the corresponding orthonormal coframe field $\{\omega_1, \omega_2\}$ is given by

$$\begin{aligned} \omega_1 &= \frac{\beta(u)+\gamma(u)}{2}\mathrm{d}x + \frac{B(u,\partial_x u,...,\partial_x^n u)+C(u,\partial_x u,...,\partial_x^n u)}{2}\mathrm{d}t, \\ \omega_2 &= A(u,\partial_x u,...,\partial_x^n u)\mathrm{d}t. \end{aligned} \tag{3.5}$$

The corresponding connection form of the metric is $\omega_{12} = p\omega_1 + q\omega_2$ with

$$p = \frac{-\beta+\gamma}{\beta+\gamma} \quad \text{and} \quad q = \frac{-\gamma B + \beta C}{(\beta+\gamma)A}, \tag{3.6}$$

and the Gaussian curvature is -1, i.e.,

$$\partial_2 p = \partial_1 q + p^2 + q^2 - 1. \tag{3.7}$$

By the first equation in (3.6), our assumption (3.3), and the Inverse Function Theorem, there exists a smooth function f such that

$$u = f(p). \tag{3.8}$$

Hence, the first equation in (3.4) becomes

$$\frac{\partial}{\partial x} = \tilde{\alpha}(p)\, e_1, \tag{3.9}$$

where

$$\tilde{\alpha}(p) = \frac{\beta(f(p))+\gamma(f(p))}{2}, \tag{3.10}$$

Thus, by the second equation in (3.6), (3.8) and (3.9), the foliation consisting of the x-lines satisfies

$$q = \frac{-\gamma(f(p))B(f(p),...,(\tilde{\alpha}(p)\partial_1)^n f(p)) + \beta(f(p))C(f(p),...,(\tilde{\alpha}(p)\partial_1)^n f(p))}{(\beta(f(p))+\gamma(f(p)))A(f(p),...,(\tilde{\alpha}(p)\partial_1)^n f(p))}. \tag{3.11}$$

Using (3.11), one can rewrite (3.7) as

$$\partial_2 p = \sum_{j=0}^{n} \frac{\partial q(p,\partial_1 p,...,\partial_1^n p)}{\partial(\partial_1^j p)}\partial_1^{j+1} p + p^2 + q^2(p,\partial_1 p,...,\partial_1^n p) - 1. \tag{3.12}$$

By (3.8) and (3.9), the second equation in (3.4) becomes

$$\frac{\partial}{\partial t} = \tilde{\gamma}(p,\partial_1 p,...,\partial_1^n p)\, e_1 + \tilde{\delta}(p,\partial_1 p,...,\partial_1^n p)\, e_2, \tag{3.13}$$

where

$$\tilde{\gamma}(p,...,\partial_1^n p) = \frac{B(f(p),...,(\tilde{\alpha}(p)\partial_1)^n f(p)) + C(f(p),...,(\tilde{\alpha}(p)\partial_1)^n f(p))}{2},$$

$$\tilde{\delta}(p, ..., \partial_1^n p) = A(f(p), ..., (\tilde{\alpha}(p)\partial_1)^n f(p)). \tag{3.14}$$

In view of (3.9) and (3.13), $[\frac{\partial}{\partial x}, \frac{\partial}{\partial t}] = 0$ implies

$$\tilde{\alpha}\Big(\tfrac{\partial\tilde{\gamma}}{\partial p}\partial_1 p + ... + \tfrac{\partial\tilde{\gamma}}{\partial(\partial_1^n p)}\partial_1^{n+1}p\Big) - \tilde{\gamma}\tilde{\alpha}'\partial_1 p - \tilde{\delta}\tilde{\alpha}'\partial_2 p = p\tilde{\alpha}\tilde{\delta},$$
$$\tfrac{\partial\tilde{\delta}}{\partial p}\partial_1 p + ... + \tfrac{\partial\tilde{\delta}}{\partial(\partial_1^n p)}\partial_1^{n+1}p = q\tilde{\delta}. \tag{3.15}$$

Also by (3.8) and (3.9), the values of p, $\partial_1 p$, ..., $\partial_1^{n+1}p$ at a point can be treated as independent variables since the values of u, $\partial_x u$, ..., $\partial_x^{n+1}u$ at a point are independent from each other. So, substituting (3.11) and (3.12) into (3.15) yields four partial differential equations

$$\tilde{\alpha}\tfrac{\partial\tilde{\gamma}}{\partial(\partial_1^n p)} - \tilde{\delta}\tilde{\alpha}'\tfrac{\partial q}{\partial(\partial_1^n p)} = 0, \tag{3.16}$$

$$\tilde{\alpha}\textstyle\sum_{j=0}^{n-1}\tfrac{\partial\tilde{\gamma}}{\partial(\partial_1^j p)}\partial_1^{j+1}p - \tilde{\gamma}\tilde{\alpha}'\partial_1 p \tag{3.17}$$
$$-\tilde{\delta}\tilde{\alpha}'\Big(\textstyle\sum_{j=0}^{n-1}\tfrac{\partial q}{\partial(\partial_1^j p)}\partial_1^{j+1}p + p^2 + q^2 - 1\Big) = p\tilde{\alpha}\tilde{\delta},$$

$$\tfrac{\partial\tilde{\delta}}{\partial(\partial_1^n p)} = 0, \tag{3.18}$$

$$\textstyle\sum_{j=0}^{n-1}\tfrac{\partial\tilde{\delta}}{\partial(\partial_1^j p)}\partial_1^{j+1}p = q\tilde{\delta} \tag{3.19}$$

satisfied by the functions q, $\tilde{\alpha}$, $\tilde{\gamma}$, and $\tilde{\delta}$ of p, $\partial_1 p$, ..., $\partial_1^n p$ in general. Therefore, (3.15) is a consequence of (3.11) and (3.12).

We now prove the converse. Given a foliation on a pseudo-spherical surface satisfying (3.11), there hold (3.12) and, hence, (3.15). By (3.15),

$$\big[\tilde{\alpha}(p)\,e_1,\, \tilde{\gamma}(p, \partial_1 p, ..., \partial_1^n p)\,e_1 + \tilde{\delta}(p, \partial_1 p, ..., \partial_1^n p)\,e_2\big] = 0, \tag{3.20}$$

i.e., there is a coordinate system (x, t) satisfying (3.9) and (3.13). Define u by (3.8), then (3.5) and (3.6) hold. Therefore, u is a solution to the nonlinear PDE in question–the compatibility condition of (3.2).

Hence, (3.11) is a class of foliations on a pseudo-spherical surface described by the given nonlinear PDE.

Next, assume that the nonlinear PDE describes a class of foliations on a pseudo-spherical surface via an $sl(2,\mathbb{R})$-linear system (3.2). There hold (3.4)–(3.7). We want to prove that (3.2) satisfies (3.3). If not, i.e.,

$$\frac{-\beta(u) + \gamma(u)}{\beta(u) + \gamma(u)} = \text{constant}, \tag{3.21}$$

then by the first equation of (3.6), p is constant, and hence, by (3.7),

$$\partial_1 q = 1 - p^2 - q^2. \tag{3.22}$$

By the first equation of (3.4) and (3.22),

$$q_x = \frac{\beta(u)+\gamma(u)}{2}(1-p^2-q^2). \tag{3.23}$$

The invariant q can not be constant: otherwise, the u's obtained from the foliations are only constants. Thus, substituting the second equation of (3.6) into (3.23) yields a nontrivial restriction on the u's obtained. This contradicts Definition 1. Hence, (3.2) must satisfy (3.3). □

Remark. It is very lengthy and requires a lot of details about (3.2) to verify directly (3.15) or the fact that u is a solution. Therefore, the second half of the proof of the sufficiency in Theorem 3.1 is of interest itself.

Example 1. Each equation in the KdV hierarchy is the compatibility condition for an sl$(2,\mathbb{R})$-linear system (3.2) satisfying (3.3), so it describes a class of foliations on a pseudo-spherical surface. For example, the KdV equation

$$u_t = \frac{3}{2}uu_x + u_{xxx} \tag{3.24}$$

is the compatibility condition for

$$V_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ \eta - u & 0 \end{pmatrix} V,$$

$$V_t = \frac{1}{2}\begin{pmatrix} -\frac{1}{2}u_x & \eta + \frac{1}{2}u \\ (\eta-u)(\eta+\frac{1}{2}u) - u_{xx} & \frac{1}{2}u_x \end{pmatrix} V, \tag{3.25}$$

where $\eta \neq 0$ is a free parameter, and hence describes the class

$$q = \frac{\partial_1^2 p}{\partial_1 p} - \frac{3\partial_1 p}{p-1} \tag{3.26}$$

of foliations on a pseudo-spherical surface with

$$u = \frac{p+1}{p-1} + \eta. \tag{3.27}$$

As mentioned in the introduction, this example was first given by Tian in [9].

Example 2. For any smooth function $\sigma = \sigma(u, u_x, ..., u_{x\cdots x})$, the equation

$$u_t = \frac{1}{2}u_x\sigma + u\sigma_x + \sigma_{xxx} \tag{3.28}$$

is the compatibility condition for the sl$(2,\mathbb{R})$-linear system

$$
\begin{aligned}
V_x &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix} V, \\
V_t &= \frac{1}{2}\begin{pmatrix} -\frac{1}{2}\sigma_x & \frac{1}{2}\sigma \\ -\frac{1}{2}u\sigma - \sigma_{xx} & \frac{1}{2}\sigma_x \end{pmatrix} V.
\end{aligned} \tag{3.29}
$$

Thus, by Theorem 3.1, the equation (3.28) describes the class

$$q = \frac{(1-p)\sigma_{xx}}{\sigma_x} \tag{3.30}$$

of foliations on a pseudo-spherical surface, where

$$u = \frac{p+1}{p-1} \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{1}{1-p}e_1. \tag{3.31}$$

The equation (3.28) has appeared in [6] and some special cases of it follow. First of all, as shown above, the equations in the KdV hierarchy are examples of (3.28). Secondly, if we take $\sigma(u, u_x, u_{xx}) = u^2/8 + u_{xx}$ in (3.28), then we get the Sawada-Kotera equation [8]

$$u_t = \frac{5}{16}u^2u_x + \frac{5}{4}u_xu_{xx} + \frac{5}{4}uu_{xxx} + u_{xxxxx}, \tag{3.32}$$

a soliton equation closely related to the fifth order KdV equation

$$u_t = \frac{15}{8}u^2u_x + \frac{10}{2}u_xu_{xx} + \frac{5}{2}uu_{xxx} + u_{xxxxx} \tag{3.33}$$

obtained from (3.28) with $\sigma = 3u^2/4 + u_{xx}$. Hence, (3.30) with $\sigma = u^2/8 + u_{xx}$ is a class of foliations on a pseudo-spherical surface described by the Sawada-Kotera equation. Thirdly, setting $\sigma(u, u_x, u_{xx}) = 2u^2 + u_{xx}$ in (3.28) yields the Kaup-Kupershmidt equation [5]

$$u_t = 5u^2u_x + \frac{25}{2}u_xu_{xx} + 5uu_{xxx} + u_{xxxxx}, \tag{3.34}$$

another soliton equation similar to the fifth order KdV equation. Thus, (3.30) with $\sigma = 2u^2 + u_{xx}$ is a class of foliations on a pseudo-spherical surface described by the Kaup-Kupershmidt equation. Moreover, by taking $\sigma(u, u_x) = f(u) + u_x$ in (3.28) one gets the class

$$
\begin{aligned}
u_t &= \left(\frac{1}{2}f(u) + uf'(u)\right)u_x + \frac{1}{2}u_x^2 + f'''(u)u_x^3 \\
&\quad + \left(u + 3f''(u)u_x\right)u_{xx} + f'(u)u_{xxx} + u_{xxxx}
\end{aligned} \tag{3.35}
$$

of nonlinear PDE's, where f is any smooth function. So, each of these equations describes a class of foliations on a pseudo-spherical surface. Since there are few soliton equations of order 4, it is interesting to find out if (3.35) can be a soliton equation for some choice of f.

Example 3. Each equation in the MKdV hierarchy is the compatibility condition for an $\mathrm{sl}(2,\mathbb{R})$-linear system (3.2) satisfying (3.3), so it describes a class of foliations on a pseudo-spherical surface. For example, the MKdV equation (2.6) is the compatibility condition for

$$V_x = \frac{1}{2}\begin{pmatrix} 0 & \eta - u \\ \eta + u & 0 \end{pmatrix} V,$$

(3.36)

$$V_t = \frac{1}{2}\begin{pmatrix} \eta u_x & \eta^3 - \eta^2 u + \frac{1}{2}\eta u^2 - \frac{1}{2}u^3 - u_{xx} \\ \eta^3 + \eta^2 u + \frac{1}{2}\eta u^2 + \frac{1}{2}u^3 + u_{xx} & -\eta u_x \end{pmatrix} V,$$

where $\eta \neq 0$ is a free parameter, and the class of foliations on a pseudo-spherical surface yielded by Theorem 3.1 for the MKdV equation via (3.36) is (2.3).

Example 4. For any (real) constant c and any smooth function $\sigma = \sigma(u, u_x, \ldots, u_{x\cdots x})$, the equation

(3.37)

$$u_t = cu_x - \left(2u_x - 3u^2 u_x - 2u_{xxx}\right)\sigma - \left(u - u^3 - 5u_{xx}\right)\sigma_x + 4u_x\sigma_{xx} + u\sigma_{xxx}$$

is the compatibility condition for the $\mathrm{sl}(2,\mathbb{R})$-linear system

$$\begin{aligned} V_x &= \frac{1}{2}\begin{pmatrix} 0 & 1-u \\ 1+u & 0 \end{pmatrix} V, \\ V_t &= \frac{1}{2}\begin{pmatrix} 2u_x\sigma + u\sigma_x & B \\ C & -2u_x\sigma - u\sigma_x \end{pmatrix} V, \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} B &= c - cu + \left(u^2 - u^3 - 2u_{xx}\right)\sigma - 3u_x\sigma_x - u\sigma_{xx}, \\ C &= c + cu + \left(u^2 + u^3 + 2u_{xx}\right)\sigma + 3u_x\sigma_x + u\sigma_{xx}, \end{aligned} \tag{3.39}$$

and hence describes the class

$$q = \frac{2u_{xx}\sigma + 3u_x\sigma_x + u\sigma_{xx}}{2u_x\sigma + u\sigma_x} \tag{3.40}$$

of foliations on a pseudo-spherical surface, where

$$u = p \qquad \text{and} \qquad \frac{\partial}{\partial x} = e_1. \tag{3.41}$$

The equations in the MKdV hierarchy are examples of (3.37). And by choosing other σ's in (3.37), one gets many more nonlinear PDE's describing classes of foliations on a pseudo-spherical surface.

Example 5. The class of foliations on a pseudo-spherical surface described by the Calogero-Degasperis-Fokas equation (2.28) via the $\mathrm{sl}(2,\mathbb{R})$-linear system (2.26) with $\eta = 1$ is

$$q = \frac{\sin u\, B(u, u_x, u_{xx}) + \cos u\, C(u, u_x, u_{xx})}{(\sin u - \cos u) u_x}, \tag{3.42}$$

where B and C are defined by (2.27),

$$\begin{aligned} u &= \arctan \frac{p+1}{p-1}, \\ \frac{\partial}{\partial x} &= \frac{1}{\sqrt{2}}\Big(\cos \arctan \frac{p+1}{p-1} - \sin \arctan \frac{p+1}{p-1}\Big) e_1. \end{aligned} \tag{3.43}$$

Next, for the case of an su(2)-linear system, we have the following result similar to Theorem 3.1.

Theorem 3.2. *A nonlinear partial differential equation* $u_t = \mathcal{F}(u, u_x, \dots, u_{x\cdots x})$ *describes a class of foliations on a spherical surface via an* $\mathrm{su}(2)$*-linear system*

$$\begin{aligned} W_x &= \frac{1}{2}\begin{pmatrix} 0 & \beta(u) + \mathrm{i}\gamma(u) \\ -\beta(u) + \mathrm{i}\gamma(u) & 0 \end{pmatrix} W, \\ W_t &= \frac{1}{2}\begin{pmatrix} \mathrm{i}A(u, u_x, \dots, u_{x\cdots x}) & (B + \mathrm{i}C)(u, u_x, \dots, u_{x\cdots x}) \\ (-B + \mathrm{i}C)(u, u_x, \dots, u_{x\cdots x}) & -\mathrm{i}A(u, u_x, \dots, u_{x\cdots x}) \end{pmatrix} W \end{aligned} \tag{3.44}$$

for it if and only if

$$\frac{\gamma(u)}{\beta(u)} \neq \textit{constant}. \tag{3.45}$$

The proof of Theorem 3.2 is almost the same as that of Theorem 3.1, except that the basic equations now are

$$\omega_1 = \beta\, \mathrm{d}x + B\, \mathrm{d}t, \qquad \omega_2 = A\, \mathrm{d}t, \qquad \omega_{12} = \gamma\, \mathrm{d}x + C\, \mathrm{d}t,$$
$$p = \frac{\gamma}{\beta}, \qquad q = \frac{-\gamma B + \beta C}{\beta A}.$$

Example 6. Each equation in the MKdV hierarchy is the compatibility condition for an su(2)-linear system (3.44) satisfying (3.45), so it describes a class of foliations on a spherical surface. For example, the MKdV equation (2.6) is the compatibility condition for

$$W_x = \frac{1}{2}\begin{pmatrix} 0 & \eta + \mathrm{i}u \\ -\eta + \mathrm{i}u & 0 \end{pmatrix} V,$$

(3.46)

$$W_t = \frac{1}{2}\begin{pmatrix} \mathrm{i}\eta u_x & -\eta^3 - \mathrm{i}\eta^2 u + \frac{1}{2}\eta u^2 + \frac{\mathrm{i}}{2}u^3 + \mathrm{i}u_{xx} \\ \eta^3 - \mathrm{i}\eta^2 u - \frac{1}{2}\eta u^2 + \frac{\mathrm{i}}{2}u^3 + \mathrm{i}u_{xx} & -\mathrm{i}\eta u_x \end{pmatrix} V,$$

where $\eta \neq 0$ is a free parameter, and the class of foliations on a spherical surface yielded by Theorem 3.2 for the MKdV equation via (3.46) is (2.3).

4. Some Equations $u_{xt} = \mathcal{F}(u, u_x, \ldots, u_{x\cdots x})$ Describing Foliations

In this section, we consider a nonlinear PDE

$$u_{xt} = \mathcal{F}(u, u_x, \ldots, u_{x\cdots x}). \tag{4.1}$$

First, assume that it is the compatibility condition for an $\mathrm{sl}(2, \mathbb{R})$-linear system

$$\begin{aligned} V_x &= \Phi(u_x)V, \\ V_t &= \Psi(u, u_x, \ldots, u_{x\cdots x})V, \end{aligned} \tag{4.2}$$

where

$$\Phi = \frac{1}{2}\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \quad \text{and} \quad \Psi = \frac{1}{2}\begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{4.3}$$

with β, γ, A, B and C being smooth functions. Then, we have the following result.

Theorem 4.1. *A nonlinear partial differential equation $u_{xt} = \mathcal{F}(u, u_x, \ldots, u_{x\cdots x})$ describes a class of foliations on a pseudo-spherical surface via an $\mathrm{sl}(2, \mathbb{R})$-linear system (4.2) for it satisfying (4.3) if and only if*

$$\frac{-\beta(u_x) + \gamma(u_x)}{\beta(u_x) + \gamma(u_x)} \neq \textit{constant},$$

$$\frac{\partial}{\partial u}\frac{-\gamma(u_x)B(u, u_x, \ldots, u_{x\cdots x}) + \beta(u_x)C(u, u_x, \ldots, u_{x\cdots x})}{(\beta(u_x) + \gamma(u_x))A(u, u_x, \ldots, u_{x\cdots x})} \neq 0. \tag{4.4}$$

The main idea in the proof of Theorem 4.1 is the same as that in the proof of Theorem 3.1. However, since the general procedures for computing the classes of foliations produced are different, here we give the corresponding part of the proof of Theorem 4.1.

PROOF. For a general solution u, the relations between the natural frame $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\}$ and the orthonormal frame $\{e_1, e_2\}$ are

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\beta(\partial_x u)+\gamma(\partial_x u)}{2} e_1,\\ (4.5)\frac{\partial}{\partial t} &= \frac{B(u,\partial_x u,...,\partial_x^n u)+C(u,\partial_x u,...,\partial_x^n u)}{2} e_1 + A(u,\partial_x u,...,\partial_x^n u) e_2,\end{aligned}$$

and hence

$$\begin{aligned}\omega_1 &= \frac{\beta(\partial_x u)+\gamma(\partial_x u)}{2}\mathrm{d}x + \frac{B(u,\partial_x u,...,\partial_x^n u)+C(u,\partial_x u,...,\partial_x^n u)}{2}\mathrm{d}t,\\ (4.6)\qquad \omega_2 &= A(u,\partial_x u,...,\partial_x^n u)\,\mathrm{d}t.\end{aligned}$$

The corresponding connection form of the metric is $\omega_{12} = p\omega_1 + q\omega_2$ with

$$(4.7)\qquad p = \frac{-\beta+\gamma}{\beta+\gamma} \quad \text{and} \quad q = \frac{-\gamma B+\beta C}{(\beta+\gamma)A}.$$

By (4.7), our assumptions in (4.4), and the Inverse Function Theorem, there exist smooth functions f and g such that

$$(4.8)\qquad \partial_x u = f(p), \qquad u = g(q, \partial_x u, ..., \partial_x^n u).$$

Hence, the first equation in (4.5) becomes

$$(4.9)\qquad \frac{\partial}{\partial x} = \tilde{\alpha}(p)\, e_1,$$

where

$$(4.10)\qquad \tilde{\alpha}(p) = \frac{\beta(f(p))+\gamma(f(p))}{2},$$

and the second equation in (4.8) is equivalent to

$$(4.11)\quad u = g(q, f(p), \tilde{\alpha}(p)\partial_1 f(p), ..., (\tilde{\alpha}(p)\partial_1)^{n-1} f(p)) = h(q, p, \partial_1 p, ..., \partial_1^{n-1} p),$$

where h is a smooth function. Thus, by the first equation in (4.8), (4.9) and (4.11), the foliation consisting of the x-lines satisfies

(4.12)

$$\tilde{\alpha}(p)\Big(\partial_q h(q,p,\partial_1 p,...,\partial_1^{n-1}p)\partial_1 q + \sum_{j=0}^{n-1} \partial_{\partial_1^j p} h(q,p,\partial_1 p,...,\partial_1^{n-1}p)\partial_1^{j+1} p\Big) = f(p).$$

As in the proof of Theorem 3.1, we can show that (4.12) is a class of foliations described by the nonlinear PDE in question–the compatibility condition of (4.2). □

Example 7. The sine-Gordon equation

$$u_{xt} = \sin u \tag{4.13}$$

is the compatibility condition for the sl(2, $\mathbb{R}$)-linear system

$$\begin{aligned} V_x &= \frac{1}{2}\begin{pmatrix} 0 & \eta + u_x \\ \eta - u_x & 0 \end{pmatrix} V, \\ V_t &= \frac{1}{2\eta}\begin{pmatrix} \sin u & \cos u \\ \cos u & -\sin u \end{pmatrix} V, \end{aligned} \tag{4.14}$$

where $\eta \neq 0$ is a free parameter, and describes the class of foliations on a pseudo-spherical surface defined by

$$\partial_1 q = \frac{q\partial_1 p}{p} - p^2 - q^2 \tag{4.15}$$

with

$$u = -\operatorname{arccot}\frac{q}{p}, \qquad p = -\frac{1}{\eta}u_x, \qquad q = \frac{1}{\eta}u_x \cot u. \tag{4.16}$$

This example first appeared in [9], as mentioned in the introduction.

Example 8. Just like the sine-Gordon equation, every other equation in the sine-Gordon hierarchy is the compatibility condition for an sl(2, $\mathbb{R}$)-linear system (4.2) satisfying (4.3) and (4.4), so it describes a class of foliations on a pseudo-spherical surface.

Next, assume that the equation (4.1) is the compatibility condition for an su(2)-linear system

$$\begin{aligned} W_x &= \Phi(u_x)W, \\ W_t &= \Psi(u, u_x, \ldots, u_{x\cdots x})W \end{aligned} \tag{4.17}$$

of the form (2.20), i.e.,

$$\Phi = \frac{1}{2}\begin{pmatrix} 0 & \beta + \mathrm{i}\gamma \\ -\beta + \mathrm{i}\gamma & 0 \end{pmatrix} \quad \text{and} \quad \Psi = \frac{1}{2}\begin{pmatrix} \mathrm{i}A & B + \mathrm{i}C \\ -B + \mathrm{i}C & -\mathrm{i}A \end{pmatrix}, \tag{4.18}$$

where β, γ, A, B and C are smooth functions. Similar to Theorem 4.1, we have the following result.

Theorem 4.2. *A nonlinear partial differential equation* $u_{xt} = \mathcal{F}(u, u_x, \ldots, u_{x\cdots x})$ *describes a class of foliations on a spherical surface via an* su(2)*-linear system*

(4.17) for it satisfying (4.18) if and only if

$$\frac{\gamma(u_x)}{\beta(u_x)} \neq \text{constant},$$

$$\frac{\partial}{\partial u}\frac{-\gamma(u_x)B(u,u_x,\ldots,u_{x\cdots x})+\beta(u_x)C(u,u_x,\ldots,u_{x\cdots x})}{\beta(u_x)A(u,u_x,\ldots,u_{x\cdots x})} \neq 0. \tag{4.19}$$

Example 9. The sine-Gordon equation (4.13) is the compatibility condition for the su(2)-linear system

$$\begin{aligned} W_x &= \tfrac{1}{2}\begin{pmatrix} 0 & \eta - \mathrm{i}u_x \\ -\eta - \mathrm{i}u_x & 0 \end{pmatrix} W, \\ W_t &= \tfrac{1}{2\eta}\begin{pmatrix} -\mathrm{i}\sin u & -\cos u \\ \cos u & \mathrm{i}\sin u \end{pmatrix} W, \end{aligned} \tag{4.20}$$

where $\eta \neq 0$ is a free parameter, and the class of foliations on a spherical surface yielded by Theorem 4.2 for the sine-Gordon equation via (4.20) is (4.15) together with the relations in (4.16). This example first appeared in [9], as mentioned in the introduction.

Example 10. Just like the sine-Gordon equation, every other equation in the sine-Gordon hierarchy is the compatibility condition for an su(2)-linear system (4.17) satisfying (4.18) and (4.19), so it describes a class of foliations on a spherical surface.

5. Some Equations $\mathcal{F}(u, u_x, u_t, u_{xt}) = 0$ Describing Foliations

The sinh-Gordon equation

$$u_{xt} = \sinh u \tag{5.1}$$

is the compatibility condition for the sl(2, $\mathbb{R}$)-linear system

$$\begin{aligned} V_x &= \frac{1}{2\eta}\begin{pmatrix} 0 & e^{u} \\ e^{-u} & 0 \end{pmatrix} V, \\ V_t &= \frac{1}{2}\begin{pmatrix} u_t & \eta \\ \eta & -u_t \end{pmatrix} V. \end{aligned} \tag{5.2}$$

However, Theorem 4.1 does not apply to the sinh-Gordon equation via (5.2). (Even though the imaginary solutions of the sinh-Gordon equation correspond to the real solutions of the sine-Gordon equation, here we are considering real solutions of the sinh-Gordon equation.) For this kind of cases, we have the following result.

Theorem 5.1. *A nonlinear partial differential equation $u_{xt} = \mathcal{F}(u, u_x, u_t)$ describes a class of foliations on a pseudo-spherical surface via an* $\mathrm{sl}(2,\mathbb{R})$*-linear system*

$$\begin{aligned} V_x &= \frac{1}{2}\begin{pmatrix} 0 & \beta(u) \\ \gamma(u) & 0 \end{pmatrix} V, \\ V_t &= \frac{1}{2}\begin{pmatrix} A(u,u_t) & B(u,u_t) \\ C(u,u_t) & -A(u,u_t) \end{pmatrix} V \end{aligned} \tag{5.3}$$

for it if and only if

$$\frac{-\beta(u)+\gamma(u)}{\beta(u)+\gamma(u)} \neq \textit{constant},$$

$$\frac{\partial}{\partial u_t}\frac{-\gamma(u)B(u,u_t)+\beta(u)C(u,u_t)}{(\beta(u)+\gamma(u))A(u,u_t)} \neq 0. \tag{5.4}$$

The main idea in the proof of Theorem 5.1 is also the same as that in the proof of Theorems 3.1. However, since the general procedures for computing the classes of foliations produced are still quite different, the corresponding part of the proof of Theorem 4.1.

Proof. For a general solution u, define a metric on its domain by specifying an orthonormal frame field $\{e_1, e_2\}$ via

$$\frac{\partial}{\partial x} = \frac{\beta(u)+\gamma(u)}{2}e_1, \qquad \frac{\partial}{\partial t} = \frac{B(u,u_t)+C(u,u_t)}{2}e_1 + A(u,u_t)e_2, \tag{5.5}$$

then the corresponding orthonormal coframe field $\{\omega_1, \omega_2\}$ is given by

$$\omega_1 = \frac{\beta(u)+\gamma(u)}{2}dx + \frac{B(u,u_t)+C(u,u_t)}{2}dt, \qquad \omega_2 = A(u,u_t)dt. \tag{5.6}$$

The corresponding connection form of the metric is $\omega_{12} = p\omega_1 + q\omega_2$ with

$$p = \frac{-\beta+\gamma}{\beta+\gamma} \quad \text{and} \quad q = \frac{-\gamma B + \beta C}{(\beta+\gamma)A}. \tag{5.7}$$

By (5.7), our assumptions in (5.4), and the Inverse Function Theorem, there exist smooth functions f and g such that

$$u = f(p) \quad \text{and} \quad u_t = g(p,q). \tag{5.8}$$

Hence, (5.5) becomes

$$\frac{\partial}{\partial x} = \tilde{\alpha}(p)\,e_1, \qquad \frac{\partial}{\partial t} = \tilde{\gamma}(p,q)\,e_1 + \tilde{\delta}(p,q)\,e_2, \tag{5.9}$$

where

$$\begin{aligned}
\tilde{\alpha}(p) &= \frac{\beta(f(p))+\gamma(f(p))}{2}, && (5.10)\\
\tilde{\gamma}(p,q) &= \frac{B(f(p),g(p,q))+C(f(p),g(p,q))}{2}, && (5.11)\\
\tilde{\delta}(p,q) &= A(f(p),g(p,q)). && (5.12)
\end{aligned}$$

Thus, by (5.8) and the second equation of (5.9), the foliation consisting of the x-lines satisfies

$$f'(p)\big(\tilde{\gamma}(p,q)\partial_1 p+\tilde{\delta}(p,q)\partial_2 p\big)=g(p,q). \tag{5.13}$$

As in the proof of Theorem 3.1 again, we can show that (5.13) is a class of foliations described by the nonlinear PDE in question–the compatibility condition of (5.2). □

Example 11. By (5.2) and Theorem 5.1, the sinh-Gordon equation (5.1) describes the class

$$\frac{p\partial_2 p-q\partial_1 p}{p^2-1}=p \tag{5.14}$$

of foliations on a pseudo-spherical surface with

$$u=-\operatorname{arctanh} p,\qquad p=-\tanh u,\qquad q=\frac{\eta\tanh u}{u_t}. \tag{5.15}$$

The following result is similar to Theorem 5.1.

Theorem 5.2. *A nonlinear partial differential equation* $u_{xt}=\mathcal{F}(u,u_x,u_t)$ *describes a class of foliations on a spherical surface via an* su(2)*-linear system*

$$\begin{aligned}
W_x &= \frac{1}{2}\begin{pmatrix} 0 & \beta(u)+\mathrm{i}\gamma(u)\\ -\beta(u)+\mathrm{i}\gamma(u) & 0\end{pmatrix}W,\\
W_t &= \frac{1}{2}\begin{pmatrix} \mathrm{i}A(u,u_t) & B(u,u_t)+\mathrm{i}C(u,u_t)\\ -B(u,u_t)+\mathrm{i}C(u,u_t) & -\mathrm{i}A(u,u_t)\end{pmatrix}W
\end{aligned} \tag{5.16}$$

for it if and only if

$$\frac{\gamma(u)}{\beta(u)}\neq \textit{constant},$$

$$\frac{\partial}{\partial u_t}\frac{-\gamma(u)B(u,u_t)+\beta(u)C(u,u_t)}{\beta(u)A(u,u_t)}\neq 0. \tag{5.17}$$

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