HOUSTON JOURNAL OF MATHEMATICS

Volume 24, No. 2, 1998

THE DETERMINANT OF A HYPERGEOMETRIC PERIOD MATRIX

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COMMUNICATED BY JAMES DAMON

ABSTRACT. We consider a function $U = e^{-f_0}\prod_{j=1}^{p} f_j^{\alpha_j}$ on a real affine space, here $f_0, .., f_p$ are linear functions, $\alpha_1, ..., \alpha_p$ complex numbers. The zeros of the functions $f_1, ..., f_p$ form an arrangement of hyperplanes in the affine space. We study the period matrix of the hypergeometric integrals associated with the arrangement and the function U and compute its determinant as an alternating product of gamma functions and critical points of the functions $f_0, ..., f_p$ with respect to the arrangement. In the simplest example, $p = 1, f_0 = f_1 = t$, the determinant formula takes the form $\int_0^\infty e^{-t}\, t^{\alpha-1}\, dt = \Gamma(\alpha)$. We also give a determinant formula for Selberg type exponential integrals. In this case the arrangements of hyperplanes is special and admits a symmetry group, the period matrix is decomposed into blocks corresponding to different representations of the symmetry group on the space of the hypergeometric integrals associated with the arrangement. We compute the determinant of the block corresponding to the trivial representation.

1. INTRODUCTION

The Euler beta function is an alternating product of Euler gamma functions,

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1}$$

where the Euler gamma and beta functions are defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\, e^{-t}\, dt, \qquad B(\alpha,\beta) = \int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1}\, dt. \tag{2}$$

There is a generalization of formula (1) to the case of an arrangement of hyperplanes in an affine space, see [V1, V2, DT].

*Supported in part by NSF grant DMS-9501290 .

Example. Consider an arrangement of three points z_1, z_2, z_3 in a line. The point z_j is the zero of the function $f_j = t - z_j$. Set

$$\Delta_1 = [z_1, z_2], \qquad \Delta_1 = [z_2, z_3],$$
$$U_\alpha = (t-z_1)^{\alpha_1}(t-z_2)^{\alpha_2}(t-z_3)^{\alpha_3},$$
$$\omega_1 = \alpha_1 U_\alpha dt/(t-z_1), \qquad \omega_2 = \alpha_2 U_\alpha dt/(t-z_2),$$

then

$$\det\Big(\int_{\Delta_i}\omega_j\Big) = \frac{\Gamma(\alpha_1+1)\,\Gamma(\alpha_2+1)\,\Gamma(\alpha_3+1)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+1)}\prod_{i\neq j} f_i^{\alpha_i}(z_j).$$

In this paper we describe a generalization of the first formula in (2) to the case of an arrangement of hyperplanes in an affine space.

Example. Consider an arrangement of two points z_1, z_2 in a line. Let $f_j = t - z_j$. Set

$$\Delta_1 = [z_1, z_2], \qquad \Delta_2 = [z_2, \infty],$$
$$U_\alpha = (t-z_1)^{\alpha_1}(t-z_2)^{\alpha_2},$$
$$\omega_1 = \alpha_1 U_\alpha dt/(t-z_1), \qquad \omega_2 = \alpha_2 U_\alpha dt/(t-z_2).$$

Let a be a positive number. Then

$$\det\Big(\int_{\Delta_i} e^{-at}\omega_j\Big) = \Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\, e^{-a(z_1+z_2)}\, a^{-(\alpha_1+\alpha_2)}\prod_{i\neq j} f_i^{\alpha_i}(z_j).$$

The determinant formulas are useful, in particular in applications to the Knizhnik-Zamolodchikov type of differential equations when a determinant formula allows one to conclude that a set of solutions to the equation given by suitable multidimensional hypergeometric integrals forms a basis of solutions, cf. [SV], [TV], [V3], see also [L], [LS], [V4], [V5].

The paper is organized as follows. Sections 2 – 5 contain definitions of the main objects: arrangements, critical values, and hypergeometric period matrices. The main result of the paper is Theorem 6.2. The proofs of all statements are presented in Section 7. In Sections 8 and 9 we discuss two determinant formulas for Selberg type integrals. In this case the configuration of hyperplanes is special and admits a symmetry group. The symmetry group acts on the domains of the configuration and on the hypergeometric differential forms associated with the configuration. Therefore the period matrix of the configuration $(\int_{\Delta_i}\omega_j)$ splits into blocks according to different representations of the symmetry group. We compute the determinant of the block corresponding to the trivial representation.

2. Arrangements

In this section we review results from [FT] and [V1].

2.1. Let $f_1, \dots, f_p$ be linear polynomials on a real affine space V. Let I denote $\{1, \dots, p\}$ and let A be the arrangement $\{H_i\}_{i \in I}$, where $H_i = \ker f_i$ is the hyperplane defined by f_i.

An *edge* of A is a nonempty intersection of some of its hyperplanes. A *vertex* is a 0-dimensional edge. Let $L(A)$ denote the set of all edges.

An arrangement A is said to be *essential* if it has vertices. Until the end of this paper we suppose that A is essential.

An arrangement A is said to be in *general position* if, for all subarrangements $\{H_{i_1}, \cdots, H_{i_k}\}$ of A, we have $\operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_k}) = k$ if $1 \le k \le \dim V$ and $H_{i_1} \cap \cdots \cap H_{i_k} = \emptyset$ if $k > \dim V$.

Let

$$M(A) = V - \cup_{i \in I} H_i. \tag{3}$$

The topological space $M(A)$ has finitely many connected components, which are called *domains*. Domains are open polyhedra, not necessary bounded. Their faces are precisely the domains of the arrangements induced by A on the edges of A. More generally, in any subspace $U \subset V$ the arrangement A cuts out a new arrangement A_U consisting of the hyperplanes $\{H_i \cap U \,|\, H_i \in A,\ U \not\subset H_i\}$. A_U is called a *section* of the arrangement. For every edge F of A the domains of the section A_F are called the *faces* of the arrangement A.

Let F be an edge of A and $I(F)$ the set of all indices i for which $F \in H_i$. The arrangement A^F in V consisting of the hyperplanes $\{H_i \,|\, H_i \in A,\ i \in I(F)\}$, is called the *localization of the arrangement at the edge F*.

Every edge F of codimension l is associated to an arrangement in an $(l-1)$–dimensional projective space. Namely, let L be a normal subspace to F of the complementary dimension. Consider the localization at this edge and its section by the normal subspace. All of the hyperplanes of the resulting arrangement $(A^F)_L$ pass through the point $v = F \cap L$. We consider the arrangement which $(A^F)_L$ induces in the tangent space T_vL. It determines an arrangement in the projectivization of the tangent space, which is called the *projective normal arrangement* and denoted PA^F. The arrangements corresponding to different normal subspaces are naturally isomorphic.

A face of an arrangement is said to be *bounded relative to a hyperplane* if the closure of the face does not intersect the hyperplane. It is known [V1, Theorem

1.5] that if $A = \{H_i\}_{i \in I}$ is an arrangement in a real projective space, then the number of domains bounded with respect to H_i does not depend on i. This number is called the *discrete length of the arrangement.* The discrete length of the empty arrangement is set to be equal to 1.

Let F be an edge of an arrangement A in a projective space. The *discrete length* of the edge is defined as the discrete length of the arrangement A_F; the *discrete width* of the edge is the discrete length of the arrangement PA^F; and the *discrete volume* of the edge is the product of its discrete length and discrete width. These numbers are denoted $l(F)$, $s(F)$, and $vol(F)$, respectively.

If F is a k-dimensional edge of an arrangement A in an affine space, then its *discrete length* is the number of bounded k-dimensional faces of the arrangement A_F. Its *discrete width* is the discrete length of the arrangement PA^F, and its *discrete volume* is the product of its discrete length and discrete width.

Another more invariant definition of the above quantities could be given as follows (see [OT]). Consider the complexification and then the projectivization of the affine space V. Denote it $\mathbb{P}V$. Let $\mathbb{H}_i = \{f_i = 0\}_{i \in I}$ be hyperplanes in $\mathbb{P}V$, $\mathbb{H}_\infty$ the infinite hyperplane and $\mathbb{A}$ the arrangement in $\mathbb{P}V$ defined by all these hyperplanes. Let $\chi(\mathbb{A})$ denote the Euler characteristic of $\mathbb{P}V - \cup_{i \in \bar{I}} \mathbb{H}_i$, where $\bar{I} = 1, \dots, p, \infty$. If F is an edge of $\mathbb{A}$, then

$$l(F) = |\chi(\mathbb{A}_F)|, \quad s(F) = |\chi(P\mathbb{A}^F)|, \quad vol(F) = l(F)s(F).$$

2.2. **The beta-function of an arrangement.** An arrangement is called *weighted* if a complex number is assigned to every hyperplane of the arrangement. The complex numbers are called *weights.* The weight of a hyperplane H_i is denoted α_i.

The *weight* of an edge F of a weighted arrangement is the sum, $\alpha(F)$, of the weights of the hyperplanes which contain F.

Let A be a weighted arrangement in an affine space V. Make V into a projective space by adding the hyperplane H_∞ at infinity: $\overline{V} = V \cup H_\infty$. For all $i \in I$ denote $\overline{H_i}$ the projective closure of H_i in $\overline{V}$. Let $\overline{A} = \{\overline{H_i}\}_{i \in I} \cup \{H_\infty\}$ be the corresponding projective arrangement in $\overline{V}$. The arrangement $\overline{A}$ is called the *projectivization* of A. Set $\alpha_\infty = -(\alpha_1 + \cdots + \alpha_p)$. Let L_- denote the set of all edges at infinity of the arrangement $\overline{A}$ and L_+ the set consisting of all the other edges.

Definition 1. (*i*) Let the weights $\alpha_1, \dots, \alpha_p$ of the hyperplanes be complex numbers with positive real part. The *beta-function of an affine arrangement A*

is defined by

$$B(A;\alpha) = \prod_{F\in L_+} \Gamma(\alpha(F)+1)^{vol(F)} / \prod_{F\in L_-} \Gamma(-\alpha(F)+1)^{vol(F)};$$

(ii) In addition, let H_0 be a hyperplane in V and $\overline{H_0}$ its closure in $\overline{V}$. The *beta-function of an affine arrangement A relative to the hyperplane H_0* is defined by

$$B(A;\alpha;H_0) = \prod_{F\in L_+} \Gamma(\alpha(F)+1)^{vol(F)} / \prod_{F\in L_-,\, F\subset\overline{H_0}} \Gamma(-\alpha(F)+1)^{vol(F)}.$$

EXAMPLE. Let $n = \dim V$. For an arrangement A of p hyperplanes in general position the above formulas take the form

$$B(A;\alpha) = \left(\frac{\Gamma(\alpha_1+1)\dots\Gamma(\alpha_p+1)}{\Gamma(\alpha_1+\dots+\alpha_p+1)}\right)^{\binom{p-2}{n-1}};$$

$$B(A;\alpha;f_0) = (\Gamma(\alpha_1+1)\dots\Gamma(\alpha_p+1))^{\binom{p-2}{n-1}}.$$

2.3. **Trace of an arrangement at infinity relative to a hyperplane.** Assume that an additional non-constant linear function f_0 on V is given. Denote the hyperplane $\{f_0 = 0\}$ by H_0.

Let A be an affine arrangement in the affine space V . Consider the projectivized arrangement $\overline{A}$ in $\overline{V}$ and its section $\overline{A}_{H_\infty}$. The intersection of $\overline{A}$ with the affine space $W = H_\infty - \overline{H_0}\cap H_\infty$ is called the *trace of the arrangement A at infinity* relative to the hyperplane H_0 and is denoted $tr(A)_{H_0}$.

Lemma 2.1. *If the affine arrangement A is given by the linear functions $\{f_i\}_{i\in I}$ on V, then the affine arrangement $tr(A)_{H_0}$ is given by the linear functions $\{h_i = f_i^0/f_0^0 \mid i\in I;\ f_i^0/f_0^0 \neq const\}$ on W, where f_i^0 denotes the homogeneous part of f_i.*

3. Properties of an arrangement

In this section we examine the properties of the unbounded domains of an arrangement A on which an additional linear function f_0 tends to $+\infty$.

Let A be an arrangement in an affine space V. Consider its projectivization $\overline{A}$ in the projective space $\overline{V}$. Let Δ be an unbounded face of the arrangement A. Take the closure $\overline{\Delta}$ of Δ in $\overline{V}$. Consider the intersection $\overline{\Delta}\cap H_\infty$. It is a union of faces of the arrangement $\overline{A}_{H_\infty}$. There is a unique one of highest dimension. We call it the *trace* of Δ at infinity and denote $tr(\Delta)$.

An unbounded face Δ of A is called a *growing face with respect to* f_0 if $f_0(x)$ tends to $+\infty$ whenever x tends to infinity in Δ. A face at infinity Σ is called a *bounded face at infinity with respect to* f_0 if $\overline{\Sigma} \cap \overline{H_0}$ is empty. In other words, Σ is a bounded face at infinity if and only if it is a bounded face of the affine arrangement $tr(A)_{H_0}$.

If Σ is a face of A, denote F_Σ the unique edge of the smallest dimension which contains Σ. Define the discrete length, the discrete width and the volume of the face as the same quantities for the corresponding edge.

Theorem 3.1. *The trace map from the unbounded faces of A to the faces of $\overline{A}$ in H_∞ has the following properties:*

(i) The trace of a growing face is a bounded face at infinity.

(ii) For any bounded face at infinity, Σ, there exist exactly $s(\Sigma)$ growing domains with trace Σ, where $s(\Sigma)$ denotes the discrete width of this face.

(iii) The number of growing domains of A is equal to the sum of the volumes of all edges of $\overline{A}$ at infinity which do not lie in $\overline{H_0}$:

$$\#\,\text{growing domains} = \sum_{F\subseteq H_\infty,\, F\not\subseteq \overline{H_0}} vol(F).$$

Theorem 3.1 is proved in Section 7.1

4. Critical values

The aim of this section is to define the critical values of the functions $f_1^{\alpha_1}, \dots , f_p^{\alpha_p}$ on the bounded domains of an arrangement A and the critical values of the same functions, with respect to an additional linear function f_0, on the bounded and growing domains of A.

Let an arrangement A be given by linear functions $\{f_i\}_{i\in I}$ and let $\alpha = \{\alpha_i\}_{i\in I}$ be a corresponding set of weights.

For every $i \in I$, a face of the arrangement A on which f_i is constant is called a *critical face* with respect to f_i and the value of f_i on that face is called a *critical value*. In particular, each vertex is a critical face for every function f_i.

Assume that a function $|f_i|$ is bounded on a face Σ of A. The subset of $\overline{\Sigma}$ on which $|f_i|$ attains its maximum is a union of critical faces. Among them, there is a unique one of highest dimension. It is called the *external support* of the face Σ with respect to f_i.

Denote $\mathsf{Ch}(A)$ the set of all bounded domains of A. Let $\beta(A) = |\mathsf{Ch}(A)|$. Enumerate the bounded domains by numbers $1, \dots , \beta(A)$. For every $i \in I$ and $j \in \{1, \dots , \beta(A)\}$, choose a branch of the multi-valued function $f_i^{\alpha_i}$ on the domain Δ_j and denote it $g_{i,j}$. Let $\Sigma_{i,j}$ be the external support of Δ_j with respect to f_i. Define the *extremal critical value of the chosen branch* $g_{i,j}$ *on* Δ_j as the number $c(g_{i,j}, \Delta_j) = g_{i,j}(\Sigma_{i,j})$. Denote $c(A; \alpha)$ the product of all extremal critical values of the chosen branches,

$$c(A;\alpha) = \prod_{j=1}^{\beta(A)} \prod_{i \in I} c(g_{i,j}, \Delta_j).$$

Assume that an additional non-constant linear function f_0 on V is given. Let Δ be bounded or growing domain of A. Then f_0 is bounded from below on Δ. The subset of $\overline{\Delta}$ on which e^{-f_0} attains its maximum coincides with the subset of $\overline{\Delta}$ where f_0 attains its minimum. This subset has a unique face of highest dimension; it is called the *support face* of f_0 on Δ and denoted Σ_Δ. Define the *extremal critical value of* e^{-f_0} *on* Δ as the number $c(e^{-f_0}, \Delta) = e^{-f_0(\Sigma_\Delta)}$.

Denote $\mathsf{Ch}(A; f_0)$ the set of all bounded or growing domains of A. Let $\gamma(A) = |\mathsf{Ch}(A; f_0)|$. Enumerate these domains by numbers $1, \ldots, \gamma(A)$. For every $i \in I$ and $j \in \{1, \ldots, \gamma(A)\}$, choose a branch of the multi-valued function $f_i^{\alpha_i}$ on the domain Δ_j and denote it $g_{i,j}$. Assume that $|f_i|$ is bounded on Δ_j. Let $\Sigma_{i,j}$ be the external support of Δ_j with respect to f_i. Define the *extremal critical value of the chosen branch* $g_{i,j}$ *on* Δ_j *with respect to* f_0 as the number $c(g_{i,j}, \Delta_j, f_0) = g_{i,j}(\Sigma_{i,j})$. Notice that, if Δ_j is a bounded domain of A, then $c(g_{i,j}, \Delta_j) = c(g_{i,j}, \Delta_j, f_0)$.

Now assume that $|f_i|$ is unbounded on Δ_j. Thus, Δ_j is a growing domain of A and $tr(\Delta_j)$ is a bounded face of $tr(A)_{H_0}$. Denote $M = f_0(\Sigma_{\Delta_j})$. Consider the rational function $\widetilde{h_i} = f_i/(f_0 - M)$ on Δ_j. Notice that $\widetilde{h_i}|_{tr(\Delta_j)}$ coincides with the restriction of the linear function $h_i = f_i^0/f_0^0$ to the same set $tr(\Delta_j)$. Since the sign of $\widetilde{h_i}$ on Δ_j is the same as the sign of f_i on Δ_j we can choose a branch of $\widetilde{h_i}^{\alpha_i}$ on Δ_j which has the same argument as $g_{i,j}$ and denote it $\widetilde{g_{i,j}}$. Let Σ_j be the external support of $tr(\Delta_j)$ with respect to h_i in the affine arrangement $tr(A)_{H_0}$. Define the *the extremal critical value of the chosen branch* $g_{i,j}$ *on* Δ_j *with respect to* f_0 as the number $c(g_{i,j}, \Delta, f_0) = \widetilde{g_{i,j}}(\Sigma_j)$.

Denote $c(A; \alpha; f_0)$ the product of all extremal critical values with respect to f_0 of the chosen branches,

$$c(A;\alpha;f_0) = \prod_{j=1}^{\gamma(A)} \left(e^{-f_0(\Sigma_{\Delta_j})} \prod_{i \in I} c(g_{i,j}, \Delta_j, f_0) \right).$$

5. Hypergeometric period matrix

5.1. **βnbc-bases.** Let A be an essential arrangement in an n-dimensional real affine space V. Define a linear order $<$ in A putting $H_i < H_j$ if $i < j$. A subset $\{H_i\}_{i\in J}$ of A is called *dependent* if $\cap_{i\in J} H_i \neq \emptyset$ and $\operatorname{codim}(\cap_{i\in J} H_i) < |J|$. A subset of A which has nonempty intersection and is not dependent is called *independent.* Maximal independent sets are called *bases.* An intersection of a basis defines a vertex.

A k-tuple $S = (H_1, \cdots, H_k)$ is called a *circuit* if $(H_1, \cdots, H_k)$ is dependent and if for each l, $1 \le l \le k$, the $(k-1)$-tuple $(H_1, \cdots, \widehat{H_l}, \cdots, H_k)$ is independent. A k-tuple S is called a *broken circuit* if there exists $H < min(S)$ such that $\{H\} \cup S$ is a circuit, where $min(S)$ denotes the minimal element of S for $<$.

The collection of subsets of A having nonempty intersection and containing no broken circuits is denoted **BC**. **BC** consists of independent sets. Maximal (with respect to inclusion) elements of **BC** are bases of A called **nbc**-bases. Recall that n is the dimension of the affine space.

An **nbc**-basis $B = (H_{i_1}, \cdots, H_{i_n})$ is called *ordered* if $H_{i_1} < H_{i_2} < \cdots < H_{i_n}$. The set of all ordered **nbc**-bases of A is denoted **nbc**(A)

A basis B is called a β**nbc**–*basis* if B is an **nbc**-basis and if

$$\forall H \in B\, \exists H' < H \text{ such that } (B - \{H\}) \cup \{H'\} \text{ is a base.} \tag{4}$$

Denote β**nbc**(A) the set of all ordered β**nbc**-bases. Put the lexicographic order on β**nbc**(A)

The definition and basic properties of the β**nbc**-bases are due to Ziegler [Z].

For a basis $B = (H_{i_1}, \cdots, H_{i_n})$, let $F_j = \bigcap_{k=j+1}^{n} H_{i_k}$ for $0 \le j \le n-1$ and $F_n = V$. Then $\xi(B) = (F_0 \subset F_1 \subset \cdots \subset F_n)$ is a flag of affine subspaces of V with $\dim F_j = j$ $(0 \le j \le n)$. This flag is called the *flag associated with* B.

For an edge F of A, remember that $I(F) = \{i \in I \mid F \subseteq H_i\}$. Introduce a differential one-form

$$\omega_\alpha(F, A) = \sum_{i\in I(F)} \alpha_i \frac{df_i}{f_i}.$$

For a basis $B = (H_{i_1}, \cdots, H_{i_n})$, let $\xi(B) = (F_0 \subset F_1 \subset \cdots \subset F_n)$ be the associated flag. Introduce a differential n–form $\Xi(B, A) = \omega_\alpha(F_0, A) \wedge \cdots \wedge \omega_\alpha(F_{n-1}, A)$.

If $\beta\mathbf{nbc}(A) = \{B_1, \cdots, B_{\beta(A)}\}$ and $\phi_j = \phi_j(A) = \Xi(B_j, A)$ for $j \in \{1, \dots, \beta(A)\}$, define

$$\Phi(A) = \{\phi_1, \cdots, \phi_{\beta(A)}\}. \tag{5}$$

EXAMPLE. For an arrangement A of p hyperplanes in general position, the set $\beta\mathbf{nbc}(A)$ coincides with the set $\{(H_{i_1}, \dots, H_{i_n}) \mid 2 \le i_1 < \cdots < i_n \le p\}$. The latter corresponds to all vertices of A away from the hyperplane H_1. The differential n–forms are

$$\Phi(A) = \{\alpha_{i_1} \dots \alpha_{i_n} \frac{df_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{f_{i_n}}{f_{i_n}} \mid 2 \le i_1 < \cdots < i_n \le p\}.$$

5.2. **The definition of the hypergeometric period matrix.** Let $\xi = (F_0 \subset F_1 \subset \cdots \subset F_n)$ be a flag of edges of A with dim $F_i = i$ $(i = 0, \dots, n-1;\ F_n = V)$. Let Δ be a domain of A and $\overline{\Delta}$ its closure in V. We say that the flag is *adjacent to* the domain if $\dim(F_i \cap \overline{\Delta}) = i$ for $i = 0, \dots, n$.

The following proposition from [DT, Proposition 3.1.2] allows us to enumerate the bounded domains of a configuration A by means of $\beta\mathbf{nbc}(A)$.

Proposition 5.1. *There exists a unique bijection*

$$C : \beta\mathbf{nbc}(A) \longrightarrow \mathsf{Ch}(A)$$

such that for any $B \in \beta\mathbf{nbc}(A)$, the associated flag $\xi(B)$ is adjacent to the bounded domain $C(B)$.

Let $t > 0$ be a number which is larger than the maximum of f_0 on the closure of any bounded domain of A. Then the hyperplane $H_t = \{f_0 = t\}$ does not intersect the bounded domains of A. Consider the affine arrangement $A_t = A \cup \{H_t\}$. The set of its bounded domains consists of two disjoint subsets: the first is the subset of all bounded domains of A; the second is formed by the domains of A_t which are intersections of unbounded domains of A and the half–space $\{f_0 < t\}$. Notice that the intersection of an unbounded domain Δ of A and the half–space $\{f_0 < t\}$ is nonempty and bounded if and only if Δ is a growing domain. Thus $\beta(A_t) = \gamma(A)$.

Define an order $<$ on A_t as $H_t < H_1 < \dots < H_p$. Consider the set $\beta\mathbf{nbc}(A_t)$ with respect to this order. If $B \in \beta\mathbf{nbc}(A_t)$, then $H_t \notin B$ because of condition (4) and the minimality of H_t with respect to the order $<$. This observation implies that $\beta\mathbf{nbc}(A_t)$ and $\Phi(A_t)$ do not depend on t. Denote them $\beta\mathbf{nbc}(A; f_0)$ and $\Phi(A; f_0)$ respectively. Notice that $\beta\mathbf{nbc}(A)$ and $\Phi(A)$ are subsets of $\beta\mathbf{nbc}(A; f_0)$ and $\Phi(A; f_0)$ respectively because the order on A is a restriction of the order on A_t to its subset A and because of condition (4). We also have an analog of Proposition 5.1.

Proposition 5.2. *There exists a unique bijection*

$$\overline{C} : \beta\mathbf{nbc}(A; f_0) \longrightarrow \mathsf{Ch}(A; f_0)$$

such that for any $B \in \beta\mathbf{nbc}(A; f_0)$, the associated flag $\xi(B)$ is adjacent to the domain $\overline{C}(B)$. Moreover, $\overline{C}|_{\beta\mathbf{nbc}(A)} = C$.

Let the set $\beta\mathbf{nbc}(A; f_0) = \{B_1, \dots, B_\gamma\}$ be lexicographically ordered as in section 5.1. For $i = 1, \dots, \gamma$, define a domain $\Delta_i \in \mathsf{Ch}(A; f_0)$ by $\Delta_i = C(B_i)$. This gives us an order on the set of the growing and bounded domains of A. The order is called the $\beta\mathbf{nbc}$*–order.*

We give an orientation to each domain $\Delta \in \mathsf{Ch}(A; f_0)$ as follows. Let $\Delta = C(B)$ with $B \in \beta\mathbf{nbc}(A; f_0)$. Let $\xi(B) = (F_0 \subset F_1 \subset \dots \subset F_n)$ be the associated flag. The flag $\xi(B)$ is adjacent to the domain B and defines its *intrinsic orientation* [V2, 6.2]. The intrinsic orientation is defined by the unique orthonormal frame $\{e_1, \dots, e_n\}$ such that each e_i is a unit vector originating from the point F_0 in the direction of $F_i \cap \overline{\Delta}$.

Let $\beta = \beta(A)$. Assume that $\mathsf{Ch}(A) = \{\Delta_1, \dots, \Delta_\beta\}$ is the $\beta\mathbf{nbc}$-ordered set of the bounded domains of A and $\Phi(A) = \{\phi_1, \dots, \phi_\beta\}$ is the $\beta\mathbf{nbc}$-ordered set of differential n–forms constructed in Section 5.1. Assume that the weights $\{\alpha_i\}_{i\in I}$ have positive real parts. For every $i \in I$ and $j \in \{1, \dots, \beta\}$, choose a branch of $f_i^{\alpha_i}$ on the domain Δ_j and the intrinsic orientation of the domain Δ_j. Let $U_\alpha := f_1^{\alpha_1} \cdots f_p^{\alpha_p}$. The choice of branches of the functions $f_i^{\alpha_i}$ on all bounded domains defines a choice of branches of the function U_α on all bounded domains. Define the *hypergeometric period matrix* by

$$\mathsf{PM}(A; \alpha) = \left[\int_{\Delta_j} U_\alpha \phi_k\right]_{k,j=1}^{\beta}. \tag{6}$$

Since $\operatorname{Re}\alpha_i > 0$, all elements of the period matrix are well defined.

Let $\gamma = \gamma(A)$. Let $\mathsf{Ch}(A; f_0) = \{\Delta_1, \dots, \Delta_\gamma\}$ be the $\beta\mathbf{nbc}$-ordered set of the bounded and growing domains of A and let $\Phi(A; f_0) = \{\phi_1, \dots, \phi_\gamma\}$ be the $\beta\mathbf{nbc}$-ordered set of differential n–forms constructed in Section 5.2. Assume that the weights $\{\alpha_i\}_{i\in I}$ have positive real parts. For every $i \in I$ and $j \in \{1, \dots, \gamma\}$, choose a branch of $f_i^{\alpha_i}$ on the domain Δ_j and the intrinsic orientation of each domain Δ_j. The choice of branches of the functions $f_i^{\alpha_i}$ on all bounded and growing domains defines a choice of branches of the function U_α on all bounded and growing domains. Define the *hypergeometric period matrix with respect to* f_0

by

$$\mathsf{PM}(A;\alpha;f_0) = \left[\int_{\Delta_j} e^{-f_0} U_\alpha \phi_k\right]^{\gamma}_{k,j=1}. \tag{7}$$

Since $\operatorname{Re}\alpha_i > 0$ and f_0 tends to $+\infty$ on the growing domains of A, all elements of the period matrix are well defined.

6. The main theorem

In [DT], Douai and Terao proved the following theorem, cf. also [V1,V2].

Theorem 6.1. *Let A be a weighted arrangement given by functions $\{f_i\}_{i\in I}$ and weights $\alpha = \{\alpha_i\}_{i\in I}$ such that $\operatorname{Re}\alpha_i > 0$ for all $i \in I$. Fix branches of the multivalued functions $\{f_i^{\alpha_i}\}_{i\in I}$ on all bounded domains of A. Then*

$$\det \mathsf{PM}(A;\alpha) = c(A;\alpha)B(A;\alpha). \tag{8}$$

The main result of this paper is the following theorem.

Theorem 6.2. *Let A be a weighted arrangement given by functions $\{f_i\}_{i\in I}$ and weights $\alpha = \{\alpha_i\}_{i\in I}$ such that $\operatorname{Re}\alpha_i > 0$ for all $i \in I$. Let an additional non-constant linear function f_0 be given. Denote H_0 the hyperplane $\{f_0 = 0\}$. Fix branches of the multivalued functions $\{f_i^{\alpha_i}\}_{i\in I}$ on all bounded and growing domains of A. Then*

$$\det \mathsf{PM}(A;\alpha;f_0) = c(A;\alpha;f_0)B(A;\alpha;H_0). \tag{9}$$

We will deduce this formula for the determinant of the period matrix with respect to f_0 from Theorem 6.1 by passing to a limit.

7. Proofs

7.1. Proof of Theorem 3.1.

Lemma 7.1. *Let Δ be a growing face. Then $\overline{tr(\Delta)} \cap \overline{H_0} = \emptyset$, i.e. $tr(\Delta)$ is a bounded face at infinity.*

Proof. Let $x_0, \dots, x_{n-1}$ be affine coordinates on V such that $f_0(x) = x_0$. Let $(t_0 : t_1 : \cdots : t_n)$ be the corresponding projective coordinates in $\overline{V}$: $\{x_i = t_i/t_n\}_{i=0}^{n-1}$. Let Δ be a growing face. Assume that $\overline{tr(\Delta)} \cap \overline{H_0} \neq \emptyset$ and $P = (p_0 : p_1 : \cdots : p_n)$ is a point of this intersection. Thus, $p_0 = p_n = 0$. Let $Q = (q_0 : q_1 : \cdots : q_n)$ be any point inside Δ. Thus $q_n \neq 0$. Since $\overline{\Delta}$ is a closed polyhedron in $\overline{V}$ it contains the segment PQ. This segment is parametrized by the points $P_\lambda = (\lambda p_0 + (1-\lambda)q_0 : \lambda p_1 + (1-\lambda)q_1 : \cdots : \lambda p_n + (1-\lambda)q_n)$, for $\lambda \in [0,1]$. The point P_λ tends to $P \in H_\infty$ when $\lambda \mapsto 1$. We have

$$f_0(P_\lambda) = \frac{\lambda p_0 + (1-\lambda)q_0}{\lambda p_n + (1-\lambda)q_n} = \frac{q_0}{q_n} = \text{constant}.$$

This contradicts to the assumption that Δ is a growing face. So $\overline{tr(\Delta)} \cap \overline{H_0} = \emptyset$. Part (i) of Theorem 3.1 is proved. □

Lemma 7.2. *Let Δ be an unbounded domain of A. Let $tr(\Delta)$ be a bounded face at infinity with respect to f_0. Let f_0 be unbounded on $\Delta \cap \{f_0 > 0\}$. Then Δ is a growing domain of A.*

PROOF. Since $tr(\Delta)$ is bounded at infinity, we have $\overline{\Delta} \cap H_\infty \cap \overline{H_0} = \emptyset$. For a real t, let $H_t = \{f_0 = t\}$. Then

$$\overline{\Delta} \cap H_\infty \cap \overline{H_t} = \emptyset. \tag{10}$$

Let $\{x_i\}_{i=1}^\infty$ be a sequence of points in Δ such that x_i tends to ∞ when $i \mapsto \infty$. Choose a positive T. Assume that T is larger than the supremum of f_0 on all bounded domains of A. Consider the arrangement $A_T = A \cup \{H_T\}$. Formula (10) implies that $H_T \cap \Delta$ is a bounded domain of the section $(A_T)_{H_T}$. Since A_T is essential, Proposition 9.9 [BBR] is applicable. It implies that there is a bounded domain Δ_T of the arrangement A_T, such that $H_T \cap \Delta$ is a subset of the boundary of Δ_T. This bounded domain must be $\Delta \cap \{f_0 < T\}$, because of the choice of T.

Since Δ_T is bounded, there exists a positive integer N_T such that for every integer $n \geq N_T$ we have $x_n \in \Delta - \Delta_T$. Since $\Delta - \Delta_T = \Delta \cap \{f_0 \geq T\}$, we have $f_0(x_n) \geq T$ for all $n \geq N_T$. This proves that Δ is a growing domain. □

Lemma 7.3. *For any bounded face at infinity, Σ, there exist exactly $s(\Sigma)$ growing domains with trace Σ, where $s(\Sigma)$ denotes the discrete width of this face.*

PROOF. Let Σ be a bounded face at infinity of codimension k. Choose projective coordinates $(t_0 : t_1 : \cdots : t_n)$ on $\overline{V}$ such that $\overline{H_0} = \{t_0 = 0\}$, $H_\infty = \{t_n = 0\}$, and F_Σ is given by $t_1 = \cdots = t_{k-1} = t_n = 0$.

Let v be a point in Σ and B an open ball around v. If the ball is sufficiently small, then the domains of A which intersect B are precisely those for which v belongs to their closure in $\overline{V}$ and the hyperplanes of $\overline{A}$ which intersect B are exactly those belonging to $\overline{A}^{F_\Sigma}$. Local affine coordinates on B are given by $\{y_i = t_i/t_0\}_{i=1}^n$. Since F_Σ is given by the equations $y_1 = \cdots = y_{k-1} = y_n = 0$, the subspace L through v spanned by the coordinate vectors $e_1, \ldots, e_{k-1}, e_n$ is a normal subspace to F_Σ. Then the number of open domains in B is equal to the number of open domains of the arrangement induced in the tangent space $T_v L$ by the arrangement $\overline{A}^{F_\Sigma}$. On B the function f_0 has the form $f_0(y) = 1/y_n$. We are interested in the domains in B on which $f_0 \mapsto +\infty$ when $y_n \mapsto 0$. So, on this domains we must have $y_n > 0$. If the codimension of Σ in H_∞ is 0, then the number of such domains is equal to 1, which is exactly the discrete width of the empty configuration. Assume that the above codimension is positive. Then the number of domains in B on which $y_n > 0$ is equal to the number of the domains of the projective normal arrangement $P\overline{A}^{F_\Sigma}$. Finally, we want to count only those domains for which $\overline{\Sigma}$ is the only part of their closure in $\overline{V}$, lying in H_∞. Thus, they are the projective domains away from the hyperplane $y_n = 0$. Their number is equal to the discrete length of $P\overline{A}^{F_\Sigma}$. By definition this number is equal to the discrete width of Σ. Lemma 7.2 implies that the corresponding domains of A are growing. □

Lemma 7.4. *The number of growing domains of A is equal to the sum of the volumes of all edges of $\overline{A}$ at infinity which do not lie in $\overline{H_0}$.*

PROOF. Let F be an edge at infinity with non-zero volume which do not lie in $\overline{H_0}$. Then, by definition, there are exactly $l(F)$ bounded faces at infinity which generate F. For each of them, Σ, there exist exactly $s(F)$ growing domains of A with trace Σ. Thus there exist exactly $vol(F) = l(F)s(F)$ growing domains whose traces generate F. Finally, in order to count all growing domains of A, we have to sum over all edges at infinity which have non-zero volume and do not lie in $\overline{H_0}$. Theorem 3.1 is proved. □

7.2. **Asymptotic behavior of critical values.** Let A be an arrangement in the affine space V. Let f_0 be an additional non–constant linear function on V. Define $f_t = 1 - \frac{f_0}{t}$ and $H_t = \{f_t = 0\}$. Consider a new weighted arrangement $A_t = A \cup \{H_t\}$ where we assume that the weight of H_t is equal to t. For a sufficiently big t, the hyperplane H_t intersects only some of the unbounded domains of the arrangement A. Moreover, the intersection creates a new bounded domain if and only if the intersected domain is a growing one. So if Δ is a growing domain, we will denote the corresponding bounded domain of A_t by Δ_t and will call it a *growing bounded domain.* If Δ is a bounded domain of A, then it is also a bounded domain of A_t. This correspondence between the bounded domains of A_t and the bounded or growing domains of A is a bijection.

Lemma 7.5. *Let Δ_t be a bounded domain of the arrangement A_t. Let Δ be the corresponding bounded or growing domain of A. If $t > 0$ and $(1 - f_0/t)$ is positive on Δ_t choose the positive branch g_t of $(1 - f_0/t)^t$ on Δ_t.*

Then the external support of Δ_t with respect to f_t is a face of the arrangement A. For every big enough t this external support coincides with the support face, Σ_Δ, of f_0 on Δ. Moreover, $\lim_{t\mapsto+\infty} c(g_t, \Delta_t) = e^{-f_0(\Sigma_\Delta)}$.

PROOF. For a fixed t, the external support of Δ_t with respect to f_t lies outside H_t. Thus, it is a face of the arrangement A.

The set of all critical faces of A with respect to f_0 is finite. Let M be the maximum of f_0 on this set. Assume that $t > M$. Then all bounded domains of A_t lie inside the positive half–space with respect to f_t. Let $f_t = 1 - f_0/t$ attains its maximum on a critical face Σ_Δ of $\overline{\Delta_t}$. This is equivalent to the condition that f_0 attains its minimum on the same face. So Σ_Δ is the support face of Δ with respect to f_0. On the other side, it is the external support of Δ_t with respect to f_t. Hence

$$\lim_{t\mapsto+\infty} c(g_t, \Delta_t) = \lim_{t\mapsto+\infty} \left(1 - \frac{f_0(\Sigma_\Delta)}{t}\right)^t = e^{-f_0(\Sigma_\Delta)}.$$

□

Lemma 7.6. *Let the hyperplane $H = \{f = 0\}$ belongs to the arrangement A. Let Δ be a growing domain of A and Δ_t the corresponding growing bounded domain of A_t. Let $|f|$ be unbounded on the growing domain Δ.*

Then there exists a unique face Σ of highest dimension, belonging to the closure of Δ, such that for every big enough t, the external support of the face Δ_t with respect to f is $\Sigma_t = \Sigma \cap \Delta_t$. Moreover, $tr(\Sigma)$ is the external support of $tr(\Delta)$ with respect to h in the affine space $W = H_\infty - \overline{H_0} \cap H_\infty$, where $h = f^0/f_0^0$. The asymptotic behavior of $f(\Sigma_t)$ when t tends to $+\infty$ is given by $f(\Sigma_t) = h(tr(\Sigma))t(1+o(1))$.

PROOF. The set of critical faces with respect to f of the arrangement A is finite. $|f|$ is bounded on this set. Since $|f|$ is unbounded on Δ the external support of Δ_t with respect to f lies on H_t for t big enough.

Let Σ_{t_1} be a critical face of Δ_{t_1} which lies on H_{t_1} for some t_1 fixed. Then $\Sigma_{t_1} = H_{t_1} \cap \Sigma$, where Σ is a face of $\overline{\Delta}$. Consider the face $\Sigma_t = H_t \cap \Sigma$ of Δ_t for an arbitrary t. It is a critical face of Δ_t because Σ_t is parallel to Σ_{t_1} and the latter is parallel to H. Let us compute the asymptotic behavior of $f(\Sigma_t)$ when t tends to $+\infty$.

Choose affine coordinates $\{x_j\}_{j=0}^{n-1}$ in V such that $f_0(x) = x_0$. Let $f = f^0 + b$ be the sum of the homogeneous part of f and the constant term. Then $f =$

$x_0(f^0/x_0 + b/x_0) = x_0(f^0/f_0^0 + b/x_0)$. Since $H_t = \{x_0 = t\}$, $f(\Sigma_t) = t(h(\Sigma_t) + b/t) = h(tr(\Sigma))t(1 + o(1))$.

Let Σ' be the external support of $tr(\Delta)$ relative to $tr(H)$. Let Σ is the face of Δ for which $\Sigma' = tr(\Sigma)$. Then the previous computation shows that for every t big enough Σ_t is the external support of Δ_t with respect to f. □

Corollary 7.7. *Let the conditions be as in Lemma 7.6. In addition, assume that α is a complex number. Fix a branch of f^α on Δ and denote it g. Fix a branch of $(f/f_0)^\alpha$ on Δ as in Section 4 and denote it $\tilde{g}$. Fix branches of t^α and $(1 + o(1))^\alpha$ using the branch of the logarithm with zero argument. Then the asymptotic behavior of $c(g, \Delta_t)$ when t tends to $+\infty$ is $c(g, \Delta_t) = c(g, \Delta, f_0)t^\alpha(1 + o(1))$.*

PROOF. Use the notation of the previous proof. Since $c(g, \Delta_t) = g(\Sigma_t)$, $c(g, \Delta, f_0) = \tilde{g}(tr(\Sigma))$ and the arguments of g and $\tilde{g}$ are the same on Δ, the asymptotic formula for f implies the statement of the corollary. □

Lemma 7.8. *Let the hyperplane $H = \{f = 0\}$ belongs to the arrangement A. Let α be a complex number. Let Δ be a growing domain of A and Δ_t the corresponding growing bounded domain of A_t. Fix a branch of f^α on Δ and denote it g. Then*

$|f|$ is bounded on Δ if and only if $tr(\Delta) \subset tr(H)$. The latter condition is equivalent to the equation $h(tr(\Delta)) = 0$ where $h = f^0/f_0^0$ is a linear function of the arrangement $tr(A)_{H_0}$. Moreover, if $|f|$ is bounded on Δ, then for every big enough t, $c(g, \Delta_t)$ equals the constant $c(g, \Delta, f_0)$.

PROOF. $|f|$ is bounded on Δ if and only if Δ is placed between two hyperplanes H' and H'' parallel to H. Denote the domain between these two hyperplanes by D. Since $\Delta \subset D$ we have $tr(\Delta) \subset tr(D) = tr(H)$.

The reverse part is a consequence of Lemma 7.6.

Let Σ be the external support of Δ with respect to f. Then for every big enough t, $c(g, \Delta_t) = g(\Sigma)$. The latter equals $c(g, \Delta, f_0)$. □

7.3. **Proof of Theorem 6.2.** We prove Theorem 6.2 applying Theorem 6.1 to the arrangement A_t and then passing to the limit when $t \mapsto +\infty$.

First study $B(A_t; \alpha, t)$.

Lemma 7.9. (*i*) *The only factor in the numerator of $B(A_t; \alpha, t)$ depending on t, when t is big enough, is the factor corresponding to the edge H_t. It contributes $\Gamma(t + 1)^\#$, where $\#$ is the number of growing domains of A.*

(*ii*) *The factors in the denominator depending on t come from the edges at infinity with non-zero volume which do not lie in $\overline{H_0}$. Each of them, F, contributes $\Gamma(t + 1 + \alpha'(F))^{vol(F)}$, where $\alpha'(F) = \sum_{H \in A;\, F \not\subset \overline{H}} \alpha_H$.*

(iii) The asymptotic behavior of $B(A_t;\alpha,t)$ when t tends to $+\infty$ is given by

$$B(A_t;\alpha,t) = B(A;\alpha;H_0) \prod_{F\in L_-;\, F\not\subset \overline{H_0}} t^{-\alpha'(F)vol(F)}(1+o(1)).$$

PROOF. Recall that

$$B(A_t;\alpha,t) = \prod_{F\in L_{t+}} \Gamma(\alpha(F)+1)^{vol(F)} / \prod_{F\in L_{t-}} \Gamma(-\alpha(F)+1)^{vol(F)},$$

$$B(A;\alpha;H_0) = \prod_{F\in L_+} \Gamma(\alpha(F)+1)^{vol(F)} / \prod_{F\in L_-; F\subset H_0} \Gamma(-\alpha(F)+1)^{vol(F)},$$

where L_{t-}, L_- denote the set of all edges at infinity of the arrangements $\overline{A_t}$ and $\overline{A}$, respectively, and L_{t+}, L_- denote the set consisting of all the other edges of the same arrangements.

(i) Since the only weight depending on t corresponds to H_t, the factors in the denominator that depend on t correspond to the edges of A_t lying in H_t. If such an edge F is a proper subspace of H_t, then it is decomposable [STV, Section 2], that is the localization of the arrangement A_t at the edge F is a product of two nonempty subarrangements where one of the subarrangements is equal to $\{H_t\}$. According to [STV, Proposition 7], the discrete width of a decomposable edge is zero. Thus its discrete volume is zero.

The volume of H_t is the number of bounded domains of the section arrangement $(A_t)_{H_t}$ which is exactly the number of growing domains of the arrangement A.

(ii) If F is an edge at infinity of A_t, then $\alpha(F) = -t - \sum_{i\in I}\alpha_p + \sum_{H\in A_t;\, F\subset\overline{H}}\alpha_H$. The last sum depends on t if and only if $\sum_{H\in A_t;\, F\subset \overline{H}}\alpha_H$ does not depend on t, i.e. if and only if $F\not\subset \overline{H_t}$. Since $\overline{H_t}\cap H_\infty = \overline{H_0}\cap H_\infty$, the weight $\alpha(F)$ depends on t if and only if $F\not\subset\overline{H_0}$. So $\alpha(F) = -t-\alpha'(F)$. Notice that such an edge is also an edge of the arrangement A.

(iii) According to Theorem 3.1 the number of growing domains of the arrangement A is equal to the sum of the volumes of all edges at infinity of the arrangement A which do not lie in $\overline{H_0}$. Thus the number of factors in the numerator and in the denominator containing t is equal. Sterling's formula gives us $\Gamma(t+1)/\Gamma(t+1+a) = t^{-a}(1+o(1))$ when t tends to $+\infty$. So we obtain the required formula.

Now consider the limit of the product of the critical values, $c(A_t;\alpha,t)$. For every $i\in I$ and every bounded or growing domain Δ of the arrangement A, choose a branch of $f_i^{\alpha_i}$ on Δ and denote it $g_{i,\Delta}$. This also fixes branches of $f_i^{\alpha_i}$

on the bounded domains of A_t independently on t. Notice that for every big enough t, f_t is positive on all bounded domains of the arrangement A_t. Choose the positive branch of $(f_t)^t$ on this domains and denote it g_t. □

Lemma 7.10. *$c(A_t;\alpha,t)$ has the following asymptotic behavior when t tends to $+\infty$:*

$$c(A_t;\alpha,t) = c(A;\alpha;f_0) \prod_{F\in L_-;F\not\subset H_0} t^{\alpha'(F)vol(F)}(1+o(1)).$$

PROOF.

$$\begin{aligned} C(A_t;\alpha;t) &= \prod_{\Delta\in Ch(A_t)} \left(c(g_t,\Delta)\prod_{i\in I} c(g_{i,\Delta},\Delta)\right) = \left(\prod_{\Delta\in Ch(A_t)} c(g_t,\Delta)\right) \\ &\times \left(\prod_{\Delta\in Ch(A)}\prod_{i\in I} c(g_{i,\Delta},\Delta)\right)\left(\prod_{\Delta_t}\prod_{i\in I} c(g_{i,\Delta_t},\Delta_t)\right), \end{aligned} \tag{11}$$

where Δ_t in the last product ranges over the growing bounded domains of A_t.

We are going to describe the asymptotic behavior of each of the three products in formula (11).

Assume that Δ is a bounded domain of A_t. Lemma 7.5 asserts that $\lim_{t\to+\infty} c(g_t,\Delta) = e^{-f_0(\Sigma_{\Delta'})}$, where Δ' is the domain of A (bounded or growing) which corresponds to the domain Δ of A_t and $\Sigma_{\Delta'}$ is the support face of f_0 on Δ'.

If Δ is a bounded domain of A and $i\in I$, then $c(g_{i,\Delta},\Delta) = c(g_{i,\Delta},\Delta,f_0)$ by definition.

Let Δ_t be a growing bounded domain of A_t and Δ the corresponding growing domain of A. Let $i\in I$. If $tr(\Delta)\not\subset \overline{H_i}$, then $c(g_{i,\Delta_t},\Delta_t) = c(g_{i,\Delta},\Delta,f_0)t^{\alpha_i}(1+o(1))$, by Corollary 7.7. If $tr(\Delta)\subset\overline{H_i}$, then $c(g_{i,\Delta_t},\Delta_t) = c(g_{i,\Delta},\Delta,f_0)$ by Lemma 7.8. Let F be an edge at infinity. Assume that F does not lie in $\overline{H_0}$ and has a non-zero volume. According to Theorem 3.1, there exist exactly $vol(F)$ growing domains of the arrangement A, whose traces generate F. Every term in the last product of formula (11) depends on a growing bounded domain. Collect all the terms such that the trace of the corresponding growing domain generates F.

Then for the product of the chosen factors we have $\prod_{\Delta,\, F_{tr(\Delta)}=F}\prod_{i\in I} c(g_{i,\Delta_t},\Delta_t)$

$$
\begin{aligned}
&= \left(\prod_{\Delta,\, F_{tr(\Delta)}=F}\prod_{i\in I} c(g_{i,\Delta},\Delta,f_0)\right)\left(\prod_{\Delta,\, F_{tr(\Delta)}=F}\prod_{i,\, F\not\subset H_i} t^{\alpha_i}(1+o(1))\right)\\
&= \left(\prod_{\Delta,\, F_{tr(\Delta)}=F}\prod_{i\in I} c(g_{i,\Delta},\Delta,f_0)\right)\left(\prod_{\Delta,\, F_{tr(\Delta)}=F} t^{\alpha'(F)}(1+o(1))\right)\\
&= t^{\alpha'(F)vol(F)}(1+o(1))\prod_{\Delta,\, F_{tr(\Delta)}=F}\prod_{i\in I} c(g_{i,\Delta},\Delta,f_0)
\end{aligned}
$$

Collecting the asymptotic behavior for the three products in (11), we obtain the statement of the lemma.

PROOF OF THEOREM 6.2: Apply formula (8) to the weighted arrangement A_t. Lemmas 7.9 and 7.10 show that the terms dependent on t in the asymptotic formulas for $B(A_t;\alpha,t)$ and $c(A_t;\alpha,t)$ cancel out. Thus,

$$\lim_{t\mapsto+\infty} c(A_t;\alpha,t)B(A_t;\alpha,t) = c(A;\alpha;f_0)B(A;\alpha;H_0),$$

which gives us the right hand side of formula (9).

Let us study the entries of the period matrix $\mathsf{PM}(A_t;\alpha,t)$: $\mathsf{PM}_{k,j}(t) = \int_{\Delta_j} U_{\alpha,t}\phi_k(A_t)$, where Δ_j is a bounded domain of A_t and $\phi_k(A_t)$ is one of the n–forms constructed in Section 5.1. Remind that for a fixed k and a big enough t, the form $\phi_k(A_t)$ is independent on t and equals $\phi_k(A;f_0)$. Since $U_{\alpha,t} = (1-f_0/t)^t U_\alpha$, we have $\lim_{t\mapsto+\infty} U_{\alpha,t} = e^{-f_0}U_\alpha$.

Since Δ_j is a bounded domain of A_t, there exists a unique bounded or growing domain, Δ, of A such that $\Delta_j = \Delta\cap\{f_0 < t\}$. Extend f_t as zero on $\Delta-\Delta_j$. Then $\mathsf{PM}_{k,j}(t) = \int_\Delta (f_t)^t U_\alpha\,\phi_k(A;f_0)$. Since $(f_t)^t < e^{-f_0}$ on $\Delta\cap\{f_0>0\}$, Lebesgue's convergence theorem is applicable and $\lim_{t\mapsto+\infty}\mathsf{PM}_{k,j}(t) = \int_\Delta e^{-f_0}U_\alpha\,\phi_k(A;f_0) = \mathsf{PM}_{k,j}(A;\alpha;f_0)$. These limits give us $\lim_{t\mapsto+\infty}\mathsf{PM}(A_t;\alpha,t) = \mathsf{PM}(A;\alpha;f_0)$. Theorem 6.2 is proved.

8. DETERMINANT FORMULAS FOR SELBERG TYPE INTEGRALS

Let $z_1 < \cdots < z_p$ be real numbers. Let $\alpha_1,\ldots,\alpha_p,\gamma$ be complex numbers with positive real parts. For $t\in\mathbb{R}^n$ define

$$\Phi(t,z) = \prod_{s=1}^{p}\prod_{i=1}^{n}(t_i-z_s)^{\alpha_s}\prod_{1\le i<j\le n}(t_j-t_i)^{2\gamma}.$$

The branches of x^{α_s} and $x^{2\gamma}$ are fixed by $-\pi/2 < \arg x < 3\pi/2$ for all $s \in \{1,\dots,p\}$.

Let $\mathcal{Z}_n^p = \{\mathbf{l} = (l_1,\dots,l_p) \in \mathbb{Z}^p \,|\, l_i \geq 0,\ l_1+\dots+l_p = n\}$. For every $s \in \{1,\dots,p\}$ denote $\mathbf{l}^s = \sum_{i=1}^s l_i$, $\mathbf{l}^0 = 0$. Let $\mathbf{m} \in \mathcal{Z}_n^p$ and $s \in \{1,\dots,p\}$. Denote $\Gamma_{\mathbf{m},s}$ the set of integers $\{\mathbf{m}^{s-1}+1,\dots,\mathbf{m}^s\}$ and $d^nt = dt_1 \wedge \dots \wedge dt_n$. Define the following n–forms

$$\omega_{\mathbf{m}}(t,z) = \left(\sum_{\sigma\in\mathbb{S}^n}\prod_{s=1}^p \frac{1}{m_s!}\prod_{j\in\Gamma_{\mathbf{m},s}}\frac{1}{(t_{\sigma_j}-z_s)}\right) d^nt.$$

If $\mathbf{m} \in \mathcal{Z}_n^{p-1}$, then we identify $\mathbf{m}$ with the p–tuple $(\mathbf{m},0) \in \mathcal{Z}_n^p$.

For $\mathbf{l} \in \mathcal{Z}_n^{p-1}$, let

$$\mathbb{U}_{\mathbf{l}} = \{t = (t_1,\dots,t_n) \in \mathbb{R}^n \,|\, z_s \leq t_{\mathbf{l}^{s-1}+1} \leq \dots \leq t_{\mathbf{l}^s} \leq z_{s+1} \text{ for } s = 1,\dots,p-1\}.$$

Assume that all domains in the formulas below inherit the standard orientation from $\mathbb{R}^n$. □

Theorem 8.1. , cf [V6].

$$\det\left[\int_{\mathbb{U}_{\mathbf{l}}}\Phi(t,z)\omega_{\mathbf{m}}(t,z)\right]_{\mathbf{l},\mathbf{m}\in\mathcal{Z}_n^{p-1}} = \tag{12}$$

$$\prod_{s=0}^{n-1}\left[\frac{\Gamma((s+1)\gamma)^{p-1}}{\Gamma(\gamma)^{p-1}}\frac{\Gamma(1+\alpha_p+s\gamma)\prod_{j=1}^{p-1}\Gamma(\alpha_j+s\gamma)}{\Gamma(1+\sum_{j=1}^p\alpha_j+(2n-2-s)\gamma)}\right]^{\binom{p+n-s-3}{p-2}}$$
$$\exp\left(i\pi\binom{p+n-2}{p-1}\sum_{s=1}^p(s-1)\alpha_s\right)\prod_{1\leq a<b\leq p}(z_b-z_a)^{(\alpha_a+\alpha_b)\binom{p+n-2}{p-1}+2\gamma\binom{p+n-2}{p}}.$$

Notice, that formula (12) is not symmetric with respect to $\alpha_1,\dots,\alpha_p$. To make it symmetric we introduce new differential n–forms, $\widetilde{\omega_{\mathbf{m}}}$, for $\mathbf{m} \in \mathcal{Z}_n^{p-1}$. Namely

$$\begin{aligned}\widetilde{\omega_{\mathbf{m}}}(t,z) &= [\prod_{s=1}^{p-1}(m_s!)\alpha_s(\alpha_s+\gamma)\dots(\alpha_s+(m_s-1)\gamma)]\omega_{\mathbf{m}}\\ &= \sum_{\sigma\in\mathbb{S}^n}\prod_{s=1}^{p-1}\alpha_s(\alpha_s+\gamma)\dots(\alpha_s+(m_s-1)\gamma)\prod_{j\in\Gamma_{m,s}}\frac{1}{(t_{\sigma_j}-z_s)}d^nt.\end{aligned}$$

Theorem 8.1 implies

$$\det\left[\int_{\mathbb{U}_{\mathbf{l}}}\Phi(t,z)\widetilde{\omega_{\mathbf{m}}}(t,z)\right]_{\mathbf{l},\mathbf{m}\in\mathcal{Z}_n^{p-1}} = \tag{13}$$

$$\prod_{s=0}^{n-1}\left[\frac{\Gamma((s+1)\gamma+1)^{p-1}}{\Gamma(\gamma+1)^{p-1}}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+s\gamma+1)}{\Gamma(1+\sum_{j=1}^{p}\alpha_j+(2n-2-s)\gamma)}\right]^{\binom{p+n-s-3}{p-2}}$$
$$\exp\Big(i\pi\binom{p+n-2}{p-1}\sum_{s=1}^{p}(s-1)\alpha_s\Big)\prod_{1\le a<b\le p}(z_b-z_a)^{(\alpha_a+\alpha_b)\binom{p+n-2}{p-1}+2\gamma\binom{p+n-2}{p}}.$$

Lemma 8.2. *For every* $\mathbf{l}\in\mathcal{Z}_n^{p-1}$, $i,j\in\{1,\dots,n\}$, $s\in\{1,\dots .p\}$, *fix branches* $g_{j,s}$, $h_{j,i}$ *of the multivalued functions* $(t_j-z_s)^{\alpha_s}$ *and* $(t_j-t_i)^{2\gamma}$ *,respectively, on the domain* $\mathbb{U}_\mathbf{l}$ *as at the beginning of the current section. Then the product*

$$\exp\Big(i\pi\binom{p+n-2}{p-1}\sum_{s=1}^{p}(s-1)\alpha_s\Big)\prod_{1\le a<b\le p}(z_b-z_a)^{(\alpha_a+\alpha_b)\binom{p+n-2}{p-1}+2\gamma\binom{p+n-2}{p}}$$

equals the product of critical values of the chosen branches

$$\prod_{\mathbf{l}\in\mathcal{Z}_n^{p-1}}\left[\prod_{j=1}^{n}\prod_{s=1}^{p}c(g_{j,s},\mathbb{U}_\mathbf{l})\prod_{1\le i<j\le n}c(h_{j,i},\mathbb{U}_\mathbf{l})\right]. \tag{14}$$

The critical values were defined in Section 4.

Lemma 8.2 allow us to replace the last lines in formulas (12) and (13) by the product of critical values (14), cf. [V6].

For $\mathbf{l}\in\mathcal{Z}_n^p$, let $z_0=-\infty$ and

$\widetilde{\mathbb{U}_\mathbf{l}}=\{t=(t_1,\dots,t_n)\in\mathbb{R}^n\mid z_{s-1}\le t_{\mathbf{l}^{s-1}+1}\le\dots\le t_{\mathbf{l}^s}\le z_s$ for all $s=1,\dots,p\}$.

Theorem 8.3. *Let a be a complex number with positive real part. Then*

$$\det\left[\int_{\widetilde{\mathbb{U}_\mathbf{l}}}\exp\Big(a\sum_{j=1}^{n}t_j\Big)\Phi(t,z)\omega_\mathbf{m}(t,z)\right]_{\mathbf{l},\mathbf{m}\in\mathcal{Z}_n^p}=(-1)^{n\binom{p+n-1}{p-1}} \tag{15}$$
$$\prod_{s=0}^{n-1}\left[\frac{\Gamma((s+1)\gamma)^p}{\Gamma(\gamma)^p}\prod_{j=1}^{p}\Gamma(\alpha_j+s\gamma)\right]^{\binom{p+n-s-2}{p-1}}$$
$$\prod_{1\le a<b\le p}(z_b-z_a)^{(\alpha_a+\alpha_b)\binom{p+n-1}{p}+2\gamma\binom{p+n-1}{p+1}}\exp\Big(i\pi\Big[\binom{p+n-1}{p}\sum_{s=1}^{p}s\alpha_s\Big]\Big)$$
$$\exp\Big(a\pi\binom{p+n-1}{p}\sum_{s=1}^{p}z_s\Big)\,a^{-\binom{p+n-1}{p}\sum_{s=1}^{p}s\alpha_s-2p\binom{p+n-1}{p+1}\gamma}$$

The next lemma allow us to replace the last line in formula (15) by the product of critical values (16).

Lemma 8.4. *For every* $\mathbf{l} \in \mathcal{Z}_n^p$, $i, j \in \{1, \dots, n\}$, $s \in \{1, \dots .p\}$, *fix branches* $g_{j,s}$, $h_{j,i}$ *of the multivalued functions* $(t_j - z_s)^{\alpha_s}$ *and* $(t_j - t_i)^{2\gamma}$ *respectively, on the domain* $\widetilde{\mathbb{U}_\mathbf{l}}$ *as in the beginning of the current section. Then the product*

$$\exp\Big(i\pi\Big[\binom{p+n-1}{p}\sum_{s=1}^{p} s\alpha_s\Big]\Big)$$

$$\exp\Big(a\pi\binom{p+n-1}{p}\sum_{s=1}^{p} z_s\Big)\, a^{-\binom{p+n-1}{p}\sum_{s=1}^{p} s\alpha_s - 2p\binom{p+n-1}{p+1}\gamma}$$

equals the product of critical values of the chosen branches with respect to the linear function $-at_1$.

$$(16)\qquad \prod_{\mathbf{l}\in\mathcal{Z}_n^p}\left[c(e^{a\sum_{j=1}^{n} t_j}, \widetilde{\mathbb{U}_\mathbf{l}})\prod_{j=1}^{n}\prod_{s=1}^{p} c(g_{j,s}, \widetilde{\mathbb{U}_\mathbf{l}}, -at_1)\prod_{1\le i<j\le n} c(h_{j,i}, \widetilde{\mathbb{U}_\mathbf{l}}, -at_1)\right].$$

The critical values were defined in Section 4.

9. Proofs of Theorem 8.1 and Theorem 8.3

Theorem 8.1 is a direct corollary of Theorems 5.15 and 7.8 [TV]. The computations are long but straightforward.

The correspondence in notation between the current paper and [TV] is as follows:

Object	current notation	[TV] – article
Dimension of the vector space	n	1
Number of points	p	n
Coordinates	t	u
Weights	α	$2\Lambda/p$
	γ	$-1/p$
Points	$z \in \mathbb{R}$	$y/h = z \in i\mathbb{R}$
Parameter	a	$i\eta/p$

For $\mathbf{l} \in \mathcal{Z}_n^p$, let $z_0 = -\infty$ and

$$\mathbb{V}_\mathbf{l} = \{t = (t_1, \dots, t_n) \in \mathbb{R}^n \mid z_{s-1} \le t_{j_s} \le z_s \forall s = 1, \dots, p \text{ and } j_s \in \Gamma_{\mathbf{l},s}\}.$$

Theorems 5.15 and 7.6 [TV] imply the following formula

(17)

$$\det\left[\int_{\mathbb{V}_{\mathbf{l}}}\exp\Big(a\sum_{j=1}^{n}t_j\Big)\prod_{s=1}^{p}\prod_{i=1}^{n}(t_i-z_s)^{\alpha_s}\prod_{1\le i<j\le n}(t_i-t_j)^{2\gamma}\omega_{\mathbf{m}}(t,z)\right]_{\mathbf{l},\mathbf{m}\in\mathcal{Z}_n^p}=$$

$$(-1)^{n\binom{p+n-1}{p-1}}\prod_{\mathbf{l}\in\mathcal{Z}_n^p}\prod_{j=1}^{p}\prod_{s=1}^{l_j}\frac{\sin(-s\pi\gamma)}{\sin(-\pi\gamma)}\prod_{s=0}^{n-1}\left[\frac{\Gamma((s+1)\gamma)^p}{\Gamma(\gamma)^p}\prod_{j=1}^{p}\Gamma(\alpha_j+s\gamma)\right]^{\binom{p+n-s-2}{p-1}}$$

$$\exp\Big(a\pi\binom{p+n-1}{p}\sum_{s=1}^{p}z_s\Big)\,a^{-\binom{p+n-1}{p}\sum_{s=1}^{p}s\alpha_s-2p\binom{p+n-1}{p+1}\gamma}$$

$$\exp\Big(i\pi\Big[\binom{p+n-1}{p}\sum_{s=1}^{p}s\alpha_s+p^2\binom{p+n-1}{p+1}\gamma\Big]\Big)$$

$$\prod_{1\le a<b\le p}(z_b-z_a)^{(\alpha_a+\alpha_b)\binom{p+n-1}{p}+2\gamma\binom{p+n-1}{p+1}}.$$

In order to obtain Theorem 8.3 we have to pass from "rectangular" domains $\mathbb{V}_{\mathbf{l}}$ to "triangular" domains $\widetilde{\mathbb{U}_{\mathbf{l}}}$. For any $\mathbf{l}$, $\mathbf{m}\in\mathcal{Z}_n^p$ we have

$$\int_{\mathbb{V}_{\mathbf{l}}}\exp\Big(a\sum_{j=1}^{n}t_j\Big)\prod_{s=1}^{p}\prod_{i=1}^{n}(t_i-z_s)^{\alpha_s}\prod_{1\le i<j\le n}(t_i-t_j)^{2\gamma}\omega_{\mathbf{m}}=$$

$$=\sum_{\sigma\in\mathbb{S}^{l_1}\times\cdots\times\mathbb{S}^{l_p}}\int_{\sigma\widetilde{\mathbb{U}_{\mathbf{l}}}}\exp\Big(a\sum_{j=1}^{n}t_j\Big)\prod_{s=1}^{p}\prod_{i=1}^{n}(t_i-z_s)^{\alpha_s}\prod_{1\le i<j\le n}(t_i-t_j)^{2\gamma}\omega_{\mathbf{m}}$$

$$=e^{n(n-1)i\pi\gamma}[\prod_{j=1}^{p}1(1+e^{-2\pi i\gamma})\cdots(1+e^{-2\pi i\gamma}+\cdots+e^{-2\pi i\gamma(l_j-1)})]$$

$$\int_{\widetilde{\mathbb{U}_{\mathbf{l}}}}\exp\Big(a\sum_{j=1}^{n}t_j\Big)\Phi(t,z)\omega_{\mathbf{m}}(t,z)$$

$$=e^{n(n-1)i\pi\gamma}\prod_{j=1}^{p}\prod_{s=1}^{l_j}\frac{\sin(-s\pi\gamma)}{\sin(-\pi\gamma)}\prod_{j=1}^{p}e^{-i\pi\gamma l_j(l_j-1)/2}$$

$$\int_{\widetilde{\mathbb{U}_{\mathbf{l}}}}\exp\Big(a\sum_{j=1}^{n}t_j\Big)\Phi(t,z)\omega_{\mathbf{m}}(t,z)$$

This proves Theorem 8.3.

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Received October 27, 1997

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