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TWISTED SUMS, FENCHEL-ORLICZ SPACES AND PROPERTY (M)

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ABSTRACT. We study certain twisted sums of Orlicz spaces with non-trivial type which can be viewed as Fenchel-Orlicz spaces on $\mathbf{R}^2$. We then show that a large class of Fenchel-Orlicz spaces on $\mathbf{R}^n$ can be renormed to have property (M). In particular this gives a new construction of the twisted Hilbert space Z_2 and shows it has property (M), after an appropriate renorming.

1. INTRODUCTION

A twisted sum Z of two Banach spaces X and Y is defined (see [11]) through a short exact sequence: $0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0$. These short exact sequences in the category of (quasi-)Banach spaces are considered naturally in the investigation of three space properties (a property P in the category of quasi-Banach spaces is called a three space property if for every short exact sequence as above, Z has property P whenever X and Y have it). The roots of this theory go to Enflo, Lindenstrauss and Pisier's solution [3] to Palais' problem: the property of being isomorphic to a Hilbert space is not a three space property. The first systematic study of twisted sums of quasi-Banach spaces appears in [11]. In that paper twisted sums of quasi-Banach spaces X and Y are associated to quasi- linear maps from Y to X and the Banach spaces $Z_p, 1 < p < \infty$, are studied as examples of twisted sums of ℓ_p's. In particular, Z_2 is a reflexive Banach space with a basis which has a closed subspace X isometric to ℓ_2 with Z_2/X also isometric to ℓ_2. Z_2 is isomorphic to its dual, yet Z_2 is not isomorphic to ℓ_2. Furthermore, Z_2 has no complemented subspace with an unconditional basis: in particular it has no complemented subspace isomorphic to ℓ_2. Z_2 has an unconditional finite dimensional Schauder decomposition into two dimensional

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spaces (2-UFDD), yet Johnson, Lindenstrauss and Schechtman [6] showed that it fails to have local unconditional structure (l.u.st.). Twisted sums appear also in a natural way in complex interpolation [9]. There are several open problems on twisted sums and in particular on Z_2, (see [8]), which make the study of these spaces very interesting.

We will use the class of Fenchel-Orlicz spaces. These spaces were introduced by Turett [16] and they form a natural generalization of Orlicz spaces. A main difference between Orlicz spaces and Fenchel-Orlicz spaces is the replacement of the Orlicz function defined on $\mathbf{R}_+$ by a Young's function defined on a given normed linear space. The elements of a Fenchel-Orlicz sequence space will then be sequences in the given normed linear space.

Property (M) was introduced in [10] as a tool in the study of M-ideals of compact operators. In that paper it is proved that for a separable Banach space X, the compact operators form an M-ideal in the space of bounded operators if and only if X has property (M) and there is a sequence of compact operators K_n such that $K_n \to I$ strongly, $K_n^* \to I$ strongly and $\lim_{n\to\infty} \|I - 2K_n\| = 1$. For a detailed study of M-ideals we refer to [5].

We now give a brief overview of the paper. In Section 2 we introduce quasi-convex functions. A function on $\mathbf{R}^n$ is quasi-convex if and only if it is equivalent to a convex function (Proposition 2.1). We construct a large class of examples of quasi-convex maps on $\mathbf{R}^2$(Theorem 2.3). In Section 3 we show how quasi-convex maps can replace Young's functions in generating Fenchel-Orlicz spaces on $\mathbf{R}^n$. The main result of the section is that a twisted sum of an Orlicz space with type $p > 1$ with itself can be represented as a Fenchel-Orlicz space over $\mathbf{R}^2$ (Theorem 3.1). This includes the case of the spaces $Z_p, 1 < p < \infty$. In Section 4 we use a method of [10] to prove that if ϕ is a Young's function on $\mathbf{R}^n$ which is 0 only at 0, then the Fenchel-Orlicz space h_ϕ can be renormed to have property (M) (Theorem 4.1). Combining results of the last two sections we see that the spaces $Z_p, 1 < p < \infty$, have property (M) after renorming.

2. Quasi-convex maps

Let $\mathbf{R}_+$ (respectively $\overline{\mathbf{R}}_+$) denote the set of non-negative (respectively extended) real numbers.

Definition 1. A function $\phi : \mathbf{R}^n \to \mathbf{R}_+$ is *quasi-convex* if there exists $L > 0$ such that for every $x_1, x_2 \in \mathbf{R}^n$ and for every $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 + \lambda_2 = 1$ we have

$$\phi(\lambda_1 x_1 + \lambda_2 x_2) \leq L(\lambda_1 \phi(x_1) + \lambda_2 \phi(x_2)).$$

Note that quasi-convex maps can be defined on a vector space and that quasi-norms are quasi-convex. The reader should also note that the name *quasiconvex* is used in the literature with different meanings. In order to give a characterization of quasi-convex maps on $\mathbf{R}^n$ we introduce an equivalence relation, standard in the study of Orlicz spaces (cf. [12]).

Definition 2. Two functions ϕ and $\psi : \mathbf{R}^n \to \mathbf{R}_+$ are *equivalent ($\phi \sim \psi$)* if there exists $M > 0$ such that $\frac{1}{M}\phi(x) \le \psi(x) \le M\phi(x)$ for all $x \in \mathbf{R}^n$.

We shall say that two functions are equivalent on a set B if the above inequalities hold for all $x \in B$. We recall that the convex envelope of a function $\phi : \mathbf{R}^n \to \mathbf{R}_+$ is defined by:

$$\mathrm{co}\,\phi(\mathrm{t}) \overset{\mathrm{def}}{=} \inf\{\sum_{\mathrm{i}} \alpha_{\mathrm{i}}\phi(\mathrm{t_i}) : \mathrm{t} = \sum_{\mathrm{i}} \alpha_{\mathrm{i}}\mathrm{t_i}, \ \text{where } \mathrm{t_i} \in \mathbf{R}^n, \sum_{\mathrm{i}} \alpha_{\mathrm{i}} = 1, \alpha_{\mathrm{i}} \ge 0\}.$$

It is easy to see that

- $\mathrm{co}\,\phi(t) \le \phi(t)$ for all $t \in \mathbf{R}^n$ and
- if $\psi : \mathbf{R}^n \longrightarrow \mathbf{R}_+$ is a convex function with $\psi(t) \le \phi(t)$ for all $t \in \mathbf{R}^n$, then $\psi(t) \le \mathrm{co}\phi(t)$ for all $t \in \mathbf{R}^n$.

Proposition 2.1. *Let $\phi : \mathbf{R}^n \longrightarrow \mathbf{R}_+$. The following are equivalent:*

1. *ϕ is quasi-convex.*
2. *$\phi \sim \mathrm{co}\,\phi$.*
3. *There exists $\psi : \mathbf{R}^n \longrightarrow \mathbf{R}_+$ convex such that $\phi \sim \psi$.*

PROOF. 1⇒2. Suppose ϕ is quasi-convex. It suffices to show that there exists $M > 0$ such that $\phi \le M \mathrm{co}\,\phi$. Note that the quasi-convexity of ϕ gives that for every $N \ge 2$ there exists $L_N > 0$ such that for every $\{t_i\}_{i=1}^N$ in $\mathbf{R}^n$ and for every $\{\lambda_i\}_{i=1}^N$ in $[0,1]$ with $\sum_{i=1}^N \lambda_i = 1$ we have:

$$\phi\left(\sum_{i=1}^N \lambda_i t_i\right) \le L_N \sum_{i=1}^N \lambda_i \phi(t_i). \tag{1}$$

The proof goes by induction upon N. For example, for $N=3$:

$$\begin{aligned}\phi\left(\sum_{i=1}^{3}\lambda_i t_i\right) &= \phi\left(\lambda_1 t_1 + (\lambda_2+\lambda_3)\left(\frac{\lambda_2}{(\lambda_2+\lambda_3)}t_2 + \frac{\lambda_3}{(\lambda_2+\lambda_3)}t_3\right)\right)\\ &\leq L\left(\lambda_1\phi(t_1) + (\lambda_2+\lambda_3)\phi\left(\frac{\lambda_2}{(\lambda_2+\lambda_3)}t_2 + \frac{\lambda_3}{(\lambda_2+\lambda_3)}t_3\right)\right)\\ &\leq L(\lambda_1\phi(t_1) + L(\lambda_2\phi(t_2)+\lambda_3\phi(t_3)))\\ &\leq L_3\left(\sum_{i=1}^{3}\lambda_i\phi(t_i)\right), \text{ where } L_3 = L^2.\end{aligned}$$

Note that $\operatorname{co}(\mathrm{M}\phi) = \mathrm{M}\operatorname{co}\phi$. We will show $\phi \leq \operatorname{co}(\mathrm{M}\phi)$ with $M = L_{n+1}$. Let $t \in \mathbf{R}^n$ and let $\alpha_i \geq 0, i = 1,\dots,m$ such that $\sum_{i=1}^m \alpha_i = 1$ and $\sum_{i=1}^m \alpha_i t_i = t$. The point $(t, \sum_{i=1}^m \alpha_i(L_{n+1}\phi(t_i))) = \sum_{i=1}^m \alpha_i(t_i, L_{n+1}\phi(t_i))$ is contained inside the convex hull of $\{(t_i, L_{n+1}\phi(t_i)) | i = 1,\dots,m\}$. Therefore, by Caratheodory's Theorem (see for example [15]), there exist $n+1$ indices $i_1,\dots,i_{n+1}$ and $\lambda_1,\dots,\lambda_{n+1} \geq 0$ with $\sum_{i=1}^{n+1}\lambda_i = 1$ such that:

$$t = \sum_{j=1}^{n+1}\lambda_j t_{i_j} \text{ and}$$

$$\sum_{j=1}^{n+1}\lambda_j L_{n+1}\phi(t_{i_j}) \leq \sum_{i=1}^{m}\alpha_i L_{n+1}\phi(t_i). \tag{2}$$

By applying (1) for $N = n+1$ we see that

$$\phi(t) \leq L_{n+1}(\sum_{j=1}^{n+1}\lambda_j\phi(t_{i_j})).$$

Hence, by (2) we get

$$\phi(t) \leq \sum_{i=1}^{m}\alpha_i L_{n+1}\phi(t_i).$$

By taking the infimum over all convex combinations $t = \sum_i \alpha_i t_i$ we get $\phi(t) \leq \operatorname{co}(\mathrm{L}_{n+1}\phi)(\mathrm{t})$.

2$\Rightarrow$3 is trivial, just let $\psi = \operatorname{co}\phi$.

3$\Rightarrow$1. Suppose ψ is convex and let $M > 0$ such that $\frac{1}{M}\psi(x) \leq \phi(x) \leq M\psi(x)$ for all $x \in \mathbf{R}^n$. Then:

$$\begin{aligned}\phi(\lambda_1 x_1 + \lambda_2 x_2) &\leq M\psi(\lambda_1 x_1 + \lambda_2 x_2) \leq M(\lambda_1\psi(x_1) + \lambda_2\psi(x_2))\\ &\leq M^2(\lambda_1\phi(x_1) + \lambda_2\phi(x_2)).\end{aligned}$$

Thus ϕ is quasi-convex. □

The next theorem will give examples of quasi-convex functions on $\mathbf{R}^2$ (which are not convex). These examples will play an important role in the next section. We recall that ϕ is an Orlicz function if it is a convex, non-decreasing function on $[0,\infty)$ such that $\phi(0)=0$ and $\lim_{t\to\infty}\phi(t)=\infty$. For more information on Orlicz spaces see [12]. The functions we shall consider will be *finite valued* and *non-degenerate*, that is 0 only at 0. We say ϕ satisfies the Δ_2 *condition at zero* if $\limsup_{x\to 0}\frac{\phi(2x)}{\phi(x)}<\infty$ and, respectively, ϕ satisfies the Δ_2 *condition* if there exists $C>0$ such that for all $x\geq 0$, $\phi(2x)\leq C\phi(x)$. We will extend an Orlicz function on the whole real line by $\phi(x)=\phi(-x)$ if $x<0$ and, abusing the language, we will still call the extension an Orlicz function. We start with the following simple

Observation 2.2. Let ϕ be an Orlicz function such that

$$\text{(3)} \qquad \exists p>1,\ \exists M>0 \text{ such that } \forall\lambda\in(0,1],\ \forall s>0,\ \frac{\phi(\lambda s)}{\lambda^p\phi(s)}\leq M.$$

Then

$$\text{(4)} \qquad \exists M'>0 \text{ such that } \forall\lambda\in(0,1], \forall y>0 \text{ we have } \frac{\phi(\lambda|\log(\lambda)|y)}{\lambda\phi(y)}\leq M'.$$

Indeed, suppose (3) holds. Note that for $\lambda\in[0,1]$, $\lambda|\log\lambda|\in[0,\frac{1}{e}]$. Therefore if $\lambda\in(0,1]$ and $y>0$ we have:

$$\frac{\phi(\lambda|\log\lambda|y)}{\lambda\phi(y)}=\frac{\phi(\lambda|\log\lambda|y)}{\lambda^p|\log\lambda|^p\phi(y)}\cdot\lambda^{p-1}|\log\lambda|^p\leq MS<\infty$$

where $S=\sup_{\lambda\in[0,1]}\lambda^{p-1}|\log\lambda|^p$. We are now ready to state the main result of this section.

Theorem 2.3. *Let ϕ be an Orlicz function satisfying* (3) *and the Δ_2 condition. Let $\theta:\mathbf{R}\to\mathbf{R}$ be a Lipschitz map. Then $\Phi:\mathbf{R}^2\to\mathbf{R}_+$ defined by*

$$\Phi(x,y)=\begin{cases}\phi(y)+\phi(x-y\theta(\log\frac{1}{|y|})) & ,\ \textit{if } y\neq 0\\ \phi(x) & ,\ \textit{if } y=0\end{cases}$$

is quasi-convex.

PROOF. By Observation 2.2 the hypothesis (3) gives (4). Using the Δ_2 condition and the increasingness of ϕ one can easily prove that there exists $C>0$ such that for all $x,y\in\mathbf{R}$ we have

$$\text{(5)} \qquad \phi(x+y)\leq C(\phi(x)+\phi(y))$$

and that for all $B>0$ there exists $C_B>0$ such that for all $x\geq 0$

$$\phi(Bx)\leq C_B\phi(x). \tag{6}$$

Let $t_i=(x_i,y_i)\in\mathbf{R}^2$ and $\lambda_i\in[0,1]$, $i=1,2$ with $\lambda_1+\lambda_2=1$. Without loss of generality we may assume that $\lambda_1 y_1\neq 0$ and $\lambda_2 y_2\neq 0$. Then

$$\begin{aligned}
\Phi\left(\sum_{i=1}^2\lambda_i t_i\right) &= \phi\left(\sum_{i=1}^2\lambda_i y_i\right)+\phi\left(\sum_{i=1}^2\lambda_i x_i-\sum_{i=1}^2\lambda_i y_i\theta\left(\log\frac{1}{|\sum_{i=1}^2\lambda_i y_i|}\right)\right)\\
&= \phi\left(\sum_{i=1}^2\lambda_i y_i\right)+\phi\left(\sum_{i=1}^2\lambda_i x_i-\sum_{i=1}^2\lambda_i y_i\theta\left(\log\frac{1}{|\lambda_i y_i|}\right)\right.\\
&\quad\left.+\sum_{i=1}^2\lambda_i y_i\theta\left(\log\frac{1}{|\lambda_i y_i|}\right)-\sum_{i=1}^2\lambda_i y_i\theta\left(\log\frac{1}{|\sum_{i=1}^2\lambda_i y_i|}\right)\right)\\
&\leq \phi\left(\sum_{i=1}^2\lambda_i y_i\right)+C\phi\left(\sum_{i=1}^2\lambda_i x_i-\sum_{i=1}^2\lambda_i y_i\theta\left(\log\frac{1}{|\lambda_i y_i|}\right)\right)\\
&\quad+C\phi\left(\sum_{i=1}^2\lambda_i y_i\theta\left(\log\frac{1}{|\lambda_i y_i|}\right)-\sum_{i=1}^2\lambda_i y_i\theta\left(\log\frac{1}{|\sum_{i=1}^2\lambda_i y_i|}\right)\right)
\end{aligned}$$

using (5). For the last term in the sum we apply the inequality

$$|t\theta(\log\frac{1}{|t|})+s\theta(\log\frac{1}{|s|})-(t+s)\theta(\log\frac{1}{|t+s|})|\leq K(|s|+|t|)$$

where K is the Lipschitz constant of θ. This inequality shows that the map $t\mapsto t\theta(\log\frac{1}{|t|})$ is quasi-additive (see [11], Theorem 3.7). Since ϕ is increasing on the positive axis, we obtain:

$$\begin{aligned}
&\Phi(\sum_{i=1}^2\lambda_i t_i)\leq\phi(\sum_{i=1}^2\lambda_i y_i)+C\phi\left(\sum_{i=1}^2\lambda_i x_i-\sum_{i=1}^2\lambda_i y_i\theta\left(\log\frac{1}{|\lambda_i y_i|}\right)\right)\\
&+C\phi(K\sum_{i=1}^2\lambda_i|y_i|)\\
\leq\ &(1+CC_K)\sum_{i=1}^2\lambda_i\phi(y_i)+C\phi\left(\sum_{i=1}^2\lambda_i x_i-\sum_{i=1}^2\lambda_i y_i\theta\left(\log\frac{1}{|\lambda_i y_i|}\right)\right)
\end{aligned}$$

by using the convexity of ϕ and (6). Thus,

$$\begin{aligned}
\Phi\left(\sum_{i=1}^{2}\lambda_i t_i\right) &\leq (1+CC_K)\sum_{i=1}^{2}\lambda_i\phi(y_i) + C\phi\left(\sum_{i=1}^{2}\lambda_i x_i - \right.\\
&\left. -\sum_{i=1}^{2}\lambda_i y_i\theta\left(\log\frac{1}{|y_i|}\right) + \sum_{i=1}^{2}\lambda_i y_i\theta\left(\log\frac{1}{|y_i|}\right) - \sum_{i=1}^{2}\lambda_i y_i\theta\left(\log\frac{1}{|\lambda_i y_i|}\right)\right)\\
&\leq (1+CC_K)\sum_{i=1}^{2}\lambda_i\phi(y_i) + C^2\phi\left(\sum_{i=1}^{2}\lambda_i x_i - \sum_{i=1}^{2}\lambda_i y_i\theta\left(\log\frac{1}{|y_i|}\right)\right)\\
&\quad +C^2\phi\left(\sum_{i=1}^{2}\lambda_i y_i\theta\left(\log\frac{1}{|y_i|}\right) - \sum_{i=1}^{2}\lambda_i y_i\theta\left(\log\frac{1}{|\lambda_i y_i|}\right)\right)\\
&\leq (1+CC_K)\sum_{i=1}^{2}\lambda_i\phi(y_i) + C^2\sum_{i=1}^{2}\lambda_i\phi\left(x_i - y_i\theta\left(\log\frac{1}{|y_i|}\right)\right)\\
&\quad +C^2\phi\left(\sum_{i=1}^{2}\lambda_i|y_i|\left|\theta\left(\log\frac{1}{|y_i|}\right) - \theta\left(\log\frac{1}{|\lambda_i y_i|}\right)\right|\right)\\
&\leq (1+CC_K)\sum_{i=1}^{2}\lambda_i\phi(y_i) + C^2\sum_{i=1}^{2}\lambda_i\phi\left(x_i - y_i\theta\left(\log\frac{1}{|y_i|}\right)\right)\\
&\quad +C^3\sum_{i=1}^{2}\phi\left(\lambda_i|y_i|K\left|\log\frac{1}{|y_i|} - \log\frac{1}{|\lambda_i y_i|}\right|\right)\\
&\leq (1+CC_K)\sum_{i=1}^{2}\lambda_i\phi(y_i) + C^2\sum_{i=1}^{2}\lambda_i\phi\left(x_i - y_i\theta\left(\log\frac{1}{|y_i|}\right)\right)\\
&\quad +C^3C_K\sum_{i=1}^{2}\phi(\lambda_i|y_i||\log\lambda_i|).
\end{aligned}$$

By applying (4) to the last term we obtain

$$\begin{aligned}
\Phi\left(\sum_{i=1}^{2}\lambda_i t_i\right) &\leq (1+CC_K)\sum_{i=1}^{2}\lambda_i\phi(y_i) + C^2\sum_{i=1}^{2}\lambda_i\phi\left(x_i - y_i\theta\left(\log\frac{1}{|y_i|}\right)\right)\\
&\quad +C^3C_KM'\sum_{i=1}^{2}\lambda_i\phi(|y_i|)\\
&\leq \max(1+CC_K+C^3C_KM', C^2)\sum_{i=1}^{2}\lambda_i\Phi(t_i)
\end{aligned}$$

which ends the proof. $\square$

Remark. If ℓ_ϕ is an Orlicz space with type greater than 1 then there exists an Orlicz function $\tilde{\phi}$ satisfying (3) and the Δ_2 condition such that $\tilde{\phi}$ coincides with ϕ on $[0,1]$.

Indeed, it is well-known that the space ℓ_ϕ has non-trivial type if and only if $\alpha_\phi > 1$ and $\beta_\phi < \infty$, where α_ϕ and β_ϕ are the lower and the upper indices:

$$\alpha_\phi = \sup\{q;\ \sup_{0<\lambda,t\leq 1} \frac{\phi(\lambda t)}{\phi(\lambda)t^q} < \infty\} \text{ and}$$

$$\beta_\phi = \inf\{q;\ \inf_{0<\lambda,t\leq 1} \frac{\phi(\lambda t)}{\phi(\lambda)t^q} > 0\}$$

(cf. [13] p.140 and [12] p.143). Moreover, $\beta_\phi < \infty$ is equivalent to ϕ satisfying the Δ_2 condition at zero (see [12]). Note that $\alpha_\phi > 1$ means:

$$\exists p > 1,\ \exists M > 0 \text{ such that } \forall \lambda \in (0,1],\ \forall s \in (0,1],\ \frac{\phi(\lambda s)}{\lambda^p \phi(s)} \leq M. \tag{7}$$

Define

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & \text{, if } x \leq 1 \\ \phi(1)x^q & \text{, if } x > 1 \end{cases}$$

where $q = \max\{\frac{\phi'(1)}{\phi(1)}, p\}$ and $\phi'(1)$ denotes the left derivative of ϕ at 1. Clearly $\tilde{\phi}$ is Orlicz. Since ϕ satisfies the Δ_2 condition at zero, $\tilde{\phi}$ satisfies the Δ_2 condition (note that we don't use the full assumption of the existence of type for this part of the argument). Let us check that $\tilde{\phi}$ satisfies (3). Let $\lambda \in (0,1)$ and $s \in (0,\infty)$. If $s \leq 1$ the inequality in (3) is given by (7). If $s > 1$ we consider two cases: if $\lambda s \leq 1$ then

$$\frac{\tilde{\phi}(\lambda s)}{\lambda^p \tilde{\phi}(s)} = \frac{\phi(\lambda s)}{\lambda^p \phi(1)s^q} \leq \frac{(\lambda s)^p M \phi(1)}{\lambda^p \phi(1)s^q} = \frac{M}{s^{q-p}} \leq M$$

and if $\lambda s > 1$ then

$$\frac{\tilde{\phi}(\lambda s)}{\lambda^p \tilde{\phi}(s)} = \frac{\phi(1)(\lambda s)^q}{\lambda^p \phi(1)s^q} = \lambda^{q-p} \leq 1.$$

Clearly if an Orlicz function ϕ satisfies (3) and the Δ_2 condition, then ℓ_ϕ has non-trivial type. Finally, we note that Theorem 2.3 implies that for an Orlicz space ℓ_ϕ with non-trivial type, there exists an Orlicz function $\tilde{\phi}$ generating ℓ_ϕ such that $\Phi : \mathbf{R}^2 \to \mathbf{R}_+$ defined by

$$\Phi(x,y) = \begin{cases} \tilde{\phi}(y) + \tilde{\phi}(x - y\theta(\log \frac{1}{|y|})) & , \text{if } y \neq 0 \\ \tilde{\phi}(x) & , \text{if } y = 0 \end{cases}$$

is quasi-convex.

3. Twisted Sums and Fenchel-Orlicz Spaces

Let c_{00} denote the space of real sequences with finite support. Let $\theta : \mathbf{R} \to \mathbf{R}$ be a Lipschitz map. Let ϕ be an Orlicz function satisfying the Δ_2 condition such that ℓ_ϕ has non-trivial type. Let $\|\cdot\|_\phi$ denote the norm of the Orlicz space ℓ_ϕ:

$$\|(x_n)_n\|_\phi = \inf\{\rho > 0 : \sum_n \phi\left(\frac{x_n}{\rho}\right) \leq 1\}.$$

Let $F : c_{00} \longrightarrow c_{00}$ be defined by:

$$(F(y_m)_m)_n = \begin{cases} y_n \theta\left(\log \frac{\|(y_m)_m\|_\phi}{|y_n|}\right) & , \text{if } y_n \neq 0 \\ 0 & , \text{if } y_n = 0 \end{cases}$$

It is proved in [11] that F is a quasi-linear map, i.e. for all $\lambda \in \mathbf{R}$ and for all $x, y \in c_{00}$ we have:

$$F(\lambda y) = \lambda F(y) \text{ and}$$
$$\|F(x+y) - F(x) - F(y)\| \leq c(\|x\| + \|y\|)$$

where c is a constant independent of x and y. We define a quasi-norm on $c_{00} \times c_{00}$ by

$$\|(x_n, y_n)_n\| = \|(y_n)_n\|_\phi + \|(x_n)_n - F((y_m)_m)_n\|_\phi.$$

The twisted sum $\ell_\phi \bigoplus_F \ell_\phi$ is defined as the completion of $c_{00} \times c_{00}$ with respect to the quasi-norm $\|\cdot\|$. In other words, $\ell_\phi \bigoplus_F \ell_\phi$ consists of all sequences $(x_n, y_n)_n$ such that $\|(x_n, y_n)_n\| < \infty$. The fact that $\ell_\phi \bigoplus_F \ell_\phi$ is a Banach space follows from Theorem 2.6 in [7] which implies that a twisted sum of two B-convex Banach spaces is (after renorming) a B-convex Banach space and Pisier's result [14] that a Banach space X has type greater than 1 if and only if it is B-convex.

Definition 3 ([16]). *A Young's function* on $\mathbf{R}^n$ is an even, convex function $\Phi : \mathbf{R}^n \to \overline{\mathbf{R}}_+$ with $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(tx) = \infty$ for all $x \in \mathbf{R}^n \setminus \{0\}$.

We set (cf. [12])

$$(8)\quad \ell_\Phi = \{(x_k^1, \dots, x_k^n)_k; \exists \rho > 0 \text{ such that } \sum_k \Phi\left(\frac{1}{\rho}(x_k^1, \dots, x_k^n)\right) < \infty\}$$

and for $(x_k^1, \dots, x_k^n)_k \in \ell_\Phi$ we define

$$\|(x_k^1, \dots, x_k^n)_k\|_\Phi = \inf\{\rho > 0 : \sum_k \Phi\left(\frac{1}{\rho}(x_k^1, \dots, x_k^n)\right) \le 1\}.$$

Then ℓ_Φ is a vector space and $(\ell_\Phi, \|\cdot\|_\Phi)$ is called a Fenchel-Orlicz space. If Φ is *finite* on $\mathbf{R}^n$, ℓ_Φ is complete in $\|\cdot\|_\Phi$ (see Corollary 2.23 in [16]). For a detailed study of (more general) Fenchel-Orlicz spaces and their completeness we refer to Turett [16]. Note that for $n = 1$ we retrieve the Orlicz spaces. We also define h_Φ to be the vector subspace of ℓ_Φ consisting of all sequences $(x_k^1, \dots, x_k^n)_k$ such that $\sum_k \Phi(\frac{1}{\rho}(x_k^1, \dots, x_k^n)) < \infty$ for every $\rho > 0$. With a slight abuse of notation we will use (8) to define ℓ_Φ for any quasi-convex function $\Phi : \mathbf{R}^n \to \mathbf{R}_+$; similarly for h_Φ. We will say that a quasi-convex map $\Phi : \mathbf{R}^n \to \mathbf{R}_+$ satisfies the Δ_2 *condition* if there exists $M > 0$ such that $\Phi(2x) \le M\Phi(x)$ for all $x \in \mathbf{R}^n$. Note that if $\Phi : \mathbf{R}^n \to \mathbf{R}_+$ is a quasi-convex even function with $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(tx) = \infty$ for all $x \in \mathbf{R}^n \setminus \{0\}$ then Proposition 2.1 implies that, as sets,

$$(9)\qquad \ell_\Phi = \ell_{\operatorname{co}\Phi} \text{ and } h_\Phi = h_{\operatorname{co}\Phi}.$$

The main result of this section is:

Theorem 3.1. *If ℓ_ϕ is an Orlicz space with non-trivial type then the twisted sum $\ell_\phi \bigoplus_F \ell_\phi$ is a Fenchel-Orlicz space on $\mathbf{R}^2$. More precisely,there exists a Young's function Ψ on $\mathbf{R}^2$ such that $\ell_\phi \bigoplus_F \ell_\phi = \ell_\Psi$ (as sets) and the identity map is an isomorphism.*

The rest of this section will be devoted to the proof of this result. By Theorem 2.3 and the remarks following it we see that, without loss of generality, we may assume that ϕ satisfies the Δ_2 condition and that the map $\Phi : \mathbf{R}^2 \longrightarrow \mathbf{R}_+$ defined by

$$\Phi(x,y) = \begin{cases} \phi(y) + \phi(x - y\theta(\log\frac{1}{|y|})) & \text{, if } y \ne 0 \\ \phi(x) & \text{, if } y = 0 \end{cases}$$

is quasi-convex. We shall show that the function $\Psi = \operatorname{co}\Phi$ is a Young's function on $\mathbf{R}^2$ with the property mentioned in the theorem.

We first prove the set equality between the two spaces. We start with a

Remark. Φ satisfies the Δ_2 condition.

Indeed, for $y \neq 0$ we have:

$$\begin{aligned}
&\phi\left(2x - 2y\theta\left(\log\frac{1}{2|y|}\right)\right) \leq C\phi\left(x - y\theta\left(\log\frac{1}{2|y|}\right)\right) \\
= \;& C\phi\left(x - y\theta\left(\log\frac{1}{|y|}\right) + y\theta\left(\log\frac{1}{|y|}\right) - y\theta\left(\log\frac{1}{2|y|}\right)\right) \\
\leq \;& C^2\phi\left(x - y\theta\left(\log\frac{1}{|y|}\right)\right) + C^2\phi(yK\log 2) \\
\leq \;& C^2\phi\left(x - y\theta\left(\log\frac{1}{|y|}\right)\right) + C^2 C_{K\log 2}\phi(y)
\end{aligned}$$

where K is the Lipschitz constant of θ while C and $C_{K\log 2}$ are given by (5) and (6) respectively. Note that if a quasi-convex function $\psi : \mathbf{R}^n \to \mathbf{R}$ satisfies the Δ_2 condition then $\ell_\psi = h_\psi$ (cf.[12] Proposition 4.a.4). Therefore,

$$\ell_\Phi = h_\Phi. \tag{10}$$

The following notation will simplify further computations. For a given sequence $(x_j, y_j)_j$ let

$$S(k) = \sum_j \phi(y_j) + \sum_j \phi\left(x_j - y_j\theta\left(\log\frac{k}{|y_j|}\right)\right)$$

for $k > 0$. It is easy to see that

$$\ell_\Phi = \{(x_j, y_j)_j | \text{ there exists } \rho > 0, S(\rho) < \infty\}$$

and

$$h_\Phi = \{(x_j, y_j)_j | \text{ for all } \rho > 0, S(\rho) < \infty\}.$$

Indeed, if $(x_j, y_j)_j \in \ell_\Phi$ there exists $\rho > 0$ such that

$$\sum_j \phi\left(\frac{y_j}{\rho}\right) + \sum_j \phi\left(\frac{x_j - y_j\theta\left(\log\frac{\rho}{|y_j|}\right)}{\rho}\right) < \infty.$$

But then, since ϕ satisfies the Δ_2 condition,

$$\begin{aligned}
S(\rho) &= \sum_j \phi\left(\rho\frac{y_j}{\rho}\right) + \sum_j \phi\left(\rho\frac{x_j - y_j\theta\left(\log\frac{\rho}{|y_j|}\right)}{\rho}\right) \\
&\leq C_\rho\left(\sum_j \phi\left(\frac{y_j}{\rho}\right) + \sum_j \phi\left(\frac{x_j - y_j\theta\left(\log\frac{\rho}{|y_j|}\right)}{\rho}\right)\right) < \infty.
\end{aligned}$$

Conversely, if $(x_j, y_j)_j$ is such that $S(\rho) < \infty$ then

$$\sum_j \phi\left(\frac{y_j}{\rho}\right) + \sum_j \phi\left(\frac{x_j - y_j\theta\left(\log \frac{\rho}{|y_j|}\right)}{\rho}\right) \leq C_{\frac{1}{\rho}} S(\rho) < \infty$$

and thus $(x_j, y_j)_j \in \ell_\Phi$. Moreover, note that for $\|(y_j)_j\|_\phi > 0$ we have

(11) $\quad \|(x_j, y_j)_j\| < \infty$ if and only if $(\|(y_j)_j\|_\phi < \infty$ and$)$ $S(\|(y_j)_j\|_\phi) < \infty$.

We now show that $\ell_\phi \bigoplus_F \ell_\phi = \ell_\Psi$ as sets. Let $(x_j, y_j)_j \in \ell_\phi \bigoplus_F \ell_\phi$. Then $\|(x_j, y_j)\| < \infty$, which implies $(y_j)_j \in \ell_\phi$. If $\|(y_j)_j\|_\phi > 0$ then by (11) we get $S(\|(y_j)_j\|_\phi) < \infty$. This shows that $(x_j, y_j)_j \in \ell_\Phi$. If $\|(y_j)_j\|_\phi = 0$ then $(x_j)_j \in \ell_\phi$ and again $(x_j, y_j)_j \in \ell_\Phi$. Hence, by (9), $(x_j, y_j)_j \in \ell_\Psi$. Conversely if $(x_j, y_j)_j \in \ell_\Psi$, by (9) and (10), $(x_j, y_j)_j \in h_\Phi$, which implies $(y_j)_j \in \ell_\phi$. If $(y_j)_j \neq 0$ then $S(\|(y_j)_j\|_\phi) < \infty$. Therefore $\|(x_j, y_j)\| < \infty$ and $(x_j, y_j)_j \in \ell_\phi \bigoplus_F \ell_\phi$. If $(y_j)_j = 0$ then $(x_j)_j \in \ell_\phi$ and again $(x_j, y_j)_j \in \ell_\phi \bigoplus_F \ell_\phi$.

Note that Ψ is a finite Young's function and thus ℓ_Ψ is a Banach space. Indeed, we only need to show that

$$\lim_{t\to\infty} \Phi(t(x, y)) = \infty, \text{ for all } (x, y) \in \mathbf{R}^2 \setminus \{(0,0)\}$$

since then Proposition 2.1 will give the same result for Ψ. Let $(x, y) \neq (0, 0)$. If $y \neq 0$ then $\Phi(t(x, y)) \geq \phi(ty) \to \infty$ as $t \to \infty$ since ϕ is an Orlicz function. If $y = 0$ then $x \neq 0$ and $\Phi(t(x, y)) = \phi(tx) \to \infty$ as $t \to \infty$ since ϕ is an Orlicz function.

The next two propositions will show that the identity mapping is an isomorphism between $\ell_\phi \bigoplus_F \ell_\phi$ and ℓ_Ψ.

Proposition 3.2. *Let X be a sequence space, complete in $\|\cdot\|_1$ and $\|\cdot\|_2$, such that the coordinate functionals are continuous. Then the identity $\imath : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is an isomorphism.*

PROOF. It is easy to see that $(X, \|\cdot\|_1 + \|\cdot\|_2)$ is complete. Therefore the identity maps

$$\imath_1 : (X, \|\cdot\|_1 + \|\cdot\|_2) \to (X, \|\cdot\|_1) \text{ and } \imath_2 : (X, \|\cdot\|_1 + \|\cdot\|_2) \to (X, \|\cdot\|_2)$$

are continuous and, hence, by the Inverse Mapping Theorem, isomorphisms. Therefore $\imath = \imath_2 \circ \imath_1^{-1}$ is an isomorphism. □

Proposition 3.3. *The coordinate functionals on $\ell_\phi \bigoplus_F \ell_\phi$ and ℓ_Ψ are continuous.*

PROOF. In both cases we will show that projections $P_i((x_j,y_j)_j) = (x_i,y_i)$ are continuous for all i. The result will follow immediately since the coordinate functionals on 2-dimensional Banach spaces are continuous.

For ℓ_Ψ note that if $\|(x_j,y_j)_j\|_\Psi = 1$ then by the continuity of Ψ we get $\sum_j \Psi(x_j,y_j) \leq 1$ and hence $\Psi(x_i,y_i) \leq 1$ for all i. Therefore, for all i, $(x_i,y_i) \in \{(x,y) \in \mathbf{R}^2; \Psi(\mathrm{x},\mathrm{y}) \leq 1\}$ which is a bounded set and thus the projection P_i is continuous since all norm-topologies on a 2-dimensional Banach space are equivalent.

For $\ell_\phi \bigoplus_F \ell_\phi$ we show that there exists M such that if $\|(x_j,y_j)_j\| \leq 1$ then $\Phi(x_i,y_i) \leq M$. The boundedness of the set $\{(x,y) \in \mathbf{R}^2; \Phi(\mathrm{x},\mathrm{y}) \leq M\}$ finishes the proof as before. Indeed, note that if $\|(x_j,y_j)_j\| \leq 1$ then $S(\|(y_j)_j\|_\phi) \leq 1$. Moreover, if C is given by (5) and K is the Lipschitz constant of θ then

$$\begin{aligned}
\Phi(x_i,y_i) \;\leq\;& S(1) = \sum_j \phi(y_j) + \sum_j \phi\left(x_j - y_j\theta\left(\log\frac{\|(y_n)_n\|_\phi}{|y_j|}\right)\right. \\
&\left. + y_j\theta\left(\log\frac{\|(y_n)_n\|_\phi}{|y_j|}\right) - y_j\theta\left(\log\frac{1}{|y_j|}\right)\right) \\
\leq\;& \sum_j \phi(y_j) + C\sum_j \phi\left(x_j - y_j\theta\left(\log\frac{\|(y_n)_n\|_\phi}{|y_j|}\right)\right) \\
&+ C\sum_j \phi\left(|y_j K \log\|(y_n)_n\|_\phi|\right) \\
\leq\;& (1+C)S(\|(y_n)_n\|_\phi) + C\sum_j \phi\left(y_j K\log\|(y_n)_n\|_\phi\right) \\
\leq\;& (1+C) + C\sum_j \phi\left(\frac{y_j}{\|(y_n)_n\|_\phi}\|(y_n)_n\|_\phi K\log\|(y_n)_n\|_\phi\right) \\
\leq\;& 1 + C + M'C\sum_j \phi\left(\frac{y_j}{\|(y_n)_n\|_\phi}K\right)\|(y_n)_n\|_\phi
\end{aligned}$$

where M' is given by (4). Thus, if C_K is given by (6), we have:

$$\Phi(x_i,y_i) \;\leq\; 1 + C + M'CC_K\sum_j \phi\left(\frac{y_j}{\|(y_n)_n\|_\phi}\right) \leq 1 + C + M'CC_K.$$

The proof of the proposition is complete. □

This concludes the proof of Theorem 3.1. In particular, by choosing $\phi(x) = |x|^p$ for $1 < p < \infty$ and θ to be the identity map, we see that the spaces Z_p introduced in [11] can be viewed as Fenchel-Orlicz spaces.

We end this section with two questions which arise naturally, in view of Theorem 3.1: For what Banach spaces X can a twisted sum of X with itself be represented as a Fenchel-Orlicz space? Note that this can not be done for $X = \ell_1$ as $\ell_1 \bigoplus_F \ell_1$ is not a Banach space. If ℓ_ϕ is an Orlicz space with non-trivial type for which quasi-linear maps G is the twisted sum $\ell_\phi \bigoplus_G \ell_\phi$ a Fenchel-Orlicz space?

4. Fenchel-Orlicz spaces with property (M)

Recall the definition of property (M) [10] (see also [5]):

Definition 4. A Banach space X has *property (M)* if whenever $u, v \in X$ with $\|u\| = \|v\|$ and $(x_n)_n$ is a weakly null sequence in X then

$$\limsup_{n\to\infty} \|u + x_n\| = \limsup_{n\to\infty} \|v + x_n\|$$

A large class of spaces with property (M) can be generated as follows: Let $(n_k)_k$ be a sequence of natural numbers. For every k let N_k be a norm on $\mathbf{R}^{n_k+1}$ such that

$$0 \le x_0 \le x_0' \Rightarrow N_k(x_0, x_1, \dots, x_{n_k}) \le N_k(x_0', x_1, \dots, x_{n_k})$$

and

$$N_k(1, 0, \dots, 0) = 1.$$

Define inductively a sequence of norms on $\mathbf{R}^s$ with $s = \sum_{i=1}^k n_i$ by:

$$N_1 * N_2(x_1, x_2, \dots, x_{n_1+n_2}) = N_2(N_1(0, x_1, \dots, x_{n_1}), x_{n_1+1}, \dots, x_{n_1+n_2})$$

and once $N_1 * \cdots * N_{k-1}$ is defined,

$$\begin{aligned}&N_1 * \cdots * N_k(x_1, \dots, x_{\sum_{i=1}^k n_i}) = \\ &N_k(N_1 * \cdots * N_{k-1}(x_1, \dots, x_{\sum_{i=1}^{k-1} n_i}), x_{\sum_{i=1}^{k-1} n_i + 1}, \dots, x_{\sum_{i=1}^k n_i})\end{aligned}$$

It can be easily checked that each $N_1 * \cdots * N_k$ is a norm. For a sequence of finite sequences $\xi = ((\xi_i)_{i=1}^{n_1}, (\xi_i)_{i=n_1+1}^{n_2}, \dots, (\xi_i)_{i=n_k+1}^{n_{k+1}}, \dots)$ let

$$\|\xi\|_{\tilde\Lambda(N_k)} = \sup_k (N_1 * \cdots * N_k)(\xi_1, \dots, \xi_{\sum_{i=1}^k n_i})$$

and let $\tilde\Lambda(N_k)$ be the space of all sequences of finite sequences ξ such that $\|\xi\|_{\tilde\Lambda(N_k)} < \infty$. Then $\|\cdot\|_{\tilde\Lambda(N_k)}$ is a norm and $\tilde\Lambda(N_k)$ is a Banach space. Define $\Lambda(N_k)$ to be the closed linear span of the basis vectors $(e_k)_k$ in $\tilde\Lambda(N_k)$. A simple gliding hump argument shows that $\Lambda(N_k)$ has property (M) (see [10]). The above technique is used in [10] to show that the closed linear span of the basis of modular spaces can be renormed to have property (M). If $N_k = N$ for all k we write $\tilde{\Lambda}(N)$ for $\tilde{\Lambda}(N_k)$ and $\Lambda(N)$ for $\Lambda(N_k)$.

For the rest of the section $n \in \mathbf{N}$ will be fixed and for Fenchel-Orlicz spaces ℓ_Φ on $\mathbf{R}^n$ we shall assume that *the Young's function* Φ *is finite and* 0 *only at* 0. Our main result in this section is the following

Theorem 4.1. *Every Fenchel-Orlicz space* h_Φ *on* $\mathbf{R}^n$ *can be equivalently renormed to have property (M).*

The theorem will be proved once we show that if $\Phi : \mathbf{R}^n \to \mathbf{R}_+$ is a Young's function there exists a norm N on $\mathbf{R}^{n+1}$ such that $\ell_\Phi = \tilde{\Lambda}(N)$ (and thus $h_\Phi = \Lambda(N)$). Sufficient conditions for this last claim are given in the following

Lemma 4.2. *If* Φ *is a Young's function on* $\mathbf{R}^n$ *and* N *is a norm on* $\mathbf{R}^{n+1}$ *such that*

$$0 \le x_0 \le x_0' \Rightarrow N(x_0, x_1, \ldots, x_n) \le N(x_0', x_1, \ldots, x_n)$$

and

$$N(1, x_1, \ldots, x_n) = 1 + \Phi(x_1, \ldots, x_n)$$

then $\ell_\Phi = \tilde{\Lambda}(N)$.

PROOF. Let $(x_k^1, \ldots, x_k^n)_k \in \tilde{\Lambda}(N)$ with $\|(x_k^1, \ldots, x_k^n)_k\|_{\tilde{\Lambda}(N)} \le 1$. Let h be the first index such that $(x_h^1, \ldots, x_h^n) \neq 0$. For $k \ge h+1$ we have

$$\begin{aligned}
& \underbrace{N * \cdots * N}_{k}(x_1^1, \ldots, x_1^n, \ldots, x_k^1, \ldots, x_k^n) \\
= \;& N * \cdots * N(x_1^1, \ldots, x_{k-1}^n) \left(1 + \Phi(\frac{x_k^1}{N * \cdots * N(x_1^1, \ldots, x_{k-1}^n)}, \ldots, \right. \\
& \left. \ldots, \frac{x_k^n}{N * \cdots * N(x_1^1, \ldots, x_{k-1}^n)})\right) \\
\ge \;& N * \cdots * N(x_1^1, \ldots, x_{k-1}^n)(1 + \Phi(x_k^1, \ldots, x_k^n)).
\end{aligned}$$

The inequality holds as Φ is increasing on each ray starting from 0 and $\|(x_k^1, \ldots, x_k^n)_k\|_{\tilde{\Lambda}(N)} \le 1$. Thus,

$$\prod_{k=1}^{\infty} (\Phi(x_k^1, \ldots, x_k^n) + 1) < \infty$$

and, hence, $\sum_{k=1}^{\infty} \Phi(x_k^1, \ldots, x_k^n) < \infty$. In particular $(x_k^1, \ldots, x_k^n)_k \in \ell_\Phi$.

Conversely, if $(x_k^1, \ldots, x_k^n)_k \notin \tilde{\Lambda}(N)$ then there exists h such that $\underbrace{N * \cdots * N}_{h}(x_1^1, \ldots, x_h^n) > 1$. By a similar argument we see that

$$\prod_{k=h+1}^{\infty} (\Phi(x_k^1, \ldots, x_k^n) + 1) = \infty$$

which concludes the proof. □

The next Proposition 4.3 and Lemmas 4.7 and 4.8 show how the conditions of Lemma 4.2 can be satisfied. Let $B(x, r)$ denote the ball in $\mathbf{R}^n$ (with the Euclidean norm $\|\cdot\|_2$) centered at x with radius r.

Proposition 4.3. *If $\phi : \mathbf{R}^n \to \mathbf{R}_+$ is a Young's function there exists $\Phi : B(0,1) \to \mathbf{R}_+$ convex, even, C^1 on $B(0,1) \setminus \{0\}$ with $\Phi \sim \phi$ on $B(0,1)$.*

We begin the proof of the proposition with

Observation 4.4. If $\phi : \mathbf{R}^n \to \mathbf{R}_+$ is a Young's function there exists $\tilde{\phi} : \mathbf{R}^n \to \mathbf{R}_+$ continuous, equal to ϕ on $B(0,1)$ such that

$$\lim_{\|x\|_2 \to \infty} \frac{\tilde{\phi}(x)}{\|x\|_2} = \infty. \tag{12}$$

Indeed, just let

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & , \text{ if } \|x\|_2 \leq 1 \\ \phi(\frac{x}{\|x\|_2}) + (\|x\|_2 - 1)^2 & , \text{ if } \|x\|_2 > 1 \end{cases}$$

The proof of the proposition will follow from the next two lemmas.

Lemma 4.5. *If $\phi : \mathbf{R}^n \to \mathbf{R}_+$ is continuous and $\phi(x) = 0 \Leftrightarrow x = 0$ then there exists $\tilde{\phi} : \mathbf{R}^n \to \mathbf{R}_+$, which is C^1 on $\mathbf{R}^n \setminus \{0\}$ with $\frac{1}{2}\phi \leq \tilde{\phi} \leq 2\phi$.*

PROOF. As ϕ is continuous

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} \phi \to \phi(x) \text{ as } r \to 0.$$

Hence $\forall x \in \mathbf{R}^n \setminus \{0\}$, there exists $r(x) > 0$ such that

- for $0 < r < r(x)$ we have $\frac{1}{2}\phi \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} \phi \leq 2\phi$, and
- the map $x \mapsto r(x)$ is continuous.

Moreover there exists a function $\tilde{r}$, which is C^1 on $\mathbf{R}^n \setminus \{0\}$, such that $0 < \tilde{r}(x) \leq r(x)$. Indeed, if

$$f(x) = \min\{r(y), y \in \overline{B}(0, \frac{1}{2^n}) \setminus B(0, \frac{1}{2^{n+1}})\} \text{ for } x \in \overline{B}(0, \frac{1}{2^n}) \setminus B(0, \frac{1}{2^{n+1}})$$

then f_1, the restriction of f to the positive x_1 axis, is a positive step function and we can easily choose a C^1 function g on the positive x_1 axis such that $0 < g \leq f_1$. Then the radial extension of g gives such an $\tilde{r}$.
Define

$$\tilde{\phi}(x) = \begin{cases} \frac{1}{|B(x,\tilde{r}(x))|} \int_{B(x,\tilde{r}(x))} \phi(y)dy & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Clearly $\tilde{\phi}$ satisfies the properties required in the conclusion of the lemma. □

Note also that if in the previous lemma ϕ is even, so is $\tilde{\phi}$. Moreover $\tilde{\phi}$ satisfies the growth condition (12), if ϕ does.

The next lemma follows the idea of Corollary 3.1 in [4] (for a similar result in the infinite dimensional case see [2]).

Lemma 4.6. *Let $p \in \mathbf{R}^n$. Let $f : \mathbf{R}^n \to \mathbf{R}$ be differentiable on $\mathbf{R}^n \setminus \{p\}$, continuous at p, with $\lim_{\|x\|_2 \to \infty} \frac{f(x)}{\|x\|_2} = \infty$. Then* $\operatorname{co} f$ *is C^1 on $\mathbf{R}^n \setminus \{p\}$.*

Proof. Let $x \in \mathbf{R}^n \setminus \{p\}$. By Theorem 2.1 in [4] there exist $q \leq n+1$, $\lambda_1, \dots, \lambda_q > 0$ and $x_1, \dots, x_q \in \mathbf{R}^n$ such that:

$$\operatorname{co} f(x) = \sum_{i=1}^{q} \lambda_i f(x_i) \text{ with } \sum_{i=1}^{q} \lambda_i x_i = x \text{ and } \sum_{i=1}^{q} \lambda_i = 1$$

As $x \neq p$ we may assume, without loss of generality, that $x_1 \neq p$. Let U_1 be a small neighborhood of x_1. For $x_1' \in U_1$ consider $x' = \lambda_1 x_1' + \sum_{i=2}^{n+1} \lambda_i x_i$. Then $U = \{x' | x_1' \in U_1\}$ is a neighborhood of x. We can choose U_1 small enough such that $p \notin U$ and $p \notin U_1$. Let $h : U \to U_1$ by $h(x') = x_1'$. Clearly h is C^1.

$$\operatorname{co} f(x') \leq \lambda_1 f(x_1') + \sum_{i=2}^{q} \lambda_i f(x_i) = \lambda_1 f(h(x')) + \sum_{i=2}^{q} \lambda_i f(x_i)$$

The right hand side is a C^1 function of x' on U, call it $s(x')$. Note that $s(x) = \operatorname{co} f(x)$. Hence:

$$s(y) - s(x) \geq \operatorname{co} f(y) - \operatorname{co} f(x), \text{ for all } y \in U \tag{13}$$

Recall that for a convex function ψ on $\mathbf{R}^n$ the subdifferential of ψ is a map $\partial\psi : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ given by $x^* \in \partial\psi(x)$ if $\psi(z) \geq \psi(x) + \langle x^*, z - x\rangle$ for all z. As

x is in the interior of the domain of co f we have that $\partial \mathrm{co}\, \mathrm{f}(\mathrm{x})$ is nonempty (see [15], Theorem 23.4). Note that if $x^* \in \partial \mathrm{co}\, \mathrm{f}(\mathrm{x})$ then (13) shows that $x^* = \nabla s(x)$. Hence $\partial \mathrm{co}\, \mathrm{f}(\mathrm{x})$ is a singleton and therefore $\mathrm{co}\, \mathrm{f}(\mathrm{x})$ is differentiable at x (see [15], Theorem 25.1). Hence co f is differentiable on $\mathbf{R}^{\mathrm{n}} \setminus \{\mathrm{p}\}$. The conclusion of the lemma follows once we notice that if a finite convex function on $\mathbf{R}^{\mathrm{n}}$ is differentiable on a set then its gradient is continuous on that set (see [15], Theorem 25.5). □

Proof of Proposition 4.3. Let $\Phi = (\mathrm{co}\, \tilde{\phi})|_{\mathrm{B}(0,1)}$ be the restriction of $\mathrm{co}\, \tilde{\phi}$ on $B(0,1)$, with $\tilde{\phi}$ given by Lemma 4.5 satisfying growth condition (12): smoothness of Φ is provided by Lemma (4.6) and equivalence to ϕ on the unit ball is obvious. □

Lemma 4.7. *Let $\phi : B(0,1) \to \mathbf{R}_+$ be convex, even and C^1 on $B(0,1) \setminus \{0\}$ such that $\phi(x) = 0$ if and only if $x = 0$. Then there exists a Young's function $\tilde{\phi} : \mathbf{R}^{\mathrm{n}} \to \mathbf{R}_+$, which coincides with ϕ on a neighborhood of 0, such that*

- *$\forall x \in \mathbf{R}^{\mathrm{n}}$ the map $t \mapsto \frac{1+\tilde{\phi}(tx)}{t}$ is decreasing on $(0,\infty)$ and*
- *$x \mapsto \lim_{t\to\infty} \frac{1+\tilde{\phi}(tx)}{t}$ is a norm on $\mathbf{R}^{\mathrm{n}}$.*

Proof. For all $\alpha > 0$ small enough $\phi^{-1}(\{\alpha\})$ is an $n-1$ dimensional closed submanifold of $B(0,1)$. Such an α will be chosen later on. Let $|\cdot|_\alpha$ be the Minkowski norm of the set $\phi^{-1}([0,\alpha])$. Then $|\cdot|_\alpha$ is C^1 on $\mathbf{R}^{\mathrm{n}} \setminus \{0\}$ since ϕ is. An easy calculation shows that

$$\nabla|\cdot|_\alpha(x) = \frac{1}{\langle \nabla\phi(x), x\rangle}\nabla\phi(x), \text{ for all } x \in \mathbf{R}^{\mathrm{n}} \text{ with } |\mathrm{x}|_\alpha = 1 \tag{14}$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbf{R}^{\mathrm{n}}$. Indeed, for $x \in \mathbf{R}^{\mathrm{n}} \setminus \{0\}$, $|x|_\alpha = \frac{1}{\lambda(x)}$ where $\phi(\lambda(x)x) = \alpha$. Thus,

$$\nabla|\cdot|_\alpha(x) = -\frac{1}{\lambda^2(x)}\nabla\lambda(x) = -|x|_\alpha^2\nabla\lambda(x).$$

By differentiating $\phi(\lambda(x)x) = \alpha$ with respect to x_j we obtain

$$\begin{aligned}
0 &= \frac{\partial}{\partial x_j}\phi(\lambda(x)x) = \sum_{i=1}^{n}\frac{\partial\phi}{\partial x_i}(\lambda(x)x)\left(\frac{\partial\lambda}{\partial x_j}(x)x_i + \lambda(x)\frac{\partial x_i}{\partial x_j}\right)\\
&= \frac{\partial\phi}{\partial x_j}(\lambda(x)x)\lambda(x) + \sum_{i=1}^{n}\frac{\partial\phi}{\partial x_i}(\lambda(x)x)\frac{\partial\lambda}{\partial x_j}(x)x_i\\
&= \frac{\partial\phi}{\partial x_j}\left(\frac{x}{|x|_\alpha}\right)\frac{1}{|x|_\alpha} + \langle\nabla\phi\left(\frac{x}{|x|_\alpha}\right), x\rangle\frac{\partial\lambda}{\partial x_j}(x)
\end{aligned}$$

Thus

$$\frac{\partial\lambda}{\partial x_j}(x) = -\frac{1}{|x|_\alpha\langle\nabla\phi\left(\frac{x}{|x|_\alpha}\right), x\rangle}\frac{\partial\phi}{\partial x_j}\left(\frac{x}{|x|_\alpha}\right)$$

and therefore

$$\begin{aligned}\nabla|\cdot|_\alpha(x) &= -|x|_\alpha^2\left(\frac{-1}{|x|_\alpha\langle\nabla\phi\left(\frac{x}{|x|_\alpha}\right), x\rangle}\right)\nabla\phi\left(\frac{x}{|x|_\alpha}\right)\\ &= \frac{|x|_\alpha}{\langle\nabla\phi\left(\frac{x}{|x|_\alpha}\right), x\rangle}\nabla\phi\left(\frac{x}{|x|_\alpha}\right)\end{aligned}$$

which gives (14) for $|x|_\alpha = 1$. Let $M = \sup_{|x|_\alpha=1}\langle\nabla\phi(x), x\rangle$. Then for every x with $|x|_\alpha = 1$ and for every $u \in \mathbf{R}^n$ such that $\langle\nabla|\cdot|_\alpha(x), u\rangle > 0$ we have

$$\langle\nabla\phi(x), u\rangle \leq M\langle\nabla|\cdot|_\alpha(x), u\rangle$$

Thus, for all such x and u we have

$$D_u\phi(x) \leq D_u(M|\cdot|_\alpha)(x) \tag{15}$$

where D_u is the directional derivative in the direction of u.

Define $\tilde{\phi}$ on $\mathbf{R}^n$ by:

$$\tilde{\phi}(x) = \begin{cases}\phi(x) & \text{, if } |x|_\alpha \leq 1\\ \alpha + M(|x|_\alpha - 1) & \text{, otherwise}\end{cases}$$

Condition (15) provides the convexity of $\tilde{\phi}$. Clearly $\tilde{\phi}$ is a Young's function and coincides with ϕ in a neighborhood of 0.

Fix $x \in \mathbf{R}^n \setminus \{0\}$. We want to show that the mapping

$$t \mapsto \begin{cases}\frac{1+\phi(tx)}{t} & \text{, if } 0 < t \leq \frac{1}{|x|_\alpha}\\ \frac{1+\alpha+M(|tx|_\alpha-1)}{t} & \text{, if } t \geq \frac{1}{|x|_\alpha}\end{cases}$$

is decreasing. As the map is continuous, it suffices to show it is decreasing on $(0, \frac{1}{|x|_\alpha})$ and on $(\frac{1}{|x|_\alpha}, \infty)$. Since ϕ is convex and $\phi(0) = 0$ there exists $\tau = \tau(x)$ such that $t \mapsto \frac{1+\phi(tx)}{t}$ is decreasing on $(0, \tau(x))$ with $0 < \tau(x) \leq \frac{1}{\|x\|_2}$ and $\tau(\mu x) = \frac{\tau(x)}{\mu}$. By compactness of $\overline{B}(0,1)$ and the continuity of τ and ϕ we have $\inf_{\|y\|_2=\frac{1}{2}}\phi(\tau(y)y) > 0$. If

$$\alpha \leq \inf_{\|y\|_2=\frac{1}{2}}\phi(\tau(y)y) \tag{16}$$

then

$$\phi\left(\tau\left(\frac{x}{2\|x\|_2}\right)\frac{x}{2\|x\|_2}\right) > \alpha;$$

hence $\phi(\tau(x)x) > \alpha$ and thus $|\tau(x)x|_\alpha > 1$. Therefore $\frac{1}{|x|_\alpha} < \tau(x)$ and $t \mapsto \frac{1+\phi(tx)}{t}$ is decreasing on $(0, \frac{1}{|x|_\alpha})$.

Note that $\frac{1+\alpha+M(t|x|_\alpha-1)}{t} = \frac{1+\alpha-M}{t} + M|x|_\alpha$ is decreasing as a function of t exactly when $1+\alpha-M > 0$. Thus it suffices to have $M \le 1$. But

$$\begin{aligned} M &= \sup_{|x|_\alpha=1} \langle \nabla\phi(x), x\rangle = \sup_{|x|_\alpha=1} \langle \nabla\phi(x), \frac{x}{\|x\|_2}\rangle \|x\|_2 \\ &\le \left(\sup_{|x|_\alpha=1} D_{\frac{x}{\|x\|_2}}\phi(x)\right)(\sup_{|x|_\alpha=1} \|x\|_2) \to 0 \text{ as } \alpha \to 0 \end{aligned}$$

since $\sup_{|x|_\alpha=1} D_{\frac{x}{\|x\|_2}}\phi(x)$ is an increasing function of α and $\sup_{|x|_\alpha=1}\|x\|_2 \to 0$ as $\alpha \to 0$. In particular α can be chosen such that (16) holds and $M \le 1$. Then $t \mapsto \frac{1+\tilde{\phi}(tx)}{t}$ is decreasing $\forall x \in \mathbf{R}^n$.

Finally, note that

$$x \mapsto \lim_{t\to\infty} \frac{1+\tilde{\phi}(tx)}{t} = M|x|_\alpha$$

is a norm on $\mathbf{R}^n$. □

Lemma 4.8. *Let $\phi : \mathbf{R}^n \to \mathbf{R}_+$ be a Young's function such that*

- *$\forall x \in \mathbf{R}^n$ the map $t \mapsto \frac{1+\phi(tx)}{t}$ is decreasing on $(0,\infty)$ and*
- *$x \mapsto \lim_{t\to\infty} \frac{1+\phi(tx)}{t}$ is a norm on $\mathbf{R}^n$.*

Then

$$N(x_0, x_1, \dots, x_n) = \begin{cases} |x_0|(1+\phi\left(\frac{x_1,\dots,x_n}{|x_0|}\right)) & \text{if } x_0 \ne 0 \\ \lim_{t\to 0} |t|(1+\phi\left(\frac{x_1,\dots,x_n}{|t|}\right)) & \text{if } x_0 = 0 \end{cases}$$

is a norm on $\mathbf{R}^{n+1}$ such that

$$0 \le x_0 \le x_0' \Rightarrow N(x_0, x_1, \dots, x_n) \le N(x_0', x_1, \dots, x_n)$$

and

$$N(1, x_1, \dots, x_n) = 1 + \phi(x_1, \dots, x_n).$$

PROOF. We only need to check the triangle inequality for N. Let $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n) \in \mathbf{R}^{n+1}$.

If $x_0 > 0$ and $y_0 > 0$

$$
\begin{aligned}
N(x+y) &= (x_0+y_0)\left(1+\phi\left(\frac{x_1+y_1}{x_0+y_0}, \ldots, \frac{x_n+y_n}{x_0+y_0}\right)\right) \\
&= x_0+y_0+(x_0+y_0)\phi\left(\frac{x_0}{x_0+y_0}\frac{x_1,\ldots,x_n}{x_0}+\frac{y_0}{x_0+y_0}\frac{y_1,\ldots,y_n}{y_0}\right) \\
&\le x_0+y_0+x_0\phi\left(\frac{x_1,\ldots,x_n}{x_0}\right)+y_0\phi\left(\frac{y_1,\ldots,y_n}{y_0}\right) \\
&= N(x)+N(y)
\end{aligned}
$$

by the convexity of ϕ.

If $x_0 > 0$ and $y_0 = 0$

$$N(x+y) \le N(x_0-\epsilon, x_1, \ldots, x_n) + N(\epsilon, y_1, \ldots, y_n), \text{ for all } \epsilon \in (0, x_0)$$

by the previous case. Letting $\epsilon \to 0$, by the continuity of N we obtain the desired inequality.

Finally, if $x_0 > 0$ and $y_0 < 0$ we may assume $0 \le x_0 + y_0 < x_0$. Then, by the properties of ϕ and the previous case we have

$$
\begin{aligned}
N(x+y) &\le N(x_0, x_1+y_1, \ldots, x_n+y_n) \\
&\le N(x) + N(0, y_1, \ldots, y_n) \\
&\le N(x) + N(|y_0|, y_1, \ldots, y_n) = N(x) + N(y)
\end{aligned}
$$

which concludes the proof. □

The proof of theorem 4.1 is now complete. Theorems 3.1 and 4.1 give immediately the following

Corollary 4.9. *Let ℓ_ϕ be an Orlicz space with non-trivial type. Then the twisted sum $\ell_\phi \bigoplus_F \ell_\phi$ can be equivalently renormed to have property (M).*

In particular, the spaces $Z_p, 1 < p < \infty$, have property (M) after renorming.

We end this section with an application of the previous corollary and Theorem 2.4 in [10]. Recall that if X is a Banach space and E is a subspace of X then E is called an *M-ideal in X* (see [1]) if X^* can be decomposed as an ℓ_1-sum $X^* = E^\perp \bigoplus_1 V$ for some closed subspace V of X^*. For a Banach space X let $\mathcal{L}(X)$ denote the algebra of all bounded operators on X and $\mathcal{K}(X)$ the ideal of compact operators.

Corollary 4.10. *Let ℓ_ϕ be an Orlicz space with non-trivial type. Then $\ell_\phi \bigoplus_F \ell_\phi$ can be renormed so that $\mathcal{K}(\ell_\phi \bigoplus_F \ell_\phi)$ is an M-ideal in $\mathcal{L}(\ell_\phi \bigoplus_F \ell_\phi)$.*

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