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TOTALLY GEODESIC BOUNDARIES OF YANG-MILLS MODULI SPACES

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ABSTRACT. Moduli spaces $\mathcal{M}$ of self-dual $SU(2)$ connections ("instantons") over a compact Riemannian 4-manifold (M, g) carry a natural L^2 metric $\mathbf{g}$, which is generally incomplete. For instantons of Pontryagin index 1 over a compact, simply connected, oriented, positive-definite base manifold, the completion $\overline{\mathcal{M}}$ is Donaldson's compactification; in fact the boundary of the completion is an isometric copy of $(M, 4\pi^2 g)$ ([GP2]). In this paper we show that the boundary is, furthermore, a *totally geodesic* submanifold of the completion. Along the way, we prove a regularity theorem: the continuous extension of $\mathbf{g}$ to the "collar region" of $\overline{\mathcal{M}}$ is $C^{1,\alpha}$ (in the conventional scale/center coordinates) for small $\alpha > 0$. The proofs rely on some new weighted Sobolev inequalities for concentrated instantons, in which the only dependence of the Sobolev constants on the connection is through the concentration parameter λ. The exponent in the weighting function translates into the Hölder exponent in the regularity theorem.

1. INTRODUCTION

The moduli spaces $\mathcal{M}$ of self-dual connections (instantons) over a compact Riemannian 4-manifold carry a natural metric, the "L^2 metric" $\mathbf{g}$, analogous to the Weil-Petersson metric of Teichmüller theory. This article is a sequel to [G2], continuing the study of the Riemannian structure of the spaces $(\mathcal{M}, \mathbf{g})$. This study reveals a rich interplay between the elliptic analysis of self-dual connections and the geometry of the moduli spaces. In [G2] it was shown, for example, that "localization" of the covariant Green operators of concentrated self-dual connections is reflected in boundedness of the Riemannian curvature of certain moduli spaces

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near the ideal boundary (see Theorem 1.1 below). In this article we show how other features of the analysis lead to $C^{1,\alpha}$ regularity of $\mathbf{g}$ for these moduli spaces, and to a proof that the boundary $\partial\mathcal{M}$ is a totally geodesic submanifold of the completion of $(\mathcal{M}, \mathbf{g})$.

Throughout this paper, as in [G2], unless otherwise stated (M, g) denotes a C^∞ Riemannian four-manifold that we assume to be compact, simply connected, oriented, and having positive-definite intersection form. We let $\mathcal{M}$ denote the space of charge-1 $SU(2)$ instantons over M, a five-dimensional space. $\mathcal{M}$ contains a 'collar' region $\mathcal{M}_{\lambda_0}$ diffeomorphic (non-canonically) to $(0, \lambda_0] \times M$ for some $\lambda_0 > 0$; the diffeomorphism $\Psi : \mathcal{M}_{\lambda_0} \to (0, \lambda_0] \times M$ assigns to an instanton its scale $\lambda \in (0, \lambda_0]$ and its center $p \in M$ as defined in [D1]. Certain assertions we make below are true only for λ_0 sufficiently small; we will always assume, without explicit mention, that λ_0 has been chosen small enough.

Let $\overline{\mathcal{M}_{\lambda_0}}$ denote the completion of $(\mathcal{M}, \mathbf{g})$ as a metric space and let $\partial\mathcal{M}$ denote the complement of $\mathcal{M}_{\lambda_0}$ in $\overline{\mathcal{M}_{\lambda_0}}$. It was proven in [GP2] that as $\lambda \to 0$ the L^2 metric is asymptotic to a product in the C^0 topology:

$$(\Psi^{-1})^*\mathbf{g} \sim 4\pi^2(2d\lambda^2 \oplus g). \tag{1.1}$$

As a consequence, completing in the L^2 metric implements Donaldson's compactification: the collar map Ψ extends to a homeomorphism $\overline{\mathcal{M}_{\lambda_0}} \cong [0, \lambda_0] \times M$, identifying the boundary $\partial\mathcal{M}$ with $\{0\} \times M$. This result was extended by P. Feehan [F] to instantons of arbitrary charge over 1-connected definite manifolds, and work of Donaldson [D2] suggests that this relation between L^2 completion and compactifications should hold very generally.

The theorems of [GP2],[F], and [D2] asserted nothing about the asymptotic behavior of geometric invariants involving derivatives of $\mathbf{g}$, although concrete examples $M = S^4$ and $M = \mathbf{CP}^2$ suggested that the curvature of $(\mathcal{M}, \mathbf{g})$ should extend continuously to the boundary and that the second fundamental form of the boundary should be zero ([DMM],[GP1],[G1],[Hab],[K]). The first general theorems along these lines were proven in [G2] (modulo a technical assumption that can now be removed; see below):

Theorem 1.1. *(Theorems 1.1-1.2 of [G2]) Let (M, g) and $\overline{\mathcal{M}_{\lambda_0}}$ be as above. Then the sectional curvature of $(\mathcal{M}_{\lambda_0}, \mathbf{g})$ is bounded above and below, and the restriction of the Riemann tensor to the tangent bundle of the leaves $\{\lambda = \text{constant}\}$ extends continuously to the boundary.*

This theorem and the behavior of $\mathbf{g}$ in the S^4 and $\mathbf{CP}^2$ examples led the author to conjecture in [G2] that the continuous extension of $\mathbf{g}$ to the completion should be C^2 or better. Note, however, that the completion $\overline{\mathcal{M}_{\lambda_0}}$ was defined above in the sense of Cauchy sequences; not enough is known about the regularity of g near $\partial\mathcal{M}$ to complete using the geodesic equation. But Cauchy completion is only a topological operation, not a smooth one; by itself it induces no smooth structure on $\overline{\mathcal{M}_{\lambda_0}}$, which is a necessity if we are to discuss derivatives. Conjecturally, there is a 'geometrically natural' smooth structure given by the normal exponential map from the boundary, but this has yet to be proven. (In fact, *a priori* it is not obvious that this exponential map is even uniquely defined since the extended metric is not known to be C^2.) Alternatively, we can obtain a smooth structure on the completion by declaring the extension to $\overline{\mathcal{M}_{\lambda_0}}$ of the collar map Ψ (with center and scale as defined in [D1]) to be a diffeomorphism; we refer to the induced coordinate systems on $\overline{\mathcal{M}_{\lambda_0}}$ as "collar coordinates".

In this paper, the regularity of $\mathbf{g}$ will be discussed in terms of collar coordinates. We use these coordinates to extend $\mathbf{g}$ to a metric $\bar{\mathbf{g}}$ on $\overline{\mathcal{M}_{\lambda_0}}$ by using the limiting boundary values given by (1.1). The first main result of this paper is the following.

Theorem 1.2. *For any fixed collar map, there exists $\alpha > 0$ for which $\bar{\mathbf{g}}$ is $C^{1,\alpha}$ in collar coordinates.*

It should be noted that in collar coordinates for the S^4 and $\mathbf{CP}^2$ examples, $\bar{\mathbf{g}}$ is $C^{1,\alpha}$ for any $\alpha < 1$, but is not C^2 (or even $C^{1,1}$). Thus, sticking with collar coordinates, the only significant way to improve Theorem 1.2 would be to increase the Hölder exponent α. Changing the meaning of "collar coordinates" by using the alternative definition of center and scale in [F] (based on cut-off versions of center-of-mass and standard deviation) is unlikely to improve the theorem either.

Thanks to this theorem, the second fundamental form of the boundary—more precisely, of $\{0\} \times M$ relative to the continuous extension of $(\Psi^{-1})^*\bar{\mathbf{g}}$—is well-defined, as it involves only first derivatives of the metric. This allows us to state our second main theorem.

Theorem 1.3. *In the smooth structure on $\overline{\mathcal{M}_{\lambda_0}}$ induced by a collar map, the second fundamental form of the boundary $\partial\mathcal{M}$ vanishes identically. Hence the submanifold $\partial\mathcal{M}$ is totally geodesic.*

Theorems 1.2–1.3 both are corollaries of the more technical Theorem 7.1, which details the behavior of the metric coefficients (in collar coordinates) and the decay of their derivatives near $\partial\mathcal{M}$.

Although usually "vanishing second fundamental form" and " totally geodesic" are synonomous, because of insufficient regularity of the metric the latter term should be used with caution here, since another common definition of totally geodesic submanifold is that geodesics initially tangent to the submanifold stay within it. However, in general the geodesics in a $C^{1,\alpha}$ metric are not uniquely determined by their initial conditions, as Hartman's two-dimensional counterexample $(1 + 9\lambda^{4/3})(dx^2 + d\lambda^2)$ shows ([Har1]). In this example the submanifold $L = \{\lambda = 0\}$ has zero second fundamental form, but the curves $\lambda = 0$ and $\lambda = x^3$ both are geodesics, so L does not satisfy the second meaning of "totally geodesic". It turns out that this phenomenon cannot occur for the moduli spaces above; in §7 we show that the combination of bounded curvature on the complement of $\partial\mathcal{M}$ and vanishing second fundamental form of $\partial\mathcal{M}$ imply uniqueness of solutions to the geodesic equation (in Hartman's example the curvature is unbounded for $\lambda \neq 0$).

The original goal of this paper was to establish Theorem 1.3. However, without knowing that the metric is differentiable at the boundary, "second fundamental form" could only be defined in a formal, limiting sense. But the author found that the computations needed to prove Theorem 1.3 also proved Theorem 1.2, with only a little extra work.

The need to refer to a fixed collar map in Theorem 1.3 is unsatisfying. The methods of this paper suggest that different collar maps determine the same $C^{1,\alpha}$ vector fields and hence the same $C^{2,\alpha}$ atlas on $\overline{\mathcal{M}_{\lambda_0}}$. If true, this would allow us to attach meaning to "collar coordinates" independent of the collar map, and would simplify the statements of Theorems 1.2 and 1.3. However, as was shown in [G2] for the S^4 and $\mathbf{CP}^2$ examples, collar coordinates should not be expected to give optimal regularity for $\mathbf{g}$; in those examples one gets four more orders of differentiability in the "Fermi coordinates" given by the inverse of the normal exponential map—i.e. coordinates in which λ is replaced by distance to the boundary, and center-point by closest point in the boundary. (One still does not get C^∞ regularity in those coordinates, proving that the metric is not C^∞ in *any* coordinates.) Therefore the most appealing way to improve Theorems 1.2–1.3 would be to establish existence and regularity of the Fermi coordinate system more generally, eliminating the need for reference to any ad hoc collar map. At present the best we can say is that the normal exponential map is Lipschitz (see §7). With more work one can probably show that this map is at least a C^1 local diffeomorphism, but we do not attempt that here.

To put the regularity theorem 1.2 in context, it should be noted that in other settings in Riemannian geometry, global bounds on curvature and injectivity radius are known to give $C^{1,\alpha}$ bounds on the metric in *some* coordinates, namely harmonic coordinates, for any $\alpha < 1$. However, to the author's knowledge, all the theorems in the relevant regularity literature assume more regularity at the outset than we know, *a priori*, for $\mathbf{g}$ on $\overline{\mathcal{M}_{\lambda_0}}$. The original "curvature bounds give $C^{1,\alpha}$ metric bounds" result, due to Jost and Karcher ([JK]; see Theorem 4.3 of [P] for an English translation), assumes the manifold to be compact and without boundary, that the metric is regular enough that the curvature is defined everywhere, and that normal coordinate systems exist at every point; furthermore the bounds produced depend on the injectivity radius. In the setting of Theorem 1.2, however, there is a boundary, at which the metric is initially known only to be C^0 and at which the curvature and exponential map are *a priori* undefined. M. Anderson considered the case of a manifold with boundary ([A], Lemma 2.2), but his $C^{1,\alpha}$ bounds apply only at interior points and potentially depend also on distance to the boundary. It is plausible that results such as Jost and Karcher's or Anderson's (including the existence of harmonic coordinates on balls of controlled size) can be extended to our situation of a manifold with boundary, possessing a C^0 metric that is C^∞ on the interior, with bounded curvature on the interior. It is also quite likely that $C^{1,\alpha}$ compactness arguments for C^2 metrics on closed manifolds with bounds on curvature (cf. [GW], [P]) can be adapted to our situation.

There are several reasons for using collar coordinates instead of attempting an approach along the lines above. First, such an approach would require knowing boundedness of the second fundamental form of the leaves $\{\lambda = \text{const}\}$, and in the course of obtaining this knowledge we'd find that we'd already proven Theorems 1.2–1.3 directly. Second, an abstract regularity result would not by itself prove Theorem 1.3, for which one needs to know that certain derivatives vanish as $\lambda \to 0$, not merely that they are bounded. Third, although in general harmonic coordinates give optimal regularity of the metric (at least on manifolds without boundary), our geometric setup—in which the boundary is attached by forming a completion—seems to cry out for Fermi coordinates, which is what our collar coordinates approximate. Finally, in the computational approach using collar coordinates, one can see explicitly how the exponent α in the weighting function of certain weighted Sobolev inequalities for concentrated connections translates into the exponent in the $C^{1,\alpha}$ regularity statement.

Our approach to Theorems 1.2–1.3 involves the "approximate tangent space" discussed in [GP2] and [G2]. To prove $C^{1,\alpha}$ regularity of the metric we need to examine the metric coefficients in a basis of the tangent bundle that is itself at least $C^{1,\alpha}$ on $\overline{\mathcal{M}_{\lambda_0}}$. Coordinate vector fields provided by the collar map are of course C^∞, but there is no obvious way to examine the metric coefficients directly in such a basis. This is where the approximate tangent space enters. Approximate tangent vectors are elements of $\Omega^1(Ad\,P)$ with explicit local formulas that allow effective computation of metric coefficients. Furthermore these elements are close to true tangent vectors—harmonic 1-forms—in a quantifiable way. The C^0 closeness of the approximate and true tangent bundles established in [G2] is not sufficient for our purposes here; we require closeness on the level of first derivatives as well. The hardest part of the proofs of Theorems 1.2–1.3 consists of showing that the approximate and true tangent bundles are C^1-close (our method gives the extra Hölder exponent as a bonus). This allows us to construct vector fields on $\overline{\mathcal{M}_{\lambda_0}}$ that are $C^{1,\alpha}$ up to the boundary (in collar coordinates) and to prove our main theorems.

A large part of our technical work consists of the elliptic estimates relating the approximate and true tangent bundles. Since the appearance of [G2], there have been two significant improvements in the technology of these estimates. The first of these is a stronger statement concerning the pointwise decay of curvatures of concentrated self-dual connections. In [G2] a sharp decay rate was established only under a technical assumption on the base metric g, the "$\bigwedge^2_-$-condition". This condition is unnecessary; in [GP3] it is proven that the decay estimate holds for arbitrary base metrics. Consequently, *all theorems and estimates in [G2] remain true if the $\bigwedge^2_-$-condition is removed from the hypotheses.* The second improvement occurs in estimates involving the Green operator G_-^A on anti-self-dual two-forms (where A is a concentrated self-dual connection). In [G2] these estimates all involved integral powers of the distance to the center of the instanton, which in some cases led to estimates that diverged logarithmically with λ (cf. (3.1)). The techniques of this paper circumvent this critical-exponent phenomenon by showing that we can increase the exponent of the distance function by a small non-integral amount in most cases. This strengthens certain key estimates in [G2] on the difference between approximate and true tangent vectors, and simultaneously simplifies their proofs. In particular, this approach eliminates the need for the parametrix method of §§6-7 of [G2] in the derivation of these estimates (though the parametrix may still supply higher–order information).

The moduli spaces in Theorems 1.1–1.3 are rather special; the topologies of both the base manifold and the $SU(2)$-bundle over it are very constrained. More generally, the boundary of the completion is a stratified space, and an attractive generalization of Theorem 1.3 is the conjecture that this stratification is totally geodesic, at least when the strata are of the form { symmetric product of M} $\times$

{lower-degree moduli space} (as opposed to proper subsets of these spaces). This rather speculative conjecture is based on the observation that estimates similar to the ones in this paper go through for more general moduli spaces, and, as we see in this paper, one should expect certain types of estimates to be reflected in specific geometric features of the moduli space.

The outline of this paper is as follows. In §2 we review the approximate tangent space and introduce the associated notation, and begin to examine how close the approximate tangent space is to the true one. To complete this examination, certain elliptic estimates are required. Since there is a very large number of these estimates and their applications in this paper, in §3 we describe some basic principles to help the reader thread his or her way through later calculations. All but one of these principles were derived in [G2]; we derive the remaining principle—essentially a collection of weighted Sobolev inequalities—in §4. In §5 we begin to consider how close the approximate and true tangent bundles are on a C^1 level, and state the results of the fundamental calculations (Proposition 5.1) that eventually lead to Theorems 1.2–1.3. In §6, we perform the actual calculations, which are quite long. It is unfortunate that the proofs of our main results require so much calculation, but the author found no way to avoid this, and much of the calculation is nontrivial; "obvious" approaches lead to divergent upper bounds on quantities that are in fact small. We have tried to limit the details presented to a few useful lemmas (such as Lemma 6.3) and their nontrivial applications, and to enough intermediate bookkeeping to display the relative sizes of various terms and render the computations checkable. Finally, in §7, we show how Proposition 5.1 leads to Theorems 1.2–1.3, proving both of these as corollaries of the stronger Theorem 7.1. At the end, we briefly discuss the exponential map on $(\overline{\mathcal{M}_{\lambda_0}}, \mathbf{g})$.

Since the approximate tangent space at a point of the moduli space involves functions of normal coordinates based at the center of the instanton, differentiating any expression involving this approximation requires us to vary the normal coordinate systems as the center point moves. This is an interesting but lengthy exercise in pure Riemannian geometry, so we have left it to the appendix, though the results are used in §6. The formula we derive is reminiscent of the expansion of the metric in normal coordinates. Although the coefficients of the $O(r^2)$ term in this expansion are not needed for the calculations in §6, we have included a derivation of these coefficients anyway, since the formula may be of independent interest.

As in [G2], we allow constants c to have their values continually updated. All constants that could potentially depend on the connection or choice of point (or frame, etc.) in M are uniform in these parameters unless otherwise indicated. The notation $\langle\cdot,\cdot\rangle$ is used for the L^2 inner product of bundle-valued forms.

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2. The approximate tangent space

In this section we introduce notation that will be used for the rest of this paper.

Let $P \to M$ be a principal $SU(2)$ bundle of Pontryagin index 1, and let $\mathcal{A}, \mathcal{G}, Ad\,P$ denote,respectively, the space of connections on P, the group of gauge transformations, and the adjoint bundle. Let $\mathcal{M} \subset \mathcal{A}/\mathcal{G}$ be the moduli space of self-dual connections, and let $\mathcal{M}_{\lambda_0} \subset \mathcal{M}$ denote a "collar region", equipped with a diffeomorphism $\Psi : \mathcal{M}_{\lambda_0} \to (0, \lambda_0] \times M$ as in [G2, p. 140]. (Ψ here is Ψ^{-1} in [G2].) Letting $\mathcal{A}_{\lambda_0}$ denote a small neighborhood of the inverse image of $\mathcal{M}_{\lambda_0}$ under the projection $\mathcal{A} \to \mathcal{A}/\mathcal{G}$, the collar map Ψ factors through a gauge-invariant map with domain $\mathcal{A}_{\lambda_0}$, so we will write $\Psi([A]) = (\lambda(A), p(A))$; we refer to $\lambda(A)$ as the *scale* and $p(A)$ as the *center* of A. Below, $F^A \in \Omega^2(Ad\,P)$ denotes the curvature of A.

The definition of Ψ involves a cutoff function, so for the rest of this paper we fix such a function $b \in C_0^\infty(\mathbf{R})$ with $b(t) = 1$ for $0 \le t \le 1$, $b(t) = 0$ for $t \ge 2$ and $0 \le b(t) \le 1$ everywhere. The cutoff b determines a certain normalization constant K (see [GP2, p. 533] or [D, definition 15]) appearing in the definition of λ, and we set

$$\bar{\lambda} = K\lambda. \tag{2.1}$$

We also fix a number r_0 such that $3r_0$ is less than the injectivity radius of (M, g), and a smooth, nonnegative function $\rho : M \times M \to \mathbf{R}$, with $\rho > 0$ off the diagonal, such that $\rho(p,q) = \mathrm{dist}(p,q)$ whenever $\mathrm{dist}(p,q) \le 2r_0$. For each $p \in M$ we define $r_p(q) = \rho(p,q)$ and $\beta_p(q) = b(r_p(q)/r_0)$; we will usually suppress the p's to avoid clutter. We will refer to r_p as "distance to p", even though this is literally true only at small distances.

We review certain aspects of the construction in [GP2] of the "approximate tangent bundle" of the collar region in $\mathcal{M}$. The notation below is essentially the same as in [G2], except that the ϕ' of [G2] is called $\hat{\phi}$ below.

Notation. Given $[A] \in \mathcal{M}_{\lambda_0}$, a vector $\mathbf{a} \in T_pM$ (where $p = p(A)$), and $a_0 \in \mathbf{R}$, we write:

$$\hat{\phi} = \frac{1}{2}\beta_p r_p^2. \tag{2.2}$$

$$\begin{aligned} \phi_{\mathbf{a}} &= \beta_p a_i x^i. \\ \phi_{(a_0,\mathbf{a})} &= \lambda^{-1} a_0 \hat{\phi} + \phi_{\mathbf{a}}. \\ Z_{(a_0,\mathbf{a})} &= \mathrm{grad}(\phi_{(a_0,\mathbf{a})}). \\ \tilde{Z}^A &= \iota_Z F^A \text{ (for any vector field } Z \text{ on } M). \end{aligned} \tag{2.3}$$

In (2.3), $\{x^i\}_1^4$ are normal coordinates based at p, and $\mathbf{a} = a_i \frac{\partial}{\partial x^i}$, but the definition is independent of choice of normal coordinates.

For $[A] \in \mathcal{M}_{\lambda_0}$, the true tangent space $T_{[A]}\mathcal{M}$ is naturally isomorphic to the harmonic space

$$H_A := \{\eta \in \Omega^1(Ad\,P) \mid (d^A)^*\eta = 0, d_-^A\eta = 0\}. \tag{2.4}$$

(Here d^A is covariant exterior derivative; the subscript "minus" denotes projection onto anti-self-dual two-forms.) The harmonic spaces fit together to form a $\mathcal{G}$-invariant sub-bundle H of the trivial tangent bundle $T\mathcal{A}|_{\mathcal{A}_{\lambda_0}} \cong \mathcal{A}_{\lambda_0} \times \Omega^1(Ad\,P)$; vector fields on $\mathcal{M}_{\lambda_0}$ correspond to $\mathcal{G}$-invariant sections of H (restricted to the inverse image of $\mathcal{M}_{\lambda_0}$).

We define the *approximate harmonic space* $\tilde{H}_A$ by setting

$$\tilde{H}_A := \{\tilde{Z}^A_{(a_0,\mathbf{a})} \mid (a_0, \mathbf{a}) \in \mathbf{R} \times T_{p(A)}M\}. \tag{2.5}$$

These piece together to form another $\mathcal{G}$-invariant sub-bundle $\tilde{H}$ of $\mathcal{A}_{\lambda_0} \times \Omega^1(Ad\,P)$, whose restriction to the inverse image of $\mathcal{M}_{\lambda_0}$ descends to what we'll call the *approximate tangent bundle* of $\mathcal{M}_{\lambda_0}$. The fiber at $[A] \in \mathcal{M}_{\lambda_0}$, the *approximate tangent space*, consists of $\mathcal{G}$-invariant sections of $\tilde{H}$ along the $\mathcal{G}$-orbit through A. We will abuse terminology and refer to $\tilde{H}_A$ itself as the approximate tangent space and its elements as approximate tangent vectors.

Heuristically, the a_0-part of the approximate tangent vector $\tilde{Z}^A_{(a_0,\mathbf{a})}$ is related to an infinitesimal change of scale, while the $\mathbf{a}$-part is related to an infinitesimal translation of center point in the direction $\mathbf{a}$. This was made more precise in [GP2], where it was shown that the $\mathcal{G}$-equivariant map

$$\begin{aligned} \mathcal{I}_A : \mathbf{R} \times T_{p(A)}M &\to H_A \\ (a_0, \mathbf{a}) &\mapsto -\pi^A \tilde{Z}^A_{(a_0,\mathbf{a})}, \end{aligned} \tag{2.6}$$

where $\pi^A : \Omega^1(Ad\,P) \to H_A$ is the L^2-orthogonal projection, approximates the differential of Ψ^{-1} as $\lambda \to 0$. To be more quantitative, we define the "error term" $\mathcal{L}^A \in End(\mathbf{R} \times T_{p(A)}M)$ by

$$\mathcal{L}_A := \Psi_* \circ \mathcal{I}_A - Id \tag{2.7}$$

and write

$$(\hat{a}_0, \hat{\mathbf{a}}) := \mathcal{L}_A(a_0, \mathbf{a}). \tag{2.8}$$

The goal of this section is to estimate $|(\hat{a}_0, \hat{\mathbf{a}})|$, which leads us to yet more notation. The objects that encode the difference between the approximate and true tangent spaces are

$$\xi_Z^A := (\pi^A - Id)\tilde{Z}^A = -(d_-^A)^* G_-^A d_-^A \tilde{Z}^A \in \Omega^1(Ad\,P). \tag{2.9}$$

(Here $G_-^A = \left(d_-^A(d_-^A)^*\right)^{-1}$.) For $Z = \mathrm{grad} f$, Lemma 3.1b of [GP2] gives

$$d_-^A(\tilde{Z}^A) = H^0 f \natural F^A \tag{2.10}$$

where $H^0 f$ is the trace-free part of the covariant Hessian of f, and where $\natural$ is the pairing defined by $T \natural S = T_{ij} S_{jk} \theta^i \bigwedge \theta^k$ relative to an orthonormal basis $\{\theta^i\}$ of T^*M. Hence if we define

$$\omega_{(a_0,\mathbf{a})} := H^0 \phi_{(a_0,\mathbf{a})} \natural F^A \in \Omega_-^2(Ad\,P). \tag{2.11}$$

then

$$\xi_{(a_0,\mathbf{a})} := \xi^A_{Z_{(a_0,\mathbf{a})}} = -(d_-^A)^* G_-^A \omega_{(a_0,\mathbf{a})}. \tag{2.12}$$

Later we will make frequent use of the fact that

$$|\omega_{(a_0,\mathbf{a})}| \le c\chi_{[0,2r_0]}(|\mathbf{a}|r + |a_0|\lambda^{-1}r^2)|F^A|, \tag{2.13}$$

where r is distance to $p(A)$ and where $\chi_{[0,2r_0]}$ is the characteristic function of the disk $\{0 \le r \le 2r_0\}$. (For the a_0 term here, it is crucial that only the *trace-free* part of the Hessian enters in (2.10).)

Finally we can state our first basic estimate on the map $\mathcal{L}_A$. The estimate below improves Proposition 5.2 of [GP2] by a full power of λ in directions "tangential" to the boundary (i.e. to the $\{\lambda = \text{const}\}$ foliation), and by a small power of λ in the "normal" direction. Our proof draws heavily on the formulas in §§4-5 of [GP2] for differentiating the collar map, whose lengthy derivations we do not repeat here. A reader without a copy of [GP2] can derive the relevant formulas from the characterization of $\lambda(A)$ and p in terms of the function $R_A(s,x) = \int b(\rho(x,y)/s)|F^A(y)^2 dv_g(y)$, namely (i) $\overline{\lambda}(A) = \inf\{s \mid R_A(s,x) = 4\pi^2 \text{ for some } x\}$, and (ii) $\partial R_A/\partial x^i = 0$.

Proposition 2.1. *There exists $\alpha > 0$ such that for all $[A] \in \mathcal{M}_{\lambda_0}$,*

$$|\mathcal{L}_A(a_0, \mathbf{a})| = |(\hat{a}_0, \hat{\mathbf{a}})| \le c(|\mathbf{a}|\lambda^2 + |a_0|\lambda^{1+\alpha}). \tag{2.14}$$

In particular, if λ_0 is small enough then $\mathcal{I}_A$ is always an isomorphism.

We will begin the proof of this proposition here, but will not complete it until the end of section 4, when we will have certain estimates in hand (in particular bounds on L^2 inner products of the form $\langle\omega, G_-^A\omega'\rangle$).

PROOF. (Begun.) Below, we write $\Psi_* = (\lambda_*, p_*)$ and $Z = Z_{(a_0,\mathbf{a})}$. We will omit many sub- and super-scripts in this proof; e.g. $\tilde{Z}$ stands for $\tilde{Z}^A_{(a_0,\mathbf{a})}$.

First we consider the differential λ_*. From Proposition 5.1 of [GP2] (with a minor typographical error corrected), we have

$$\lambda_*\pi\tilde{Z} = -4\lambda\frac{\langle\pi\tilde{Z}, \iota_{\mathrm{grad}\gamma}F\rangle}{\int_M(\nabla\gamma, \nabla(r^2))|F|^2}. \tag{2.15}$$

where

$$\gamma = b(r/\overline{\lambda}).$$

Writing $\pi\tilde{Z} = \tilde{Z} + \xi_Z$, the proof of Proposition 5.2a of [GP2] shows that (2.15) reduces to

$$-\hat{a}_0 = \lambda_*\pi\tilde{Z} + a_0 = -4\lambda\frac{\langle\xi_Z, \iota_{\mathrm{grad}\gamma}F\rangle}{\int_M(\nabla\gamma, \nabla(r^2))|F|^2}. \tag{2.16}$$

Let $\omega_\gamma = d_-^A\iota_{\mathrm{grad}\gamma}F$ ($= H^0\gamma \natural F$ by (2.10)). Integrating by parts (see (2.12)), we can rewrite the numerator in (2.16) as

$$\langle\xi_Z, \iota_{\mathrm{grad}\gamma}F\rangle = -\langle H^0\phi_{(a_0,\mathbf{a})} \natural F, G_-^A(H^0\gamma \natural F)\rangle = -\langle G_-^A\omega_{(a_0,\mathbf{a})}, \omega_\gamma\rangle. \tag{2.17}$$

The denominator in (2.16) is bounded away from zero by some constant independent of A (see [GP2], equation 4.9), so we have

$$|\hat{a}_0| \le c\lambda|\langle G_-^A\omega_{(a_0,\mathbf{a})}, \omega_\gamma\rangle| \tag{2.18}$$

We will not have the tools to estimate this inner product optimally until after §4, so we suspend this part of the proof and begin to bound $\hat{\mathbf{a}}$.

The steps needed to derive an explicit formula for $\hat{\mathbf{a}} = p_*\pi\tilde{Z} + \mathbf{a}$ can be found in the proofs of Propositions 4.5 and 5.2b in [GP2]. To write down the formula, let $\{x^i\}$ be a normal coordinate system at p, let $g^{ij} = g(dx^i, dx^j)$, and set

$$\gamma_i = \frac{\partial\gamma}{\partial x^i} = \overline{\lambda}^{-1}b'(r/\overline{\lambda})\frac{x^i}{r},$$

$$\gamma_{ij} = \frac{\partial^2\gamma}{\partial x^i \partial x^j} = \overline{\lambda}^{-2} b''(r/\overline{\lambda}) \frac{x^i x^j}{r^2} + \overline{\lambda}^{-1} b'(r/\overline{\lambda}) \left(\frac{\delta_{ij}}{r} - \frac{x^i x^j}{r^3} \right), \tag{2.19}$$

$$m_i = \overline{\lambda}^{-1} \int_M b''(r/\overline{\lambda}) x^i |F|^2, \tag{2.20}$$

$$\zeta_{ij} = \text{the function } \{ q \mapsto \left. \frac{\partial^2 r_q}{\partial x^i \partial x^j} \right|_p - \left. \frac{\partial^2 r_p}{\partial x^i \partial x^j} \right|_q \}, \tag{2.21}$$

$$\psi_{ij} = -b'(r/\overline{\lambda})\zeta_{ij} + \overline{\lambda}\gamma_{ik}(g^{jk} - \delta_{jk}) = b'(r/\overline{\lambda}) \left(-\zeta_{ij} + r^{-1}(g^{ij} - \delta_{ij}) \right), \tag{2.22}$$

$$\overline{\lambda}^2 H_{ij} = \int_M \overline{\lambda}^2 \gamma_{ij} |F|^2 + \int_M \overline{\lambda} b'(r/\overline{\lambda}) \zeta_{ij} |F|^2. \tag{2.23}$$

The functions ζ_{ij}, ψ_{ij} are discussed in greater detail in the appendix (see Definition 2 and Lemma 8.3), where it is shown that $\zeta_{ij} = O(r)$ (and hence $\psi_{ij} = b'(r/\overline{\lambda}) \cdot O(r)$). From the proofs in [GP2] cited above, one finds $\mathbf{a} + \hat{\mathbf{a}} = ((\overline{\lambda}^2 H)^{-1})_{ij} f_i \frac{\partial}{\partial x^j}$, where $f_i = m_i \overline{\lambda}_* \pi \tilde{Z} + 2\overline{\lambda}^2 \langle \xi_Z, \iota_{\nabla \gamma_i} F \rangle + \int_M \overline{\lambda}^2 (Z, \nabla \gamma_i) |F|^2$. After simplifying some expressions using the normal-coordinate identity $g^{ij} x^j = x^i$, and integrating by parts as in (2.17), we obtain

(2.24)

$$\hat{\mathbf{a}} = \left((\overline{\lambda}^2 H)^{-1} \right)_{ij} \left\{ -K m_i \hat{a}_0 + 2\overline{\lambda}^2 \langle G_-^A \omega_{(a_0, \mathbf{a})}, H^0 \gamma_i \natural F \rangle + a_j \int \overline{\lambda} \psi_{ij} |F|^2 \right\} \frac{\partial}{\partial x^j}.$$

The symmetric matrix $\overline{\lambda}^2 H$ is $\geq c \cdot I$ for some constant $c > 0$ ([GP2, Lemma 4.4]), and (since r and λ are commensurate on $\operatorname{supp}(b'(r/\overline{\lambda}))$) we have $\overline{\lambda} |\psi_{ij}| \leq c\lambda^2$, implying $\int \overline{\lambda} \psi_{ij} |F|^2 \leq c\lambda^2 \int |F|^2 \leq c\lambda^2$. Furthermore, from [GP2, Lemma 3.5],

$$m_i \to 0 \text{ as } \lambda \to 0. \tag{2.25}$$

Hence

$$|\hat{\mathbf{a}}| \leq c \left(|\hat{a}_0| + |\mathbf{a}|\lambda^2 + \lambda^2 |\langle G_-^A \omega_{(a_0, \mathbf{a})}, H^0 \gamma_i \natural F \rangle| \right). \tag{2.26}$$

This is as far as we can go towards proving (2.14) without a sharp bound on the inner products appearing in (2.26) and (2.18). We interrupt the proof of Proposition 2.1 here (to be resumed and completed at the end of §4) so that we can provide the estimates we will use to bound these and other quantities.

3. Philosophy of the estimates

To establish Proposition 2.1 and several other important bounds, we need certain elliptic estimates whose only dependence on A is through λ. To make these estimates seem less random, we provide here some basic principles (consequences of the results we will prove in §4) that should help the reader understand the strategy behind the estimation scheme used in this paper. These principles also help guide one to the sharpest estimates generally attainable.

In this game frequently wants to bound L^2 and L^4 norms of certain quantities by as high a power of λ as possible. Usually these quantities are of the form $r^n\omega$ or $r^n G_-^A\omega$, where $r = r_{p(A)}$ is distance to the center point of A, and where ω has support in a small ball centered at $p(A)$. (Powers of r arise, for example, in the pointwise bound (2.13) and through principle 4 below.) The general principles for such estimates, subject to certain limitations described below, are:

1. A power of r generally gains you an equal power of λ (i.e. $\|r^n\eta\|_q \sim \lambda^n\|\eta\|_q$), but only up to a certain exponent.
2. An L^4-norm generally costs you a power of λ relative to an L^2-norm ($\|\eta\|_4 \sim \lambda^{-1}\|\eta\|_2$).
3. A covariant derivative generally costs you a power of λ (e.g. $\|\nabla^A\eta\|_q \sim \lambda^{-1}\|\eta\|_q$).
4. A Green operator G can gain you up to two powers of r ($\|G\eta\|_q \sim \|r^m\eta\|_q$ for some $m \leq 2$), hence (by principle 3) as many as two powers of λ. The greater the order to which η vanishes at $p(A)$, the worse the gain is.

(In these principles, "$\sim$" means "has the same order in λ as $\lambda \to 0$".) The primary limitation to principle 1 in our applications comes from the pointwise norm of η. Generally, our η's will be proportional to $F = F^A$ or its covariant derivatives, and by looking at the standard instanton on $\mathbf{R}^4$ one sees that one can never expect a pointwise bound on F better than λ^2, even far away from the center point. Hence no $\|r^nF\|_q$ will ever go to zero faster than λ^2, no matter how large n is. Furthermore, by comparison to the standard instanton, one sees that estimating $\|r^nF\|_q$ there is a critical exponent $n_c(q)$ such that for $n > n_c$ one has $\|r^nF\|_q \sim \lambda^2$, while for $n < n_c$ one has $\|r^nF\|_q \sim \lambda^{2-(n_c-n)}$. At the critical exponent one always picks up logarithmic terms (essentially because $\int_1^\infty r^{-1}dr$ is logarithmically divergent): $\|r^{n_c}F\|_q \sim \lambda^2|\log\lambda|^{1/q}$. (For rigorous justification of principle 1 see §3 of [G1].)

There are similar limitations to principles 2–4, but these arise partially because of the Sobolev inequalities. For the limitations to principle 4, see Lemma 4.3.

As an illustration, we mention that starting from $\|F\|_2 \sim \mathit{const}$, one can apply the principles and discussion above, finding (correctly) that $\|\beta rF\|_2 \sim \lambda$, $\|\beta r^2F\|_2 \sim \lambda^2|\log\lambda|^{1/2}$, $\|\beta r^{2+\epsilon}F\|_2 \sim \lambda^2$ for any $\epsilon > 0$, and $\|\beta r^{5/2}\nabla^A\nabla^AF\|_4 \leq$

$c\lambda^{-1/2}$. The general rule for estimates involving only F and its covariant derivatives is Corollary 3.4 of [G2] (valid for $A \in \mathcal{M}_{\lambda_0}$ and $j \geq 0$):

(3.1)

$$\|\chi_{[0,2r_0]} r^{n+j} \nabla^j F\|_2 + \|\chi_{[0,2r_0]} r^{n+1+j} \nabla^j F\|_4 \leq c \begin{cases} \lambda^n & if\ -2 < n < 2 \\ \lambda^2 |\log \lambda|^{1/2} & if\ n = 2 \\ \lambda^2 & if\ n > 2 \end{cases}$$

Here $\chi_{[0,2r_0]}$ is the characteristic function of the disk $\{0 \leq r \leq 2r_0\}$ (the support of β). Principle 4 is less exact than the first three; for example, if $|\omega| \leq c\beta r|F|$, from the results of §4 one does not find $\|G_-^A \omega\|_4 \sim \|r^3 F\|_4 \leq c\lambda^2 |\log \lambda|^{1/4}$, but only $\|G_-^A \omega\|_4 \sim \|\beta r^{2+\alpha} F\|_4$ (for $\alpha > 0$ sufficiently small) $\leq c\lambda^{1+\alpha}$.

Another important part of our estimation scheme deals with L^2 inner products of the form $\langle \omega, G_-^A \omega' \rangle$, as arose in (2.18) and (2.26). Since the eigenvalues of G_-^A are uniformly bounded in $\mathcal{M}_{\lambda_0}$, we have the obvious bound $|\langle \omega, G_-^A \omega' \rangle| \leq c\|\omega\|_2 \|\omega'\|_2$, but this estimate can be off from best possible by as many as two powers of λ when ω is supported close to the center point of A. Instead, we use a version of principle 4 established in Corollary 4.5: there exists $\alpha_0 > 0$ such that for all $\omega, \omega' \in \Omega_-^2(Ad\,P)$, all $|\alpha| < \alpha_0$, and all $[A] \in \mathcal{M}_{\lambda_0}$,

$$|\langle \omega, G_-^A \omega' \rangle| \leq c\|r^{1+\alpha}\omega\|_2 \|r^{1-\alpha}\omega'\|_2. \tag{3.2}$$

Said another way, we have traded the two derivatives that the Green operator gains us for two powers of the distance function—but in a way that is *uniform in the connection.*

To illustrate how best to take advantage of the α in this estimate, consider an example in which $|\omega| \sim r^2|F|$ and $|\omega'| \sim r|F|$. Taking $\alpha = 0$ above we get a bound $c\lambda^4 |\log \lambda|^{1/2}$; taking $\alpha > 0$ we get the worse bound $c\lambda^{4-\alpha}$; taking $\alpha < 0$ we get the best bound $c\lambda^4$. The moral is that *by treating ω, ω' asymmetrically in (3.2), we can sometimes gain an extra (small) power of* λ. In [G2], such inner products were always treated symmetrically; that is the chief reason the estimates in the current paper are an improvement.

In the illustration above, had we taken both $|\omega|, |\omega'| \sim r|F|$, the optimal bound would have occurred with $\alpha = 0$. Correspondingly, in many examples (e.g. $\omega = \omega_{(a_0,\mathbf{a})}, \omega' = \omega_{(a_0',\mathbf{a}')}$) in order to get the sharpest estimate one must first break ω, ω' into pieces of different homogeneity in r, then apportion the powers of r differently for each corresponding term in the inner product. This trick occurs often; we will simply refer to it as "using Corollary 4.5".

Finally, we mention the source of the size restriction on α. In many of our proofs, we bound some quantity by an expression of the form $\langle r^\alpha f_1, df_2 \rangle$, which we then integrate by parts . This involves differentiating r^α, giving a term proportional to α, whose size (except for the α in front) is often the same as the object we are trying to bound. In such cases, if we simply cross out α, we get no useful bound at all. But when α is small, we can make use of this fact to reabsorb the term proportional to α into the left-hand side of the original inequality, as for example in (4.8–4.10).

4. Elliptic estimates

In this section we derive and make more precise the principles of the previous section. We then complete the proof of Proposition 2.1.

Throughout this section, given $p \in M$, the function r_p denotes distance to p in the sense of §2 (true distance near the diagonal). Constants c appearing in proofs do not depend upon any data in the hypotheses, unless explicitly stated. All objects in hypotheses are assumed smooth.

We begin with a very general lemma.

Lemma 4.1. *Let (M, g) be a compact Riemannian manifold of dimension m, and let E be a normed vector bundle over M with a metric-compatible connection A. Let $p \in M$ and let $r = r_p$ denote distance to p. Then there exists a constant c, independent of A and p, such that for any section s of E and all $\alpha < m/2$,*

$$\|r^{-\alpha}s\|_2 \le \frac{c}{m-2\alpha}(\|r^{1-\alpha}\nabla^A s\|_2 + \|r^{1-\alpha}s\|_2). \tag{4.1}$$

Proof. The proof is similar to that of [G2, Lemma 3.5], which established that under the same hypotheses

$$\|r^{-1}s\|_2 \le c(\|\nabla^A s\|_2 + \|s\|_2). \tag{4.2}$$

Mimicking the proof in [G2], one first shows that for $f \in C_0^\infty(\mathbf{R}^m)$,

$$\|r^{-\alpha}s\|_2 \le \frac{c}{m-2\alpha}\|r^{1-\alpha}\nabla^A s\|_2$$

(note that the upper bound on α ensures that $r^{-\alpha}f \in L^2$). To pass from this inequality to (4.1), insert a cutoff β as in the proof in [G2], and use the fact that $(1-\beta)/r$ is bounded. □

The hypotheses of the next lemma are motivated by considering the operator d_-^A and the quantities ξ of (2.9–2.12). To make best use of the Green operator G_-^A, we seek certain Sobolev inequalities involving the operator d_-^A rather than ∇^A, or involving the Laplacian $d_-^A(d_-^A)^*$ (on $\Omega_-^2(Ad\,P)$) rather than $(\nabla^A)^*\nabla^A$, but it is crucial that the constants in these inequalities be independent of A. Were we to use the operators ∇^A and $(\nabla^A)^*\nabla^A$, Kato's inequality would lead us to Sobolev inequalities with A-independent constants. What enables us to get the desired type of inequalities with d_-^A is essentially the fact that the Weitzenböck formula for $d_-^A(d_-^A)^*$ (when A is self-dual) does not involve F^A; the identity is of the form (4.4) with $f \equiv 2$ and B_2 proportional to the Riemann tensor of (M, g). It is also important that $d_-^A(d_-^A)^*$ is uniformly bounded below, and that d_-^A is related to ∇^A by a covariantly constant operator—orthogonal projection from $T^*M \otimes T^*M \otimes Ad\, P$ to $(\bigwedge_-^2 T^*M) \otimes Ad\, P$. Thus, what we have in mind in the lemma is the case $E = T^*M \otimes Ad\, P$, $E' = T^*M \otimes T^*M \otimes Ad\, P$, B_1 the projection just described, and $L^A = d_-^A$.

Lemma 4.2. *Let $(M, g), E, A$, and $r = r_p$ be as in Lemma 4.1, let E' be another Riemannian vector bundle over M with metric-compatible connection A', let $B_1 \in \Gamma(Hom(E \otimes T^*M), E')$ be covariantly constant, and consider the first-order linear differential operator*

$$L^A = B_1 \circ \nabla^A : \Gamma(E) \to \Gamma(E'). \tag{4.3}$$

Suppose there exist a positive function f and a section $B_2 \in \Gamma(End(E))$ (both independent of A) such that

$$\Delta^A := (\nabla^A)^*\nabla^A = f(L^A)^*L^A + B_2, \tag{4.4}$$

and that there exists $c_1 > 0$, independent of A, such that the first eigenvalue of $(L^A)^*L^A$ is $\geq c_1$. Then there exist constants $c_i, \alpha_0 > 0$, independent of A and p, such that for all $s \in \Gamma(E)$,

$$c_2\|\nabla^A s\|_2 \leq \|L^A s\|_2 \leq c_3\|\nabla^A s\|_2, \tag{4.5}$$

and if $\dim(M) \geq 2$ *and* $|\alpha| \leq \alpha_0$, *then*

$$\|r^{\alpha-1}s\|_2 + \|r^\alpha \nabla^A s\|_2 + \|r^\alpha s\|_4 \leq c_4\|r^{1+\alpha}(L^A)^*L^A s\|_2. \tag{4.6}$$

PROOF. From (4.3), the second half of (4.5) is trivial. Reciprocally, we have

$$\begin{aligned} \|\nabla^A s\|_2^2 = \langle s, f(L^A)^*L^A s + B_2(s)\rangle &\leq c(\|L^A s\|_2^2 + \|s\|_2^2) \\ &\leq c(\|L^A s\|_2^2 + c_1^{-1}\|L^A s\|_2^2), \end{aligned} \tag{4.7}$$

yielding the first half of (4.5).

Moving to (4.6), we first we use Lemma 4.1 to find

$$\|r^{\alpha-1}s\|_2 \leq c\left(\|r^\alpha \nabla^A s\|_2 + \|r^\alpha s\|_2\right). \tag{4.8}$$

Here and below, all constants c are independent of α for $|\alpha|$ sufficiently small. Next, integrating by parts we have

$$\|r^\alpha \nabla^A s\|_2^2 \le c\|r^{\alpha-1}s\|_2 \left(|\alpha|\ \|r^\alpha \nabla^A s\|_2 + \|r^{\alpha+1}\Delta^A s\|_2\right). \tag{4.9}$$

Using (4.8) and the inequality $cxy \le \frac{1}{2}x^2 + c'y^2$, we then find that, for $|\alpha|$ sufficiently small,

$$\|r^\alpha \nabla^A s\|_2 \le c\left(\|r^{\alpha+1}\Delta^A s\|_2 + \|r^\alpha s\|_2\right) \le c\left(\|r^{\alpha+1}(L^A)^* L^A s\|_2 + \|r^\alpha s\|_2\right). \tag{4.10}$$

It remains to estimate the term $\|r^\alpha s\|_2$.

Since B_1 is assumed covariantly constant, there exists c independent of A, A' for which the second-order operator $(L^A)^*L^A$ satisfies $|(L^A)^*L^A(r^\alpha s) - r^\alpha (L^A)^*L^A s| \le c|\alpha|(|r^{\alpha-1}\nabla^A s| + |r^{\alpha-2}s|)$ pointwise. Using the hypothesis on the first eigenvalue of $(L^A)^*L^A$, we therefore have

$$\begin{aligned} \|r^\alpha s\|_2^2 &\le c_1^{-1}\langle r^\alpha s, r^\alpha (L^A)^*L^A s + [(L^A)^*L^A(r^\alpha s) - r^\alpha (L^A)^*L^A s]\rangle \\ &\le c_1^{-1}\|r^{\alpha-1}s\|_2 \left(\|r^{\alpha+1}(L^A)^*L^A s\|_2 + \|r[(L^A)^*L^A(r^\alpha s) - r^\alpha (L^A)^*L^A s]\|_2\right) \\ &\le c\left(\delta^{-1}\|r^{\alpha+1}(L^A)^*L^A s\|_2^2 + (\delta + |\alpha|)\|r^{\alpha-1}s\|_2^2 + |\alpha|\ \|r^\alpha \nabla^A s\|_2^2\right). \end{aligned} \tag{4.11}$$

Now use (4.8) to bound $\|r^{\alpha-1}s\|_2$. Then, taking δ and $|\alpha|$ small enough, we obtain

$$\|r^\alpha s\|_2 \le c\left(\delta^{-1}\|r^{\alpha+1}(L^A)^*L^A s\|_2 + (\delta + |\alpha|)\ \|r^\alpha \nabla^A s\|_2\right). \tag{4.12}$$

Inserting this into (4.10) and decreasing δ and $|\alpha|$ as needed, we arrive at $\|r^\alpha \nabla^A s\|_2 \le c\|r^{\alpha+1}(L^A)^*L^A s\|_2$, which is one of the three bounds asserted in (4.6). Inserting this bound into (4.12), and the result into (4.8), we obtain another of the three asserted bounds. Finally, the Sobolev and Kato inequalities give

$$\|r^\alpha s\|_4 \le c(\|r^\alpha s\|_2 + \|\nabla^A(r^\alpha s)\|_2) \le c(\|r^{\alpha-1}s\|_2 + \|r^\alpha \nabla^A s_2\|),$$

and applying the two established parts of (4.6) we obtain the remaining part. □

For the remainder of this section, we specialize to four-dimensional manifolds M as in the Introduction and to "collar connections" in $\mathcal{M}_{\lambda_0}$. The scale λ_0 is taken small enough that the estimates of [G2] (e.g. (3.1)) apply.

The next lemma generalizes Lemma 5.3 of [G2]. It, and the subsequent corollary, quantify "principle 4" of the previous section. The exponent α_0 in this lemma is the α that eventually appears in Theorem 1.2.

Lemma 4.3. *Given $[A] \in \mathcal{M}_{\lambda_0}$, let r denote distance to the center point p_A of A. There exist $\alpha_0 > 0$, and for any $n \geq 0$ constants $c(n)$, such that for all $A \in \mathcal{M}_{\lambda_0}$, $\omega \in \Omega^2_-(Ad\,P)$, and $|\alpha|, |\alpha'| \leq \alpha_0$, we have*

(4.13)

$$(a) \quad \|r^{\alpha-1+n} G^A_- \omega\|_2 + \|r^{\alpha+n} G^A_- \omega\|_4 + \|r^{\alpha+n} \nabla^A G^A_- \omega\|_2 \leq c(n) \|r^{\alpha+1} \omega\|_2,$$

$$(4.14) \qquad (b) \quad \|r^{\alpha+1+n} \nabla^A G^A_- \omega\|_4 + \|r^{\alpha+1+n} \nabla^A \nabla^A G^A_- \omega\|_2 \leq c(n) \|r^{\alpha+1} \omega\|_2,$$

and

$$(c) \quad \|r^{\alpha-1+n} \nabla^A G^A_- \omega\|_2 + \|r^{\alpha+n} \nabla^A G^A_- \omega\|_4 + \|r^{\alpha+n} \nabla^A \nabla^A G^A_- \omega\|_2$$

$$(4.15) \qquad \leq c(n) \left(\|r^\alpha \omega\|_2 + \lambda^{-1+\alpha'} \|r^{1+\alpha-\alpha'} \omega\|_2 \right).$$

(The constants here may depend on an upper bound on α_0, but are otherwise independent of α, α'.)

Furthermore, the bounds (4.13-4.15) hold with $\nabla^A G^A_- \omega$ replaced by $(d^A_-)^* G^A_- \omega$.

PROOF. Since $r^{\alpha+n} < c(n) r^\alpha$ it suffices to prove the inequalities for $n = 0$. Below, we will write η for $G^A_- \omega$.

The operator $(d^A_-)^* : \Gamma(\bigwedge^2_- T^*M \otimes Ad\,P) \to \Gamma(T^*M \otimes Ad\,P)$ is of the form L^A considered in Lemma 4.2, and the eigenvalue hypothesis in Lemma 4.2 is also satisfied. Thus from (4.6) we obtain (4.13).

For (4.14-4.15) we start with the Sobolev and Kato inequalities, which give

$$\begin{aligned} \|r^\alpha \nabla^A \eta\|_4^2 &\leq c \left(\|r^\alpha \nabla^A \eta\|_2^2 + \|\nabla^A (r^\alpha \nabla^A \eta)\|_2^2 \right) \\ &\leq c \left(\|r^{\alpha-1} \nabla^A \eta\|_2^2 + \|r^\alpha \nabla^A \nabla^A \eta\|_2^2 \right). \end{aligned} \tag{4.16}$$

Now

$$\begin{aligned} \|r^\alpha \nabla^A \nabla^A \eta\|_2^2 &= \langle \nabla^A \eta, (\nabla^A)^* (r^{2\alpha} \nabla^A \nabla^A \eta) \rangle \\ &= \langle \nabla^A \eta, r^{2\alpha} \Delta^A (\nabla^A \eta) \rangle - \langle \nabla^A \eta, \nabla^A_{\mathrm{grad}(r^{2\alpha})} \nabla^A \eta \rangle. \end{aligned} \tag{4.17}$$

Because A is Yang-Mills, [G2, Lemma 3.1] applied to the bundle $E = \bigwedge^2_- T^*M \otimes Ad\,P$, yields

$$\Delta^A (\nabla^A \eta) = \nabla^A (\Delta^A \eta) + \mathcal{R}_1(\eta) + \mathcal{R}_2(\nabla^A \eta) + \mathcal{F}(\nabla^A \eta) \tag{4.18}$$

where the $\mathcal{R}_i$ are universal endomorphisms proportional to the Riemann tensor and its derivatives, and $\mathcal{F}$ is a universal endomorphism proportional to F_A. Now insert (4.18) into (4.17), integrate by parts the inner product involving $\nabla^A (\Delta^A \eta)$, and use the Weitzenböck formula for anti-self-dual 2-forms (see equation (2.13) of [G2]). The result is that for any α' we find

$$\begin{aligned}
\|r^\alpha \nabla^A \nabla^A \eta\|_2^2 &= \langle r^{2\alpha}(\Delta_-^A \eta + \mathcal{R}_3(\eta)) - \iota_{\mathrm{grad}(r^{2\alpha})} \nabla^A \eta, \Delta_-^A \eta + \mathcal{R}_3(\eta)\rangle \\
&\quad + \langle r^{2\alpha}\nabla^A \eta, \mathcal{R}_1(\eta) + \mathcal{R}_2(\nabla^A \eta) + \mathcal{F}(\nabla^A \eta)\rangle \\
&\quad - \langle \nabla^A \eta, \nabla^A_{\mathrm{grad}(r^{2\alpha})} \nabla^A \eta\rangle \\
&\leq c\,\big(\|r^\alpha \omega\|_2 (\|r^\alpha \omega\|_2 + \|r^\alpha \eta\|_2 + |\alpha|\ \|r^{\alpha-1}\nabla^A \eta\|_2) \\
&\quad + \|r^\alpha \nabla^A \eta\|_2 (\|r^\alpha \eta\|_2 + |\alpha|\ \|r^{\alpha-1}\eta\|_2 + \|r^\alpha \nabla^A \eta\|_2) \\
&\quad + \|r^\alpha \eta\|_2^2 + |\alpha|\ \|r^\alpha \nabla^A \nabla^A \eta\|_2 \|r^{\alpha-1}\nabla^A \eta\|_2 \\
&\quad + \|r^{1-\alpha'} F^A\|_4 \|r^{\alpha+\alpha'-1}\nabla^A \eta\|_2 \|r^\alpha \nabla^A \eta\|_4\big).
\end{aligned}$$

Take $\alpha' \geq 0$. Using (3.1) and the fact that $|F^A|$ is uniformly small for $r \geq r_0$, we have $\|r^{1-\alpha'} F^A\|_4 \leq c\lambda^{-\alpha'}$, so we can massage the above inequality into the form

$$\begin{aligned}
\|r^\alpha \nabla^A \nabla^A \eta\|_2^2 &\leq c\,\big(\|r^\alpha \omega\|_2^2 + \|r^{\alpha-1}\eta\|_2^2 + \|r^\alpha \nabla^A \eta\|_2^2 \\
&\quad + \delta_1 \|r^\alpha \nabla^A \nabla^A \eta\|_2^2 + \delta_1^{-1}|\alpha|^2 \|r^{\alpha-1}\nabla^A \eta\|_2^2 \\
&\quad + \delta_2 \|r^\alpha \nabla^A \eta\|_4^2 + \delta_2^{-1}\lambda^{-2\alpha'}\|r^{\alpha+\alpha'-1}\nabla^A \eta\|_2^2\big)
\end{aligned}$$

for any $\delta_1, \delta_2 > 0$. By taking δ_1 small enough we then obtain

$$\begin{aligned}
\|r^\alpha \nabla^A \nabla^A \eta\|_2^2 &\leq c\,\big(\|r^\alpha \omega\|_2^2 + \|r^{\alpha-1}\eta\|_2^2 + \|r^\alpha \nabla^A \eta\|_2^2 + |\alpha|^2 \|r^{\alpha-1}\nabla^A \eta\|_2^2 \\
&\quad + \delta_2 \|r^\alpha \nabla^A \eta\|_4^2 + \delta_2^{-1}\lambda^{-2\alpha'}\|r^{\alpha+\alpha'-1}\nabla^A \eta\|_2^2\big),
\end{aligned}$$

(4.19)

and inserting this bound into (4.16) we obtain a bound on $\|r^\alpha \nabla^A \eta\|_4^2$ with the same right-hand side as (4.19) (except for the actual value of c). Taking δ_2 small enough we then find

$$\begin{aligned}
\|r^\alpha \nabla^A \eta\|_4 + \|r^\alpha \nabla^A \nabla^A \eta\|_2 &\leq c\,\big(\|r^\alpha \omega\|_2 + \|r^{\alpha-1}\eta\|_2 + \|r^\alpha \nabla^A \eta\|_2 \\
&\quad + |\alpha|\ \|r^{\alpha-1}\nabla^A \eta\|_2 + \lambda^{-\alpha'}\|r^{\alpha+\alpha'-1}\nabla^A \eta\|_2\big).
\end{aligned}$$

(4.20)

If we now take $\alpha' = 0$ and α close to 1, and use (4.13), we obtain (4.14). If instead we take α close to 0, we can apply (4.1) to derive that $\|r^{\alpha-1}\nabla^A \eta\|_2 \leq c\left(\|r^\alpha \nabla^A \nabla^A \eta\|_2 + \|r^\alpha \nabla^A \eta\|_2\right)$, and (4.13) to derive that $\|r^{\alpha-1}\eta\|_2 + \|r^\alpha \nabla^A \eta\|_2 \leq c\|r^\alpha \omega\|_2$. By taking $|\alpha|$ smaller still, if necessary, we find that (4.20) implies

$$\|r^{\alpha-1}\nabla^A \eta\|_2 + \|r^\alpha \nabla^A \eta\|_4 + \|r^\alpha \nabla^A \nabla^A \eta\|_2 \leq c\left(\|r^\alpha \omega\|_2 + \lambda^{-\alpha'}\|r^{\alpha+\alpha'-1}\nabla^A \eta\|_2\right).$$

If we now take α' close enough to 1, (4.13) applies and we obtain (4.15).

The final statement of the lemma follows because, once again, $(d_-^A)^*$ is just ∇^A followed by a covariantly constant projection. □

One application of Lemma 4.3 is the following corollary, which extends [G2, Lemma 5.1] while greatly simplifying the proof of one part.

Corollary 4.4. *There exist $\alpha > 0$ and constants c such that for $A \in \mathcal{M}_{\lambda_0}$ and $\xi := \xi_{(a_0,\mathbf{a})}$ as in (2.12), $\|r^m\xi\|_2 + \|r^{m+1}\xi\|_4 + \|r^{m+1}\nabla^A\xi\|_2 \le$*

$$c\left(|a_0|\lambda + |\mathbf{a}|\begin{cases}\lambda^{2-|m|}, & -\alpha \le m < 0\\ \lambda^2|\log\lambda|^{1/2}, & m = 0\\ \lambda^2, & m > 0\end{cases}\right) \tag{4.21}$$

and

$$\|r^m\xi\|_4 + \|r^m\nabla^A\xi\|_2 \le c\left(|\mathbf{a}|\lambda^{1+m} + |a_0|\lambda^{\alpha}\right), \quad |m| \le \alpha. \tag{4.22}$$

As in Lemma 4.3, r here denotes distance to the center point p_A.

Proof. Since $\xi = -(d_-^A)^*G_-^A\omega_{(a_0,\mathbf{a})}$ and $|\omega_{(a_0,\mathbf{a})}| \le c\chi_{[0,2r_0]}(|\mathbf{a}|r + |a_0|\lambda^{-1}r^2)|F^A|$, this follows from Lemma 4.3 and (3.1). The key to obtaining the sharpest estimates in certain cases is to take $\alpha' > 0$ when applying (4.15). □

Another very important corollary of Lemma 4.3 is that when computing an inner product involving a Green operator, we gain two powers of the distance function in a very flexible way:

Corollary 4.5. *There exists $\alpha_0 > 0$ and a constant c such that if $|\alpha| \le \alpha_0$ and $\omega, \omega' \in \Omega_-^2(Ad\,P)$ then*

$$|\langle\omega, G_-^A\omega'\rangle| \le c\|r^{1+\alpha}\omega\|_2\|r^{1-\alpha}\omega'\|_2 \tag{4.23}$$

Proof. For $|\alpha|$ sufficiently small, we can use (4.13) to find

$$|\langle\omega, G_-^A\omega'\rangle| = |\langle r^{1+\alpha}\omega, r^{-1-\alpha}G_-^A\omega'\rangle| \le c\|r^{1+\alpha}\omega\|_2\|r^{1-\alpha}\omega'\|_2.$$

□

It is not at all clear how to obtain such an estimate, uniform in A as $\lambda \to 0$, using the Green *function*.

An immediate application of Corollary 4.5 is:

Proof of Proposition 2.1, completed. Applying Corollary 4.5 to (2.18), we have

$$|\hat{a}_0| \le c\lambda\|r^{1-\alpha}\omega_{(a_0,\mathbf{a})}\|_2\|r^{1+\alpha}\omega_\gamma\|_2.$$

Now use the pointwise bounds (2.13) and

$$|\nabla^j \gamma| \le c_j \lambda^{-j} \tag{4.24}$$

(which implies $|\omega_\gamma| \le c\lambda^{-2}|F|$), and the L^2 bounds (3.1), to find

$$|\hat{a}_0| \le c(|\mathbf{a}|\lambda^2 + |a_0|\lambda^{1+\alpha}). \tag{4.25}$$

The L^2 inner product in (2.26) is handled similarly, using the pointwise bound $|H^0\gamma_i| \le c\lambda^{-3}$ in place of the bound $|H^0\gamma| \le c\lambda^{-2}$ used above. The result is

$$\bar{\lambda}^2 \left|\langle G_-^A \omega_{(a_0,\mathbf{a})}, H^0\gamma_i \natural F\rangle\right| \le c(|\mathbf{a}|\lambda^2 + |a_0|\lambda^{1+\alpha}). \tag{4.26}$$

Now insert this bound and (4.25) into (2.26) to complete the proof of (2.14). □

5. How wiggly is $\tilde{H}_A$ relative to H_A?

Let $\mathbf{g}_\Psi := (\Psi^{-1})^*\mathbf{g}$, extended continuously to $[0,\lambda_0]\times M$ by setting $\mathbf{g}_\Psi|_{\{0\}\times M} = 4\pi^2(2d\lambda^2 \oplus g)$. Theorems 1.2–1.3 are really theorems about the metric $\mathbf{g}_\Psi$. To test whether $\mathbf{g}_\Psi$ is C^1 (for example), one needs to show that the functions $\mathbf{g}_\Psi(\mathbf{e}_\mu, \mathbf{e}_\nu)$ are continuously differentiable for some (local) basis of C^1 sections $\{\mathbf{e}_\mu\}$ of the tangent bundle of $[0,\lambda_0]\times M$. The problem with the obvious choice $\mathbf{e}_\mu = \frac{\partial}{\partial x^\mu}$, where $\{x^\mu\}$ are local coordinates, is that there is no obvious way to do the computation directly. This is exactly what the approximate tangent space is for: if we replace $\frac{\partial}{\partial x^\mu}$ by its image under the approximate identity map $\Psi_* \circ \mathcal{I}$, the inner products become more effectively computable. But the problem now is that *a priori* there is no guarantee that the vector fields $(\Psi_* \circ \mathcal{I})\frac{\partial}{\partial x^\mu}$, extended to the boundary by setting $\Psi_* \circ \mathcal{I} = Id$ along $\{0\}\times M$, are still C^1.

In the next two sections of this paper we will show a little more: these extended vector fields and their inner products are actually $C^{1,\alpha}$ for small α. Note that these objects are C^∞ on the interior, so the only potential problem is at the boundary.

Showing continuous differentiability of all vector fields of the form $(\Psi_* \circ \mathcal{I})\frac{\partial}{\partial x^\mu}$ is equivalent to showing that (i) $\Psi_* \circ \mathcal{I}$ extends continuously to $[0,\lambda_0)\times M$, and (ii) $\nabla(\Psi_* \circ \mathcal{I})$ extends C^α to $[0,\lambda_0)\times M$ (where ∇ is the Levi-Civita connection of the product metric), for this implies that the change-of-basis matrix relating $\{(\Psi_* \circ \mathcal{I})\frac{\partial}{\partial x^\mu}\}$ to $\{\frac{\partial}{\partial x^\mu}\}$ extends to a $C^{1,\alpha}$ function on $[0,\lambda_0)\times M$. As statement (i) follows immediately from (2.14), it remains only to establish (ii). This can be reduced to showing that $\|\nabla(\Psi_* \circ \mathcal{I})\| \le c\lambda^\alpha$. In view of (2.14) again, to bound $\nabla(\Psi_* \circ \mathcal{I})$ pointwise it suffices to bound the pointwise norm of local vector fields on $(0,\lambda_0)\times M$ of the form $\nabla_{\Psi_*\mathcal{I}_A(b_0,\mathbf{b})}(\Psi_*\mathcal{I}(a_0,\mathbf{a}))$, for all local vector fields $\mathbf{a}, \mathbf{b}$ defined near $p(A)$ with $\mathbf{a}$ covariantly constant at $p(A)$, and all constants a_0, b_0.

Equivalently, it suffices to bound the norm of $\nabla_{\Psi_* \mathcal{I}_A(b_0,\mathbf{b})}(\mathcal{L}(a_0,\mathbf{a}))$ for all such $(a_0,\mathbf{a}),(b_0,\mathbf{b})$, which is what we will do.

To get started, fix $p \in M$ and a normal coordinate system $\{x^i\}$ defined on a geodesic ball B centered at p. For each $\mathbf{a} := \mathbf{a}(p) \in T_pM$, extend $\mathbf{a}(p)$ to a vector field $\mathbf{a}(\cdot)$ on B by parallel translating $\mathbf{a}(p) := \mathbf{a}$ along radial geodesics from p. For each $p' \in B$, a function $\phi_{\mathbf{a}(p')}(\cdot)$ (essentially linear in normal coordinates based at p', not at p) is thereby determined on some neighborhood of p', as is the gradient $Z_{\mathbf{a}(p')}$ of $\phi_{\mathbf{a}(p')}$.

Given a curve $\sigma(t) = (\lambda_t, p_t) = \Psi^{-1}(A_t)$ in $(0,\lambda_0)\times M$, say with initial derivative $(\dot\lambda,\dot p) = \Psi_*\pi\tilde Z^A_{(b_0,\mathbf{b})}$, the procedure above determines a covering section $(a_{0,t}\cong a_0, \mathbf{a}_t = \mathbf{a}(p_t))$ parallel along σ. From this we obtain a parametrized family of vector fields $Z_t = Z_{\mathbf{a}_t}$ and a corresponding curve of approximate tangent vectors $\tilde Z_t = \iota_{Z_t}F_t$ in $\Omega^1(Ad\,P)$.

We will prove the following:

Proposition 5.1. *Let σ be as above and write*

$$((\hat a_0)^{\cdot},(\hat{\mathbf{a}})^{\cdot}) := \nabla_{\sigma'(t)}(\hat a_0(t),\hat{\mathbf{a}}(t))\big|_{t=0}$$

(where ∇ denotes the Levi-Civita connection of the product metric on $(0,\lambda_0)\times M$). Then

$$|((\hat a_0)^{\cdot},(\hat{\mathbf{a}})^{\cdot})| \le c(|\mathbf{b}|+|b_0|)(|\mathbf{a}|\lambda+|a_0|\lambda^\alpha). \tag{5.1}$$

Furthermore, if we set $\mathbf{X} = \pi\tilde Z_{(a_0,\mathbf{a})}, \mathbf{Y} = \pi\tilde Z_{(a_0',\mathbf{a}')}$ and vary $(a_0,\mathbf{a}),(a_0',\mathbf{a}')$ as above, then at $t=0$, the variation of $\langle\mathbf{X},\mathbf{Y}\rangle$ satisfies

$$\begin{aligned}|\langle\mathbf{X},\mathbf{Y}\rangle^{\cdot}| \quad\le\quad & c\Big(|\mathbf{a}||\mathbf{a}'|(|\mathbf{b}|\lambda^3|\log\lambda|^{1/2}+|b_0|\lambda^3|\log\lambda|)\\ &+(|\mathbf{a}||a_0'|+|a_0||\mathbf{a}'|+|a_0||a_0'|)(|\mathbf{b}|\lambda+|b_0|\lambda^\alpha)\Big).\end{aligned} \tag{5.2}$$

The proof of this proposition is long and technical, and we devote the next section to it. In the proof we estimate $|(\hat a_0)^{\cdot}|$ and $|(\hat{\mathbf{a}})^{\cdot}|$ separately, but end up with precisely the same expression (5.1) for both pieces.

It is useful to phrase (5.1) in terms of a change-of-basis matrix. Let $\{\mathbf{e}_i\}_1^4$ be a local basis of TM, covariantly constant along radial geodesics through p, and (abusing notation) extend this to a local basis $\{\mathbf{e}_\mu\}_0^4$ of $T([0,\lambda_0)\times M)$ by setting $\mathbf{e}_0 = \partial/\partial\lambda$. Then on the interior we have

$$\mathbf{e}'_\mu := (\Psi_*\circ\mathcal{I})(\mathbf{e}_\mu) = C^\nu_\mu\mathbf{e}_\nu \tag{5.3}$$

for some matrix of functions C. Together with (2.14), the bounds above imply that for $0 \le \mu \le 4$ and $1 \le i \le 4$,

$$|C_0^\mu - \delta_{\mu 0}| \le c\lambda^{1+\alpha}, \tag{5.4}$$

$$|C_i^\mu - \delta_{\mu i}| \le c\lambda^2, \tag{5.5}$$

$$|\nabla C_0^\mu| \le c\lambda^\alpha \tag{5.6}$$

$$|\nabla C_i^\mu| \le c\lambda \tag{5.7}$$

on a small enough neighborhood of $(0, p)$.

In view of the discussion above, as a corollary of Propositions 2.1 and 5.1 we immediately have the following theorem. (Of course, the anisotropic bounds above are stronger than (5.8) below.)

Theorem 5.2. *For λ_0 sufficiently small, in the collar we have the pointwise bounds*

$$||\mathcal{L}|| \le c\lambda^{1+\alpha}, \quad ||\nabla\mathcal{L}|| \le c\lambda^\alpha \tag{5.8}$$

(where α and ∇ are as above). Hence, given a smooth local basis v_i of TM, the image under $\Psi_* \circ \mathcal{I}$ of the basis $\{\frac{\partial}{\partial\lambda}, v_i\}$ of $T((0,\lambda_0)\times M)$ extends to a $C^{1,\alpha}$ local basis of $T([0,\lambda_0)\times M)$.

PROOF. The only assertion needing comment is the one concerning the Hölder exponent. Since the first derivatives of the coefficients C_ν^μ are bounded by $c\lambda^\alpha$, clearly these derivatives are Hölder continuous with exponent α at the boundary itself. On the interior, these derivatives are C^∞, so given any interior point, by taking a small enough neighborhood we obtain an α-Hölder condition with arbitrarily small constant. By local compactness, we can therefore choose the constants to be locally uniform on $[0,\lambda_0)\times M$. □

In §7 we will use Proposition 5.1, the bounds (5.4–5.7), and the proof rather than the statement of Theorem 5.2, to establish Theorems 1.2–1.3.

The proof of Proposition 5.1 involves computing the t-derivative of the vector field $\tilde{Z}_t \in \Omega^1(Ad\,P)$, for which we need the t-derivative of the vector field $Z_t \in \Gamma(TM)$. An advantage of setting up Z_t as a gradient vector field, as we've done, is that to vary the vector field we need only vary the function $\dot{Z}_t := \mathrm{grad}(\dot{\phi}_t)$, where $\phi_t = \phi_{(a_0,\mathbf{a}_t)}$.

Thus our proof entails varying the functions ϕ_t, which in turn entails varying the normal coordinate system as the base point changes. Note that this requires us to vary the frame defining the normal coordinate system as well, and there is choice here. We will choose to vary the frame by parallel translating the initial frame at $t = 0$ along radial geodesics from the initial center point. Still, the computation of the change in normal coordinates is a rather involved exercise in Riemannian geometry, which we have left to the appendix. The chief result is the following (see Propositions 8.6 and 8.7; varying the distance function is comparatively easy). Below, by "$O_{\text{strong}}(r^n)$", we mean a function that is $O(r^n)$ uniformly in the center point p of the normal coordinate system and whose m^{th} covariant derivative, holding p fixed and varying the point q at which coordinates are evaluated, is $O(r^{n-m})$ uniformly in p (see Definition 1 in §8).

Lemma 5.3. *Let $\{p_t\}$ be a curve with initial point p and initial tangent vector v, and let q be a fixed point in a normal-coordinate neighborhood of p. If we vary normal coordinates according to the prescription above, then*

$$\dot{x}^i(q) := \frac{d}{dt} x_t^i(q)\bigg|_{t=0} = -v^i + O_{\text{strong}}(r^2|v|) \tag{5.9}$$

where $r = \text{dist}(p, q)$. Furthermore

$$\dot{r} = -\frac{v^i x^i}{r} \tag{5.10}$$

exactly (for $r \neq 0$).

The "O_{strong}" assertion in this lemma is important. For example, among the quantities we need to vary is $H^0 x^i$, and we will use the fact that $(H^0 x^i)^\cdot = H^0(\dot{x}^i)$. Thus it is essential to have bounds on the space-derivatives of $\dot{x}^i$.

6. The proof of Proposition 5.1

Our strategy is not elegant; we simply differentiate (2.16) and (2.24), obtain a lot of terms, and bound them all. However, the computation is nontrivial, relying in places on some subtle localization and cancellation phenomena (see Lemmas 6.2, 6.3, and Corollary 6.4). Also we require the variation not just of normal-coordinate functions, but of the functions ζ_{ij}, ψ_{ij} of (2.21–2.22); these variations are bounded in the appendix.

Below we use a dot to denote time-derivative at $t = 0$. Throughout this section, α has the value α_0 of Lemma 4.3 and its corollaries. We note that if the initial tangent vector to the curve (λ_t, p_t) is $(\dot{\lambda}, \dot{p}) = \Psi_* \pi \tilde{Z}^{A_0}_{(b_0, \mathbf{b})} = -\mathcal{I}_A(b_0, \mathbf{b})$, then by definition (see (2.7–2.8)) we have

$$\dot{\lambda} = -(b_0 + \hat{b_0}), \tag{6.1}$$

$$\dot{p} = -(\mathbf{b} + \hat{\mathbf{b}}). \tag{6.2}$$

Hence using (2.14), we have

$$|\dot{\lambda}| \leq |b_0| + |\hat{b_0}| \quad \leq \quad c(|\mathbf{b}|\lambda^2 + |b_0|), \tag{6.3}$$

$$|\dot{p}| \leq |\mathbf{b}| + |\hat{\mathbf{b}}| \quad \leq \quad c(|\mathbf{b}| + |b_0|\lambda^{1+\alpha}). \tag{6.4}$$

6.1. **Estimating** $|(\hat{a}_0)^{\cdot}|$.

From (2.16) and (2.17) we have

$$\hat{a}_0(t) \;=\; -4\lambda_t \,\frac{\langle H^0\phi_t \natural F_t, G_-^{A_t}(H^0\gamma_t \natural F_t)\rangle}{\int(\nabla\gamma_t, \nabla(r_t^2))|F_t|^2} := -4\lambda_t \,\frac{num(t)}{denom(t)} \tag{6.5}$$

(where $\phi_t = \phi_{(a_0,\mathbf{a}_t)}$). It will take us some time to estimate the derivatives of $num(t)$ and $denom(t)$. In the end we will find that $|(num)^{\cdot}| \leq c(|\mathbf{b}| + |b_0|)(|\mathbf{a}| + |a_0|\lambda^{-1+\alpha})$ and $|(denom)^{\cdot}| \leq c(|\mathbf{b}|\lambda + |b_0|\lambda^{\alpha})$, which with (4.25) implies that the contribution to $(\hat{a}_0)^{\cdot}$ from differentiating the numerator swamps the contribution from differentiating the denominator (by more than a full power of λ).

Differentiating the numerator in (6.5).

First, we have

$$\begin{aligned}(num)^{\cdot} \;=\;& \langle H^0\dot{\phi} \natural F \,,\, G_-^A(H^0\gamma \natural F)\rangle + \langle H^0\phi \natural \dot{F} \,,\, G_-^A(H^0\gamma \natural F)\rangle \\ &+\langle H^0\phi \natural F \,,\, G_-^A(H^0\dot{\gamma} \natural F)\rangle + \langle H^0\phi \natural F \,,\, G_-^A(H^0\gamma \natural \dot{F})\rangle \\ &+\langle H^0\phi \natural F \,,\, (G_-)^{\cdot}(H^0\gamma \natural F)\rangle\end{aligned} \tag{6.6}$$

(where we have omitted numerous sub- and super-scripts at $t = 0$). We will estimate the five inner products above using (4.23). We start with the following lemma concerning the time-derivatives of ϕ_t and F_t.

Lemma 6.1. *(a) Letting $b_i, \hat{b}_i$ denote the components of $\mathbf{b}, \hat{\mathbf{b}}$ in the base frame at p, we have*

$$\dot{r} = (b_i + \hat{b}_i)x^i/r = O_{\text{strong}}\left((|\mathbf{b}| + |b_0|\lambda^{1+\alpha})\right). \tag{6.7}$$

(b) For any n,

$$\dot{\beta} = O_{\text{strong}}\left((|\mathbf{b}| + |b_0|\lambda^{1+\alpha})r^n\right) \tag{6.8}$$

(c)

$$|(H^0\gamma)^{\cdot}| \leq c\lambda^{-3}(|\mathbf{b}| + |b_0|) \tag{6.9}$$

(d) For any n,

(6.10)

$$(\hat{\phi})^{\cdot} = \phi_{\mathbf{b}} + \phi_{\hat{\mathbf{b}}} + O_{\text{strong}}((|\mathbf{b}| + |b_0|\lambda^{1+\alpha})r^n) = O_{\text{strong}}((|\mathbf{b}| + |b_0|\lambda^{1+\alpha})r)$$

and

$$\dot{\phi_{\mathbf{a}}} \quad = \quad (\mathbf{a}, \mathbf{b}+\hat{\mathbf{b}}) + O_{\mathrm{strong}}\left(|\mathbf{a}|(|\mathbf{b}|+|b_0|\lambda^{1+\alpha})r^2\right). \tag{6.11}$$

(e)

(6.12)
$$|H^0(\phi_{(a_0,\mathbf{a})})^{\cdot}| = O_{\mathrm{strong}}\left(|\mathbf{a}||\mathbf{b}| + |\mathbf{a}||b_0|\lambda^{1+\alpha} + |a_0||\mathbf{b}|\lambda^{-1}r + |a_0||b_0|(\lambda^{\alpha}r + \lambda^{-2}r^2)\right).$$

(f)

$$|\dot{F}| \le c\left\{|\mathbf{b}|(r|F| + |\nabla F|) + |\nabla\xi_{\mathbf{b}}| + |b_0|\lambda^{-1}(|F| + r|\nabla F| + |\nabla\hat{\xi}|)\right\}. \tag{6.13}$$

PROOF. (a) This follows from (5.10), with $v = \dot{p}$.

(b) We have $\dot{\beta} = \beta'(r)\dot{r}$. Since $\beta'(r) = O(r^n)$ for any n (by virtue of vanishing for $r \le r_0$), the result follows from part (a).

(c) We have

$$\begin{aligned}\dot{\gamma} = \frac{d}{dt}(b(r_t/(K\lambda_t))\Big|_{t=0} &= b'(r/\overline{\lambda})K^{-1}(\dot{r}\lambda^{-1} - r\lambda^{-2}\dot{\lambda}) \\ &= b'(r/\overline{\lambda})\cdot O_{\mathrm{strong}}(\lambda^{-1}|\dot{p}| + r\lambda^{-2}|\dot{\lambda}|)\end{aligned}$$

As in (b), for any n we have $|b'(r/\overline{\lambda})| = O_{\mathrm{strong}}((r/\lambda)^n)$, so the above implies that

$$\dot{\gamma} = O_{\mathrm{strong}}((|\dot{p}| + |\dot{\lambda}|)r^2\lambda^{-3}), \tag{6.14}$$

and hence $|(H^0\gamma)^{\cdot}| = |H^0(\dot{\gamma})| \le c(|\dot{p}| + |\dot{\lambda}|)\lambda^{-3}$. The bound (6.9) now follows from (6.3–6.4).

(d) Differentiating (2.2) and using (6.7) and (6.8) we have $(\hat{\phi})^{\cdot} = \beta(b_i + \hat{b}_i)x^i + \frac{1}{2}\dot{\beta}r^2$. Using (6.8) we obtain (6.10).

To derive (6.11) requires determining the variation of normal coordinate system as p varies. This somewhat lengthy exercise in Riemannian geometry which is done in the appendix, culminating in equations (8.20). As in part (a), the vector v of the appendix is $\dot{p} = -(\mathbf{b}+\hat{\mathbf{b}})$. Thus, using (8.20) we have

$$\dot{\phi_{\mathbf{a}}} = \dot{\beta}a^i x^i + \beta a^i\left((b_i + \hat{b}_i) + O_{\mathrm{strong}}(r^2|\dot{p}|)\right),$$

leading to (6.11).

(e) First use part (d) to compute $(\phi_{(a_0,\mathbf{a})})^{\cdot}$ (remembering to vary λ). From this compute $|(H^0\phi_{(a_0,\mathbf{a})})^{\cdot}| = |H^0((\phi_{(a_0,\mathbf{a})})^{\cdot})|$, using (6.3) and the fact that

$$|H^0\phi_{(a_0,\mathbf{a})}| = O_{\mathrm{strong}}(|\mathbf{a}|r + |a_0|\lambda^{-1}r^2). \tag{6.15}$$

(f) Since $\dot F = d^A\eta$, where η is the initial tangent vector to the curve $\Psi^{-1}(\lambda_t, p_t)$ in $\mathcal{M}$, we have

$$\dot F = d^A\pi\tilde Z = d^A\tilde Z + d^A\xi_Z,$$

where $Z = -Z_{(b_0,\mathbf{b})}$ (see the first paragraph of this section). Using the pointwise bounds

$$|Z_{(a_0,\mathbf{a})}| \leq c(|\mathbf{a}| + |a_0|\lambda^{-1}r), \tag{6.16}$$

$$|\nabla Z_{(a_0,\mathbf{a})}| \leq c(|\mathbf{a}|r + |a_0|\lambda^{-1}), \tag{6.17}$$

the result follows. □

Using the lemma above, (3.1), Corollary 4.4, and the pointwise bounds (6.16–6.17), we can deduce the bounds summarized in Table 1. (Some entries in the table are for objects we will define later.) In the table and in subsequent calculations, we use the notation

$$\omega_{(a_0,\mathbf{a})} = H^0\phi_{(a_0,\mathbf{a})} \natural F, \qquad \omega_\gamma = H^0\gamma \natural F.$$

Henceforth, we will freely use the data in this table without reference, combining these bounds with Corollary 4.5 to estimate numerous L^2 inner products.

We can now bound the first four L^2 inner products in (6.6). The result is

$$|(num)^{\cdot}| \leq c(|\mathbf{b}| + |b_0|)(|\mathbf{a}| + |a_0|\lambda^{-1+\alpha}) + |\langle\omega_{(a_0,\mathbf{a})},\ (G_-)^{\cdot}(\omega_\gamma)\rangle|, \tag{6.18}$$

the leading contributions coming from line 2 of (6.6).

It turns out that the last term in (6.18) has precisely the same bound as the first, but the analysis is more delicate. We start by noting that

$$\begin{aligned} (G_-)^{\cdot} &= -G_-^A \circ (\Delta_-^{A_t})^{\cdot} \circ G_-^A \\ &= -G_-^A \circ (P_-^\eta \circ (d_-^A)^* + d_-^A \circ (P_-^\eta)^*) \circ G_-^A, \end{aligned} \tag{6.19}$$

where $\eta = \dot A$, and where $P_-^\eta : \Omega^1(Ad\,P) \to \Omega_-^2(Ad\,P)$ is defined by $P_-^\eta(\cdot) = [\eta, \cdot]_-$. Thus, if we let

$$\xi_\gamma = -(d_-^A)^* G_-^A(\omega_\gamma)$$

(cf. (2.12)), then from (6.19) we have

$$\langle\omega_{(a_0,\mathbf{a})},\ (G_-)^{\cdot}(\omega_\gamma)\rangle = \langle\omega_{(a_0,\mathbf{a})},\ G_-^A([\dot A, \xi_\gamma]_-)\rangle + \langle\omega_\gamma,\ G_-^A([\dot A, \xi_{(a_0,\mathbf{a})}]_-)\rangle. \tag{6.20}$$

Bounding the two terms on the right requires some lengthy analysis involving an "inverted Weitzenböck identity" introduced in [G2] (Lemma 6.2 below).

Step 1: the first term in (6.20).

quantity	bound	formulas used in derivation
$\left\|r^{1+\delta}\omega_{(a_0,\mathbf{a})}\right\|$	$\|a_0\|\lambda + \|\mathbf{a}\| \cdot \begin{cases} \lambda^{2-\alpha}, & \delta=-\alpha \\ \lambda^2\|\log\lambda\|^{1/2}, & \delta=0 \\ \lambda^2, & \delta=\alpha \end{cases}$	(2.13), (3.1)
$\left\|\dot{F}\right\|$	$(\|\mathbf{b}\|+\|b_0\|)\lambda^{-1}$	(6.13)
$\left\|r^{1+\delta}H^0\dot{\phi}_{(a_0,\mathbf{a})} \natural F\right\|$	$\|a_0\|\|\mathbf{b}\| \cdot \begin{cases} \lambda^{1-\alpha}, & \delta=-\alpha \\ \lambda\|\log\lambda\|^{1/2}, & \delta=0 \\ \lambda, & \delta=\alpha \end{cases}$ $+\|a_0\|\|b_0\| + \|\mathbf{a}\|\|b_0\|\lambda^{2+\alpha+\delta} + \|\mathbf{a}\|\|\mathbf{b}\|\lambda^{1+\delta}$	(6.12), (3.1)
$\left\|r^{1+\delta}H^0\phi_{(a_0,\mathbf{a})} \natural \dot{F}\right\|$	$(\|\mathbf{b}\|\|a_0\| + \|b_0\|\|\mathbf{a}\|) \cdot \begin{cases} \lambda^{1-\alpha}, & \delta=-\alpha \\ \lambda\|\log\lambda\|^{1/2}, & \delta=0 \\ \lambda, & \delta=\alpha \end{cases}$ $+ \|b_0\|\|a_0\| + \|\mathbf{b}\|\|\mathbf{a}\|\lambda^{1+\delta}$	(6.15), (6.13), (3.1)
$\left\|r^{1+\delta}\nabla(\iota_{Z_{(b_0,\mathbf{b})}}\omega_{(a_0,\mathbf{a})})\right\|$	same bound as for $\left\|r^{1+\delta}H^0\dot{\phi}_{(a_0,\mathbf{a})} \natural F\right\|$	(6.16-6.17), (3.1)
$\left\|r^{1+\delta}\mathrm{R}((b_0,\mathbf{b}),\omega_{(a_0,\mathbf{a})})\right\|$	$\|b_0\|\|a_0\| + \|\mathbf{b}\|\|\mathbf{a}\| \cdot \left\{ \begin{matrix} \lambda^{1-\alpha}, & \delta=-\alpha \\ \lambda, & \delta=0,\alpha \end{matrix} \right\}$ $+$ $(\|b_0\|\|\mathbf{a}\| + \|\mathbf{b}\|\|a_0\|) \cdot \begin{cases} \lambda^{1-\alpha}, & \delta=-\alpha \\ \lambda\|\log\lambda\|^{1/2}, & \delta=0 \\ \lambda, & \delta=\alpha \end{cases}$	(6.24), (4.13), (4.14)
$\left\|r^m\omega_\gamma\right\|$	λ^{m-2}	(4.24), (3.1)
$\left\|r^m H^0\dot{\gamma} \natural F\right\|$	$\lambda^{m-3}(\|\mathbf{b}\|+\|b_0\|)$	(6.9), (3.1)
$\left\|r^m H^0\gamma \natural \dot{F}\right\|$	$\lambda^{m-3}(\|\mathbf{b}\|+\|b_0\|)$	(4.24), (6.13), (3.1)
$\left\|r^m\nabla(\iota_{Z_{(b_0,\mathbf{b})}}\omega_\gamma)\right\|$	$\lambda^{m-3}(\|\mathbf{b}\|+\|b_0\|)$	(6.16-6.17), (4.24), (3.1)
$\left\|r^m\mathrm{R}((b_0,\mathbf{b}),\omega_\gamma)\right\|$	$\lambda^{m-3}(\|\mathbf{b}\|+\|b_0\|)$	(6.23), (4.13), (4.14)

TABLE 1. The L^2-norm in the first column is bounded by a constant times the quantity in the second column. The third column lists formulas in the text used to derive the indicated bound. In the table m is arbitrary, $\alpha > 0$ is as in Corollary 4.5, and δ can take the values $0, \pm\alpha$.

To further streamline the notation, we will temporarily write $\omega_a = \omega_{(a_0,\mathbf{a})}$, $Z = Z_{(b_0,\mathbf{b})}$, $\tilde{Z} = \tilde{Z}^A_{(b_0,\mathbf{b})}$, $\xi_a = \xi_{(a_0,\mathbf{a})}$, and $\xi_b = \xi_{(b_0,\mathbf{b})}$.

Since $\dot{A} = \pi\tilde{Z} = \tilde{Z} + \xi_b$, the first term in (6.20) can be expanded as

$$\langle\omega_a,\ G^A_-([\dot{A},\xi_\gamma]_-)\rangle = \langle\omega_a,\ G^A_-([\tilde{Z},\xi_\gamma]_-)\rangle + \langle\omega_a,\ G^A_-([\xi_b,\xi_\gamma]_-)\rangle. \tag{6.21}$$

Focusing on the first term on the right-hand side of (6.21), we expand $G^A_-([\tilde{Z},\xi_\gamma]_-)$ as a leading-order local term plus a nonlocal remainder. (If we use Corollary 4.5 directly, we get a far worse estimate.) This expansion is given in more detail in [G2]; we summarize what we need of it in the following lemma.

Lemma 6.2. *Let $A \in \mathcal{M}_{\lambda_0}$. For any vector field X on M, let X^* denote the image of X under the metric isomorphism from TM to T^*M. Then for all $\omega \in \Omega^2_-(Ad\,P)$,*

$$G^A_-[\tilde{Z}^A_{(b_0,\mathbf{b})}, \omega]_- = p_-(Z^*_{(b_0,\mathbf{b})} \wedge \nabla^A G^A_- \omega) + G^A_-(\mathsf{R}((b_0,\mathbf{b}),\omega)), \tag{6.22}$$

where in general $\mathsf{R}(\cdot)$ satisfies the pointwise bound

$$\begin{aligned} |\mathsf{R}((b_0,\mathbf{b}),\omega)| \quad &\leq \quad c\{(|\mathbf{b}|(|\nabla^A\omega| + |\nabla^A G^A_-\omega| + r|\nabla^A\nabla^A G^A_-\omega|) \\ &\qquad + |b_0|\lambda^{-1}(|\omega| + r|\nabla^A\omega| + r|\nabla^A G^A_-\omega| + r^2|\nabla^A\nabla^A G^A_-\omega|)\}. \end{aligned} \tag{6.23}$$

In the special case $\omega = \omega_{(a_0,\mathbf{a})}$, we have the sharper bound

$$\begin{aligned} |\mathsf{R}((b_0,\mathbf{b}),\omega_{(a_0,\mathbf{a})})| \quad &\leq \quad c(|\mathbf{b}| + |b_0|\lambda^{-1}r)(|\xi_{(a_0,\mathbf{a})}| + r|\nabla\xi_{(a_0,\mathbf{a})}|) \\ &\qquad + c(|\mathbf{b}| + |b_0|\lambda^{-1}r)(|\mathbf{a}| + |a_0|\lambda^{-1}r)(|F| + r|\nabla^A F|). \end{aligned} \tag{6.24}$$

PROOF. Parts (a) and (b) are immediate consequences of [G2, Propositions 2.1 and 8.1]; part (c) uses [G2, inequality (8.9)]. □

We use this lemma to estimate the first term in (6.21). First, we have

$$|\langle\omega_a,\ G^A_-([\tilde{Z},\xi_\gamma]_-)\rangle| \leq |\langle\omega_a,\ Z^* \wedge (d^A_-)^* G^A_-\omega_\gamma\rangle| + |\langle\omega_a,\ G^A_-(\mathsf{R}((b_0,\mathbf{b}),\omega_\gamma))\rangle|. \tag{6.25}$$

In the first term, we have $\langle\omega_a,\ Z^* \wedge (d^A_-)^* G^A_-\omega_\gamma\rangle = \langle d^A_-(\iota_Z\omega_a),\ G^A_-\omega_\gamma\rangle$. Using Corollary 4.5, we then find

$$|\langle\omega_a,\ G^A_-([\tilde{Z},\xi_\gamma]_-)\rangle| \leq c(|\mathbf{b}| + |b_0|)(|\mathbf{a}| + |a_0|\lambda^{-1+\alpha}). \tag{6.26}$$

Next we turn to the second term in (6.21). Even if we bound this non-sharply, by

$$|\langle\omega_a,\ G^A_-([\xi_b,\xi_\gamma]_-)\rangle| \leq c\,\|r\omega_a\|_2\,\|r[\xi_b,\xi_\gamma]\|_2 \leq c\,\|r\omega_a\|_2\,\|\xi_b\|_4\,\left\|r\nabla^A G^A_-\omega_\gamma\right\|_4$$

(using (4.22) to bound $\|\xi_b\|_4$, and (4.14) and Table 1 to bound $\left\|r\nabla^A G^A_-\omega_\gamma\right\|_4$), we obtain an expression $\leq \lambda\cdot$right-hand side of (6.26), hence negligible. We conclude that

$$|\langle\omega_a,\ G^A_-([\dot{A},\xi_\gamma]_-)\rangle| \leq c(|\mathbf{b}| + |b_0|)(|\mathbf{a}| + |a_0|\lambda^{-1+\alpha}). \tag{6.27}$$

Step 2: The second term in (6.20).

Since this term is formally similar to the first term in (6.20), but with the roles of $\phi_{(a_0,\mathbf{a})}$ and γ interchanged, we can follow the same plan of attack as we did in Step 1. When we do, we obtain precisely the same bound as for the first term. Hence $|\langle\omega_a,\ (G_-)^{\cdot}(\omega_\gamma)\rangle| \le$ RHS of (6.27). Inserting this into (6.18), we have finally established that

$$|(num)^{\cdot}| \le c(|\mathbf{b}| + |b_0|)(|\mathbf{a}| + |a_0|\lambda^{-1+\alpha}). \tag{6.28}$$

Differentiating the denominator in (6.5).

We begin with a lemma. Notation and context are as above.

Lemma 6.3. *Let $\{A_t\}$ be a differentiable path of self-dual connections with $\dot{A} = \pi \tilde{Z}^A_{(b_0,\mathbf{b})}$. For every differentiable 1-parameter family of smooth functions $\{f_t : M \to \mathbf{R}\}$, we have*

$$\left\{\int_M f_t|F_t|^2\right\}^{\cdot} = \left\{\int_M [\dot{f} - Z(f_0)]|F|^2\right\} - 2\langle \iota_{\mathrm{grad}(f_0)}F,\ \xi_Z\rangle, \tag{6.29}$$

where $Z = Z_{(b_0,\mathbf{b})}$ and $\xi_Z = \xi_{(b_0,\mathbf{b})}$. Alternatively, integrating by parts as in (2.17), we can write this as

$$\left\{\int_M f_t|F_t|^2\right\}^{\cdot} = \left\{\int_M [\dot{f} - Z(f_0)]|F|^2\right\} + 2\langle H^0 f_0 \natural\ F, G^A_- \omega_{(b_0,\mathbf{b})}\rangle. \tag{6.30}$$

PROOF. First, we have

$$\left\{\int f_t|F_t|^2\right\}^{\cdot} = \int \dot{f}|F^2| \quad + \quad 2\int f_0(F,\dot{F}). \tag{6.31}$$

The second term can be rearranged:

$$\begin{aligned}
\int f_0(F,\dot{F}) &= \langle f_0F,\ d^A\pi^A\tilde{Z}\rangle \\
&= \langle (d^A)^*(f_0F),\ \pi^A\tilde{Z}\rangle \\
&= -\langle \iota_{\mathrm{grad}(f_0)}F,\ \tilde{Z} + \xi_Z\rangle \qquad \text{(since } (d^A)^*F = 0) \\
&= -\int(\iota_{\mathrm{grad}(f_0)}F,\ \iota_Z F) - \langle \iota_{\mathrm{grad}(f_0)}F,\ \xi_Z\rangle. \qquad (6.32)
\end{aligned}$$

Since F is self-dual, $2(\iota_{\mathrm{grad}(f_0)}F,\ \iota_Z F) = (\mathrm{grad}(f_0), Z)|F|^2 = Z(f_0)|F|^2$ pointwise (see [GP2, Lemma 3.4]). Hence the result follows from (6.31-6.32). □

Now let $h : \mathbf{R}^4 \to \mathbf{R}$ be a smooth function supported in a disk about 0 of radius less than λ_0^{-1} times the injectivity radius of M. For each $p' \in B_{r_0}(p)$, let $\{x^i_{p'}\}$ be a normal coordinate system centered at p', varying smoothly with p'. For each p' we can then define a function $f_{p'} : M \to \mathbf{R}$ by setting $f_{p'}(q) = h(x^1_{p'}(q), \ldots, x^4_{p'}(q))$. Similarly, we can define $f_{p',\lambda}(q) = h(\lambda^{-1}x^1_{p'}(q), \ldots, \lambda^{-1}x^4_{p'}(q))$. We will apply the lemma above to functions of this form.

Corollary 6.4. *In Lemma 6.3, let $f_t = h(\{x_t^i/\lambda_t\})$, where $h : \mathbf{R}^4 \to \mathbf{R}$ is as above and $\{x_t^i\}$ is the normal coordinate system whose variation is computed in Lemma 5.3. Let $x^i = x_0^i$, $\lambda = \lambda_0$ and let $\{u^i\}_1^4$ be standard coordinates on $\mathbf{R}^4$. Then the following are true.*

(a)

$$|\dot{f} - Z(f_0)| \leq c(h)(|\mathbf{b}|\lambda + |b_0|\lambda^\alpha). \tag{6.33}$$

(b)

$$|\langle \iota_{\mathrm{grad}(f_0)}F,\ \xi_Z\rangle| = |\langle H^0 f_0 \natural F, \omega_{(b_0,\mathbf{b})}\rangle| \leq c(h)\lambda^{2-\alpha})(|\mathbf{b}|\lambda + |b_0|\lambda^\alpha). \tag{6.34}$$

(c)

$$\left|\left\{\int_M f_t|F_t|^2\right\}^{\cdot}\right| \leq c(h)(|\mathbf{b}|\lambda + |b_0|\lambda^\alpha). \tag{6.35}$$

PROOF. (a) Using (5.9) and (6.1–6.2) we have

$$\begin{aligned}
\dot{f} &= \frac{\partial h}{\partial u^i}(x/\lambda)\cdot(\dot{x}^i\lambda^{-1} - x^i\lambda^{-2}\dot{\lambda}) \\
&= \frac{\partial h}{\partial u^i}(x/\lambda)\cdot\Big((b_i + \hat{b}_i + O(r^2|\mathbf{b} + \hat{\mathbf{b}}|))\lambda^{-1} + x^i\lambda^{-2}(b_0 + \hat{b}_0)\Big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
Z(f_0) &= (b_j g^{ji} + \lambda^{-1}b_0 x^i)\frac{\partial}{\partial x^i}h(x/\lambda) \\
&= \frac{\partial h}{\partial u^i}(x/\lambda)(b_j g^{ji}\lambda^{-1} + b_0 x^i\lambda^{-2}).
\end{aligned}$$

Hence, using (2.14),

$$\dot{f} - Z(f_0) = \frac{\partial h}{\partial u^i}(x/\lambda)\cdot\lambda^{-1}\cdot\Big(\hat{b}_i + x^i\lambda^{-1}\hat{b}_0 + O(r^2(|\mathbf{b}| + \lambda^{1+\alpha}|b_0|)\Big). \tag{6.36}$$

Since $|x/\lambda| \leq$ const on the support of h, we can replace $O(r^2)$ by $O(\lambda^2)$ in (6.36); similarly $x^i\lambda^{-1}$ is effectively $O(1)$. The stated bound now follows from (2.14).

(b) From Corollary 4.5 we have $|\langle H^0 f_0 \natural F, G_-^A\omega_{(b_0,\mathbf{b})}\rangle| \leq \left\|r^{1+\alpha}H f_0 \natural F\right\| \left\|r^{1-\alpha}\omega_{(b_0,\mathbf{b})}\right\|$. Note that

$$H f_0 = \nabla d f_0 = \lambda^{-2}\frac{\partial^2 h}{\partial u^i\partial u^j}(x/\lambda)dx^j\otimes dx^i + \lambda^{-1}\frac{\partial h}{\partial u^i}(x/\lambda)\nabla dx^i,$$

implying $|H f_0| \leq c(\|\nabla\nabla h\|_\infty\lambda^{-2} + \|\nabla h\|_\infty\lambda^{-1}r)$. Using (3.1), we therefore have $\left\|r^{1+\alpha}H f_0 \natural F\right\| \leq c(\|\nabla\nabla h\|_\infty\lambda^{-1+\alpha} + \|\nabla h\|_\infty\lambda)$, and, using Table 1, (6.34) follows.

(c) Follows from (6.30) and (6.33–6.34). ☐

We apply this corollary to the denominator in (6.5). For this application, we have $f_t = (\nabla\gamma_t, \nabla(r_t^2)) = \frac{2r_t}{K\lambda_t} b'(\frac{r_t}{K\lambda_t})$ (where the constant K is as in (2.1)), so we take $h(u) = |u|b'(|u|/K)$ and conclude that

$$|(denom)^{\cdot}| \leq c(|\mathbf{b}|\lambda + |b_0|\lambda^{\alpha}). \tag{6.37}$$

Estimate of $|(\hat{a}_0)^{\cdot}|$ completed.

As mentioned earlier, $denom(0)$ is bounded away from 0. Hence, from (6.5), we have

$$|(\hat{a}_0)^{\cdot}| \leq c\left\{\lambda|(num)^{\cdot}| + \ |\hat{a}_0(0)|(|(denom)^{\cdot}| + \lambda^{-1}|\dot{\lambda}|)\right\}.$$

Combining (6.3), (6.28), (6.37), and (2.14), we find that $|(\hat{a}_0)^{\cdot}|$ is bounded by the right-hand side of (5.1).

6.2. **Estimating $|(\hat{\mathbf{a}})^{\cdot}|$.**

Write $\hat{\mathbf{a}} = f^i\frac{\partial}{\partial x^i} = f^i\mathbf{e}_i$, where $\{f^i\}$ are the coefficient functions in (2.24). According to the prescription by which we are varying normal coordinate systems (see §8.3), $\mathbf{e}_i^t$ is covariantly constant at $t=0$, so $\nabla_{\sigma'(0)}\hat{\mathbf{a}} = (f^i)^{\cdot}(0)e_i$. Thus we need to bound the derivatives of all the non-constant terms in f^i. Since we have already bounded $\dot{a}_0$, that leaves us with bounding the derivatives of (i) the matrix $\bar{\lambda}^2 H$; and the functions (ii) m_i, (iii) $\bar{\lambda}^2\langle\omega_{(a_0,\mathbf{a})}, G_-^A(H^0\gamma_i \natural F)\rangle$, and (iv) $\int\bar{\lambda}\psi_{ij}|F|^2$.

(i) Differentiating the matrix $\bar{\lambda}_t^2 H_t$.

At $t=0$ the matrix $\bar{\lambda}^2 H$ is given by (2.23). Using (2.19) we can rewrite this as

$$\bar{\lambda}^2 H_{ij} = \int h_{ij}(x(q)/\bar{\lambda})|F|^2(q)dvol(q) + \int \bar{\lambda}b'(r/\bar{\lambda})\zeta_{ij}(p,q)|F|^2(q)dvol(q) \tag{6.38}$$

where $h_{ij}(u) = \left(b''(|u|/K) + \delta_{ij}K|u|^{-1}b'(|u|/K) - K|u|^{-1}b'(|u|/K)\right)u^iu^j|u|^{-2}$. Now replace p, x^i, λ by p_t, x_t^i, λ_t, and let t vary. Applying Corollary 6.4 to the first integral, we obtain

$$\left|\frac{d}{dt}\left\{\int h_{ij}(x_t(q)/\bar{\lambda}_t)|F_t|^2(q)dvol(q)\right\}\Big|_{t=0}\right| \leq c(|\mathbf{b}|\lambda + |b_0|\lambda^{\alpha}). \tag{6.39}$$

For the second integral in (6.38), define

$$(f_{ij})_t(q) = b'(r_t/\bar{\lambda}_t)\zeta_{ij}(p_t,q)$$

and consider the functions $\{\overline{\lambda}_t(f_{ij})_t\}$. Rather than using Lemma 6.3, we will directly estimate

$$\frac{d}{dt}\left\{\int \overline{\lambda}_t(f_{ij})_t|F_t|^2\right\}\bigg|_{t=0} = \int[\overline{\lambda}_t(f_{ij})_t]^{\cdot}|F|^2 + 2\int \overline{\lambda} f_{ij}(F,\dot{F}). \tag{6.40}$$

From Proposition 8.7 in the appendix, $|(\zeta_{ij})^{\cdot}| \le c|\dot{p}|$. Also, since $1 \le r/\overline{\lambda} \le 2$ on the support of $b'(r/\overline{\lambda})$, powers of r are equivalent to powers of λ in pointwise estimates involving f_{ij}. Hence, since $\zeta_{ij} = O(r)$ (Lemma 8.3), we have $|f_{ij}| \le c\lambda$ and $|(f_{ij})^{\cdot}| \le c(|\dot{r}| + |\dot{p}|)$. Using (6.3), (6.4), and (6.7) we then find $|(\lambda_t(f_{ij})_t)^{\cdot}| \le c(|\mathbf{b}| + |b_0|)$; This bounds the first integral in (6.40) since $\|F\|_2 \le c$. For the second integral in (6.40), we use Table 1 to find $\left|\int \overline{\lambda} f_{ij}(F,\dot{F})\right| \le c\lambda^2\,\|F\|_2\left\|\dot{F}\right\|_2 \le c(|\mathbf{b}| + |b_0|)\lambda$. Combining the these results we obtain

$$\left|\frac{d}{dt}\left\{\int \overline{\lambda}_t(f_{ij})_t|F_t|^2\right\}\bigg|_{t=0}\right| \le c(|\mathbf{b}| + |b_0|)\lambda. \tag{6.41}$$

From (6.38), (6.39), and (6.41) we then find $\left|(\overline{\lambda}_t^{-2}H_{ij}^t)^{\cdot}\right| \lesssim |\mathbf{b}|\lambda + |b_0|\lambda^\alpha$. Since $\overline{\lambda}^{-2}H$ is uniformly bounded below, the same estimate holds for the inverse matrix:

$$\left|\left((\overline{\lambda}_t^{-2}H_{ij}^t)^{-1}\right)^{\cdot}\right| \lesssim |\mathbf{b}|\lambda + |b_0|\lambda^\alpha. \tag{6.42}$$

(ii) Differentiating m_i.

Corollary 6.4(d) applies to m_i (see (2.20)), yielding

$$|\dot{m}_i| \lesssim |\mathbf{b}|\lambda + |b_0|\lambda^\alpha. \tag{6.43}$$

(iii) Differentiating $\overline{\lambda}^{-2}\langle\omega_{(a_0,\mathbf{a})}, G_-^A(H^0\gamma_i \natural F)\rangle$.

In §6.1 we differentiated $num = \langle\omega_{(a_0,\mathbf{a})}, G_-^A(H^0\gamma \natural F)\rangle$. Our work here is similar; we simply need to replace γ in (6.6) by γ_i. Since $\gamma_i = b'(r/\overline{\lambda})\overline{\lambda}^{-1}x^i/r$, (8.20) and (5.10) imply that $\dot{\gamma}_i = O_{\text{strong}}((|\dot{p}|+|\dot{\lambda}|)r^2\lambda^{-4})$ (see the proof of Lemma 6.1(c)). This is one power of λ worse than the corresponding bound (6.14) on $\dot{\gamma}$. Apart from this attendant factor of λ^{-1} in each estimate, the argument by which we bounded $(num)^{\cdot}$ goes through just as before, replacing γ by γ_i. Hence we find that $\langle\omega_{(a_0,\mathbf{a})}, G_-^A(H^0\gamma_i \natural F)\rangle$ is bounded by λ^{-1} times the right-hand side of (6.28). Since from (4.26) we have $\left|\langle G_-^A\omega_{(a_0,\mathbf{a})}, H^0\gamma_i \natural F\rangle\right| \le c(|\mathbf{a}| + |a_0|\lambda^{-1+\alpha})$, using (6.3) we conclude that

$$\left|\left(\overline{\lambda}^{-2}\langle\omega_{(a_0,\mathbf{a})}, G_-^A(H^0\gamma_i \natural F)\rangle\right)^{\cdot}\right| \le c(|\mathbf{b}| + |b_0|)(|\mathbf{a}|\lambda + |a_0|\lambda^\alpha). \tag{6.44}$$

(iv) Differentiating $\int \overline{\lambda}\psi_{ij}|F|^2$.

We can write $\psi_{ij} = b'(r/\overline{\lambda})\hat{\psi}_{ij}$, where, by Lemma 8.3 and Proposition 8.7 of the appendix, $|\hat{\psi}_{ij}| \le cr$ and $|(\hat{\psi}_{ij})^{\cdot}| \le c|\dot{p}|$. Hence the estimate of $\left(\int \overline{\lambda}\psi_{ij}|F|^2\right)^{\cdot}$ is identical to the one achieved in (6.41):

$$\left| \frac{d}{dt}\left\{ \int \overline{\lambda}_t(\psi_{ij})_t|F_t|^2 \right\}\bigg|_{t=0} \right| \le c(|\mathbf{b}| + |b_0|)\lambda. \tag{6.45}$$

Estimate of $|(\hat{\mathbf{a}})^{\cdot}|$ completed.

Write the coefficient f^j of $\frac{\partial}{\partial x^j}$ in (2.24) as

$$f^j = (\overline{\lambda}^{2} H)^{-1}_{ji}(\text{stuff}).$$

Then from (2.25), (2.14), (6.43–6.45), the bound on $|(\hat{a}_0)^{\cdot}|$ in (5.1), and the fact that $\overline{\lambda}^{2} H$ is uniformly bounded below, we have

$$\left|(\overline{\lambda}^{2} H)^{-1}\left((\text{stuff})^{\cdot}\right)\right| \le c(|\mathbf{b}| + |b_0|)(|\mathbf{a}|\lambda + |a_0|\lambda^{\alpha}). \tag{6.46}$$

Furthermore, the proof of Proposition 2.1 shows that $|\text{stuff}| \le$ RHS of (2.14), and multiplying this by the bound on $\left|\left((\overline{\lambda}^{2} H)^{-1}\right)^{\cdot}\right|$ in (6.42), we obtain a quantity smaller than the one in (6.46). Thus $|(f^j)^{\cdot}| \le$ RHS of (5.1), completing the proof of (5.1).

6.3. Estimating $|\langle \mathbf{X}, \mathbf{Y}\rangle^{\cdot}|$.

We start by noting that

$$\langle \mathbf{X}, \mathbf{Y}\rangle = \langle \tilde{Z}^A_{(a_0,\mathbf{a})}, \pi^A \tilde{Z}^A_{(a_0',\mathbf{a}')}\rangle = \langle \tilde{Z}^A_{(a_0,\mathbf{a})}, \tilde{Z}^A_{(a_0',\mathbf{a}')}\rangle - \langle \omega_{(a_0,\mathbf{a})}, G^A_{-}\omega_{(a_0',\mathbf{a}')}\rangle. \tag{6.47}$$

We differentiated similar objects above, so our work here is simplified. It will turn out our estimate of the derivative of the first term in (6.47) dominates our estimate of the derivative of the second term, with the exception of one coefficient, which is logarithmically larger for the second term.

Differentiating the first term in (6.47).

From the algebra of self-dual two-forms (see [GP1, Lemma 3.4]), we can write

$$\langle \tilde{Z}^A_{(a_0,\mathbf{a})}, \tilde{Z}^A_{(a_0',\mathbf{a}')}\rangle = \int (\tilde{Z}^A_{(a_0,\mathbf{a})}, \tilde{Z}^A_{(a_0',\mathbf{a}')})\; dvol = \frac{1}{2}\int (Z_{(a_0,\mathbf{a})}, Z_{(a_0',\mathbf{a}')})|F|^2.$$

Inserting the t-dependence, we therefore have to differentiate something of the form $\int f_t|F_t|^2$, for which we can use Lemma 6.3. To apply this lemma, we write $(Z_{(a_0,\mathbf{a})}, Z_{(a_0',\mathbf{a}')})$ in the form

$$(Z_{(a_0,\mathbf{a})}, Z_{(a_0',\mathbf{a}')}) = f^{(1)} + f^{(2)},$$

where

$$
\begin{aligned}
f^{(1)} &= \beta^2\left((\mathbf{a},\mathbf{a}') + \lambda^{-1}(a_i a_0' + a_i' a_0)x^i + a_0 a_0'\lambda^{-2}r^2\right),\\
f^{(2)} &= \beta^2 a_i a_j'(g^{ij}-\delta_{ij}) + \beta' r^{-1}(2\phi_{(a_0,\mathbf{a})}\phi_{(a_0',\mathbf{a}')} + \frac{1}{2}r^2(a_0\phi_{(a_0',\mathbf{a}')} + a_0'\phi_{(a_0,\mathbf{a})}))\\
&\quad +(\beta')^2(\frac{1}{2}\lambda^{-1}a_0 r^2 + a_i x^i)(\frac{1}{2}\lambda^{-1}a_0' r^2 + a_j' x^j) \qquad (6.48)
\end{aligned}
$$

We will deal first with $f^{(2)}$. Note that $g^{ij}-\delta_{ij} = O_{\text{strong}}(r^2)$ and $(g^{ij})^{\cdot} = O(r|\dot{p}|)$ (see Proposition 8.7). The t-derivatives of the terms involving β' can be calculated as in Lemma 6.1, but this time each differentiated term is proportional to β' or β''. Calculating as in the proof of Corollary 6.4(a), we therefore find

$$
\begin{aligned}
|(f_0^{(2)})^{\cdot} - Z_{(b_0,\mathbf{b})}(f_0^{(2)})| &\leq \beta\cdot O(r|\mathbf{a}||\mathbf{a}'|)\cdot O(r^2|\mathbf{b}| + \lambda^{-1}r|b_0| + |\hat{\mathbf{b}}|)\\
&\quad + c\chi_{ann}\cdot(|\mathbf{a}| + \lambda^{-1}|a_0|)(|\mathbf{a}'| + \lambda^{-1}|a_0'|)(|\mathbf{b}| + \lambda^{-1}|b_0|)
\end{aligned}
\tag{6.49}
$$

where χ_{ann} is the characteristic function of the annulus $\{r_0 \leq r \leq 2r_0\}$. On this annulus we have the uniform bound $|F| \leq c\lambda^2$ (see [GP3, Lemma 5.2]). Hence, using (3.1) and (2.14) we find

$$
\int [(f_0^{(2)})^{\cdot} - Z_{(b_0,\mathbf{b})}(f_0^{(2)})]|F|^2 \leq c\lambda(\lambda|\mathbf{b}| + |b_0|)(\lambda|\mathbf{a}| + |a_0|)(\lambda|\mathbf{a}'| + |a_0'|) \tag{6.50}
$$

As for the second term in (6.30), we have

$$
|H^0(f_0^{(2)})| \leq c\left(\bar{-}\mathbf{a}||\mathbf{a}'| + \chi_{ann}(|\mathbf{a}| + \lambda^{-1}|a_0|)(|\mathbf{a}'| + \lambda^{-1}|a_0'|)\right)
$$

Using (4.23) we therefore have $|\langle H^0 f_0^{(2)} \natural F, G_-^A\omega_{(b_0,\mathbf{b})}\rangle|$

$$
\leq c\lambda(\lambda|\mathbf{b}| + |b_0|)\left(\lambda^{1+\alpha}|\mathbf{a}||\mathbf{a}'| + (\lambda|\mathbf{a}| + |a_0|)(\lambda|\mathbf{a}'| + |a_0'|)\right). \tag{6.51}
$$

Combining this with (6.50) we find

$$
\left(\int f_t^{(2)}|F_t|^2\right)^{\cdot} \leq c\lambda(\lambda|\mathbf{b}| + |b_0|)\left(\lambda^{1+\alpha}|\mathbf{a}||\mathbf{a}'| + (\lambda|\mathbf{a}| + |a_0|)(\lambda|\mathbf{a}'| + |a_0'|)\right). \tag{6.52}
$$

Now turn to $f_t^{(1)}$, which we write as $\beta_t^2 f_t^{(3)}$ (with $\beta_t = \beta(r_t)$). As in the situation of Corollary 6.4, we can write $f_t^{(3)} = h(\{x_t^i/\lambda_t\})$ for a fixed function $h : \mathbf{R}^4 \to \mathbf{R}$, specifically

$$
h(u) = c_0 + d_i u^i + c_2|u|^2,
$$

where c_i, d_i are constants:

$$
c_0 = (\mathbf{a},\mathbf{a}'), \quad d_i = a_i a_0' + a_0 a_i', \quad c_2 = a_0 a_0'.
$$

Certain aspects of the proof of Corollary 6.4 still go through, but the presence of β and especially the noncompactness of the support of h must be dealt with.

Looking at (6.30), we see two basic terms to compute. It will turn out that in our eventual bound (6.57), the first term in (6.30) dominates for the part of the bound proportional to $|\mathbf{a}||\mathbf{a}'|$, while the second dominates for the parts proportional to $|\alpha_0||\mathbf{a}'|$, $|\mathbf{a}||a_0'|$, and $|a_0||a_0'|$.

To start with, we split up the first term in (6.30):

(6.53)
$$\dot{f^{(1)}} - Z_{(b_0,\mathbf{b})}(f_0^{(1)}) = \beta_0^2\left(\dot{f^{(3)}} - Z_{(b_0,\mathbf{b})}(f_0^{(3)})\right) + 2f_0^{(3)}\beta_0\left(\dot{\beta} - Z_{(b_0,\mathbf{b})}(\beta_0)\right).$$

From (6.36), we have

$$\begin{aligned}\left|\dot{f^{(3)}} - Z_{(b_0,\mathbf{b})}(f_0^{(3)})\right| &\le c(|\mathbf{d}| + |c_2||u|)\lambda^{-1}\left(|\hat{\mathbf{b}}| + r\lambda^{-1}|\hat{b}_0| + O(r^2(|\mathbf{b}| + \lambda^{1+\alpha}|b_0|))\right)\\ &\le c\lambda^{-1}(|\mathbf{d}| + |c_2|\lambda^{-1}r)\left(|\mathbf{b}|(\lambda^2+r^2) + |b_0|\lambda^\alpha(\lambda+r)\right)\end{aligned}$$

(using (2.14)). Since this expression is a polynomial in r of degree less than 4, integrating against $\beta^2|F|^2$ replaces each power of r by an equal power of λ (see (3.1)). Thus

$$\left|\int \beta_0^2\left(\dot{f^{(3)}} - Z_{(b_0,\mathbf{b})}(f_0^{(3)})\right)|F|^2\right| \le c(|\mathbf{d}| + |c_2|)(|\mathbf{b}|\lambda + |b_0|\lambda^\alpha). \tag{6.54}$$

Next, arguing as in the derivation of (6.36) we find

$$\dot{\beta} - Z_{(b_0,\mathbf{b})}(\beta_0) = \beta_0'\cdot O(|\mathbf{b}| + |b_0|).$$

Integrating against $|F|^2$ (after multiplying by $\beta_0 f_0^{(3)}$), the presence of β' once again effectively turns $|F|$ into λ^2, and r into a constant. Hence

(6.55)
$$\left|\int f_0^{(3)}\beta\left(\dot{\beta} - Z_{(b_0,\mathbf{b})}(\beta_0)\right)|F|^2\right| \le c\lambda^4(|\mathbf{b}| + |b_0|)(|c_0| + |\mathbf{d}|\lambda^{-1} + |c_2|\lambda^{-2}).$$

Combining this with (6.53) and (6.54), we obtain a bound on the first term that (6.30) gives for $f = f^{(1)}$:

(6.56)
$$\left|\int\left(\dot{f^{(1)}} - Z_{(b_0,\mathbf{b})}(f_0^{(1)})\right)|F|^2\right| \le c\left(|c_0|\lambda^4(|\mathbf{b}| + |b_0|) + (|\mathbf{d}| + |c_2|)(|\mathbf{b}|\lambda + |b_0|\lambda^\alpha)\right).$$

Moving on to the second term in (6.30), we have $|H^0 f_0^{(1)}| \le \beta^2|H^0 f_0^{(3)}| + c\chi_{ann}(|f_0^{(3)}| + |\nabla f_0^{(3)}|)$. Since $Hx^i = O(r)$ and $H^0(r^2) = O(r^2)$, this leads to the pointwise bound

$$|H^0 f_0^{(1)}| \le c\left(\beta^2(|\mathbf{d}|r\lambda^{-1} + |c_2|r^2\lambda^{-2}) + \chi_{ann}(|c_0| + |\mathbf{d}|\lambda^{-1} + |c_2|\lambda^{-2})\right).$$

Now use (4.23) to estimate the second term in (6.30) (with $f = f^{(1)}$). Apportioning powers of r judiciously, the principles applied above lead to

$$|\langle H^0 f_0^{(1)} \natural F, G_-^A \omega_{(b_0,\mathbf{b})}\rangle| \le c\left((|c_0|\lambda^2 + |\mathbf{d}|\lambda + |c_2|)(|\mathbf{b}|\lambda^2 + |b_0|\lambda) + |\mathbf{d}||\mathbf{b}|\ \lambda^3\ |\log\lambda|\right),$$

which is dominated by (6.56). Hence (6.30) gives

$$\left(\int f_t^{(1)}|F_t|^2\right)^{\cdot} \le \text{RHS of (6.56)}. \tag{6.57}$$

Combining this with (6.52) we arrive at

$$\begin{aligned}\langle \tilde Z^A_{(a_0,\mathbf{a})}, \tilde Z^A_{(a_0',\mathbf{a}')}\rangle^{\cdot} \quad \le \quad & c\left(|\mathbf{a}||\mathbf{a}'|(|\mathbf{b}|\lambda^{3+\alpha} + |b_0|\lambda^{2+\alpha})\right. \\ & \left.+ (|\mathbf{a}||a_0'| + |a_0||\mathbf{a}'| + |a_0||a_0'|)(|\mathbf{b}|\lambda + |b_0|\lambda^{\alpha})\right).\end{aligned} \tag{6.58}$$

Differentiating the second term in (6.47).

This term is similar to the numerator in (6.5), so the procedure for differentiating and estimating is the same here; we need only make the appropriate modifications, replacing γ by $\phi_{(a_0',\mathbf{a}')}$. Looking at (6.6), after taking into account the symmetry between $(a_0,\mathbf{a})$ and $(a_0',\mathbf{a}')$, there are three types of terms: one in which only $\phi_{(a_0,\mathbf{a})}$ or $\phi_{(a_0',\mathbf{a}')}$ gets (time-)differentiated, one in which only F gets differentiated, and one in which only the Green operator gets differentiated.

To write down the bounds, let us modify the notation $|c_0|, |\mathbf{d}|$ slightly from its usage above, and write

$$|c_0| = |\mathbf{a}||\mathbf{a}'|, \quad |\mathbf{d}| = |\mathbf{a}||a_0'| + |a_0||\mathbf{a}'|.$$

Also we temporarily abbreviate subscripts $(a_0,\mathbf{a}), (a_0',\mathbf{a}')$ and $(b_0,\mathbf{b})$ as a, a', and b respectively, as in (6.20–6.28). Using Corollary 4.5, one finds

$$\begin{aligned}|\langle (H^0\dot\phi_a) \natural F, G_-^A\omega_{a'}\rangle| + |\langle H^0\phi_a \natural \dot F, G_-^A\omega_{a'}\rangle| + (\text{same with } \ (a_0,\mathbf{a}) \leftrightarrow (a_0',\mathbf{a}'))& \\ \le c(|c_0|(|\mathbf{b}|\lambda^3 + |b_0|\lambda^3|\log\lambda|^{1/2}) \quad + \quad |\mathbf{d}|(|\mathbf{b}|\lambda^{2+\alpha} + |b_0|\lambda^2|\log\lambda|^{1/2})& \\ + \quad |c_2|(|\mathbf{b}|\lambda^2 + |b_0|\lambda)).&\end{aligned} \tag{6.59}$$

For the term in which the Green operator is differentiated, we have the analog of (6.20–6.21):

$$\begin{aligned}\langle \omega_a,\ (G_-)^{\cdot}(\omega_{a'})\rangle \quad = \quad & \langle \omega_a,\ G_-^A([\tilde Z_b, \xi_{a'}]_-)\rangle + \langle \omega_a,\ G_-^A([\xi_b, \xi_{a'}]_-)\rangle \\ & + (\text{same with } \ (a_0,\mathbf{a}) \leftrightarrow (a_0',\mathbf{a}'))\end{aligned} \tag{6.60}$$

To estimate the first term in (6.60), we break it up as in (6.25). This time we find

$$\left|\langle\omega_a,\ G_-^A([\tilde Z_b,\xi_{a'}]_-)\rangle\right| + (\text{same with }\ (a_0,\mathbf{a})\leftrightarrow(a_0',\mathbf{a}')) \leq$$

(6.61)

$$c\left(|c_0|(|\mathbf{b}|\lambda^3|\log\lambda|^{1/2}+|b_0|\lambda^3|\log\lambda|) + |\mathbf{d}|(|\mathbf{b}|\lambda^2+|b_0|\lambda^2) + |c_2|(|\mathbf{b}|\lambda^2+|b_0|\lambda)\right).$$

If use Corollary 4.5 to bound the second term in (6.60) simply by $\left|\langle\omega_a,\ G_-^A([\xi_b,\xi_{a'}]_-)\rangle\right| \leq \|r\omega_a\|_2\,\|r\xi_{a'}\|_4\,\|\xi_b\|_4$, we obtain a bound less than λ times the right-hand side of (6.61). Hence

$$|\langle\omega_a,\ (G_-)^{\cdot}(\omega_{a'})\rangle| \leq \text{RHS of (6.61)}. \tag{6.62}$$

Adding (6.59) and (6.62), we obtain $\left|\left(\langle\omega_{(a_0,\mathbf{a})}, G_-^A\omega_{(a_0',\mathbf{a}')}\rangle\right)^{\cdot}\right|$

$$\begin{aligned} &\leq\ c(|c_0|(|\mathbf{b}|\lambda^3|\log\lambda|^{1/2}+|b_0|\lambda^3|\log\lambda|) \\ &+\ |\mathbf{d}|(|\mathbf{b}|\lambda^2+|b_0|\lambda^2|\log\lambda|^{1/2}) + |c_2|(|\mathbf{b}|\lambda^2+|b_0|\lambda)). \end{aligned} \tag{6.63}$$

Estimate of $\mathbf{Z}\langle\mathbf{X},\mathbf{Y}\rangle$, completed. Finally, adding (6.63) to (6.58), we obtain (5.2).

7. First derivatives of the metric

To discuss continuity of the derivatives of $\mathbf{g}_\Psi = (\Psi^{-1})^*\mathbf{g}$ we need to look at functions of the form $(\Psi^{-1})^*([\mathbf{Z}]\mathbf{g}([\mathbf{X}],[\mathbf{Y}]))$, where $[\mathbf{X}],[\mathbf{Y}],[\mathbf{Z}]$ are (local) vector fields on $\mathcal{M}$. If $\mathbf{X},\mathbf{Y},\mathbf{Z}$ are the (equivariant) lifts of these vector fields to $H\to\mathcal{A}_{\lambda_0}$ (i.e. $\mathcal{G}$-invariant covering sections of H), then $[\mathbf{Z}](\mathbf{g}_{\mathcal{M}}([\mathbf{X}],[\mathbf{Y}]))|_{[A]} = \mathbf{Z}(\mathbf{g}_{\mathcal{A}}(\mathbf{X},\mathbf{Y}))|_A = \mathbf{Z}\langle\mathbf{X},\mathbf{Y}\rangle|_A$. It is exactly this derivative that was computed in (5.2), where we took $\mathbf{X}=\pi\tilde Z_{(a_0,\mathbf{a})}, \mathbf{Y}=\pi\tilde Z_{(a_0',\mathbf{a}')}$, and $\mathbf{Z}=\pi\tilde Z_{(b_0,\mathbf{b})}$. However, (5.2) applies only at points (p,λ) for which $\nabla\mathbf{a}|_p = \nabla\mathbf{a}'|_p = 0$, so we cannot instantly deduce Theorem 1.2 from these bounds and Theorem 5.2. When we do the work required to extend the bounds away from p, with essentially no extra effort we obtain the following stronger theorem, of which Theorems 1.2–1.3 are corollaries. Below, Latin indices always run from 1 to 4, and Greek indices from 0 to 4.

Theorem 7.1. *Let $\{x^i\}_1^4$ be local coordinates on a neighborhood $U\subset M$. Extend these to a coordinate system $\{x^\mu\}_0^4$ on $[0,\lambda_0)\times U$ by setting $x^0=\lambda$, and write $\mathbf{g}_{\mu\nu}=\mathbf{g}_\Psi(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu})$, $g_{ij}=g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})$. Assume that the metric coefficients g_{ij}*

and their first derivatives $\frac{\partial}{\partial x^k} g_{ij}$ are bounded on U. Let $\alpha = \alpha_0$ be as in Lemma 4.3. Then

$$\mathbf{g}_{ij} = 4\pi^2 g_{ij} + O(\lambda^2) \tag{7.1}$$

$$\mathbf{g}_{0\mu} = 8\pi^2 \delta_{0\mu} + O(\lambda^{1+\alpha}) \tag{7.2}$$

$$\frac{\partial}{\partial x^k}\mathbf{g}_{ij} = 4\pi^2 \frac{\partial}{\partial x^k} g_{ij} + O(\lambda) \tag{7.3}$$

$$\frac{\partial}{\partial \lambda}\mathbf{g}_{ij} = O(\lambda) \tag{7.4}$$

$$\frac{\partial}{\partial x^\nu}\mathbf{g}_{0\mu} = O(\lambda^\alpha) \tag{7.5}$$

The constants implicit in the $O(\cdot)$ terms can depend on the the upper bounds on g_{ij} and $\frac{\partial}{\partial x^k} g_{ij}$ on U. If we start with normal coordinates on a sufficiently small ball, these constants are independent of the choice of the ball and the choice of normal coordinates.

Remark. The bound on $\partial \mathbf{g}_{0\mu}/\partial x^i$ in (7.5) is not as strong as one could hope for; in the examples $M = S^4, \mathbf{CP}^2$, one loses a power of λ only differentiating in the λ-direction. Thus, given (7.2), one might expect to find $\partial \mathbf{g}_{0\mu}/\partial x^i = O(\lambda^{1+\alpha})$. The source of this worse-than-expected bound is (5.1), where one would expect to find a larger power of λ for the terms proportional to $|\mathbf{b}|$ than for those proportional to $|b_0|$. In our proof of (5.1), what puts $\mathbf{b}$ and b_0 on an equal footing is our treatment of the last three terms in (6.6), each of which our estimation scheme bounds by the right-hand side of (6.28). Conceivably, there is some hidden cancellation among these terms.

Proof of Theorem 7.1: Fix $p \in U$, set $e_i(p) = \frac{\partial}{\partial x^i}(p)$, and extend $\{e_i\}$ to a neighborhood of p by parallel translation along radial geodesics through p. Define $\{e'_\mu\}$ as in (5.3). Then (5.2) can be written as follows:

$$e'_0\left(\mathbf{g}_\Psi(e'_0, e'_\mu)\right)\big|_{(p,\lambda)} = O(\lambda^\alpha) \tag{7.6}$$

$$e'_k\left(\mathbf{g}_\Psi(e'_0, e'_\mu)\right)\big|_{(p,\lambda)} = O(\lambda) \tag{7.7}$$

$$e'_0\left(\mathbf{g}_\Psi(e'_i, e'_j)\right)\big|_{(p,\lambda)} = O(\lambda^3|\log\lambda|) \tag{7.8}$$

$$e'_k\left(\mathbf{g}_\Psi(e'_i, e'_j)\right)\big|_{(p,\lambda)} = O(\lambda^3|\log\lambda|^{1/2}) \tag{7.9}$$

Our derivation yields these bounds only at points (p, λ) for which p is the chosen point above. Below, we will always compute at such (p, λ); the reader can check that as we do the computations, the constants involved have the uniformity asserted in the theorem.

Our strategy for deriving (7.3-7.5) will be to write the vector fields $\{\frac{\partial}{\partial x^\mu}\}$ in terms of the $\{e'_\mu\}$ and then to estimate $\frac{\partial}{\partial x^\mu}\mathbf{g}_{\nu\rho}$ using (7.6–7.9). We will do this in two stages, first writing $\{e_\mu\}$ in terms of $\{e'_\mu\}$, and then writing $\{\frac{\partial}{\partial x^\mu}\}$ in terms of $\{e_\mu\}$.

In the first stage, the relevant information on the matrix relating the bases $\{e_\mu\}, \{e'_\mu\}$ is contained in (5.4–5.7), but the undifferentiated metric coefficients $\mathbf{g}_\Psi(e'_\mu, e'_\nu)$ enter the derivative computation as well. We claim that

$$\mathbf{g}_\Psi(e'_i, e'_j) = 4\pi^2 g(e_i, e_j) + O(\lambda^2), \quad \mathbf{g}_\Psi(e'_0, e'_\mu) = 8\pi^2 \delta_{\mu 0} + o(1). \tag{7.10}$$

To establish this, we use (6.47) and the following bounds from [GP2, Proposition 3.6]:

$$\begin{aligned} &\left|\langle \tilde{Z}^A_{(a_0,\mathbf{a})}, \tilde{Z}^A_{(a'_0,\mathbf{a}')}\rangle - 4\pi^2(g(\mathbf{a},\mathbf{a}') + 2a_0 a'_0)\right| \\ \leq\ & c|\mathbf{a}||\mathbf{a}'|\lambda^2 + \epsilon(\lambda)(|\mathbf{a}||a'_0| + |a_0||\mathbf{a}'| + |a_0||a'_0|) \end{aligned} \tag{7.11}$$

where $\epsilon(\lambda) \to 0$ as $\lambda \to 0$. Applying Corollary (4.5) to the second term in (6.47) gives

$$\left|\langle \omega_{(a_0,\mathbf{a})}, G^A_- \omega_{(a'_0,\mathbf{a}')}\rangle\right| \leq c\left(|\mathbf{a}||\mathbf{a}'|\lambda^4|\log\lambda| + (|a_0||\mathbf{a}'| + |\mathbf{a}||a'_0|)\lambda^3 + |a_0||b_0|\lambda^2\right). \tag{7.12}$$

From (6.47) and (7.11–7.12) we then obtain (7.10).

The next step is to replace the $\{e'_\mu\}$ in (7.10)) by $\{e_\mu\}$. First, from (5.3) we have $e_\mu = D^\nu_\mu e'_\nu$, where $D = C^{-1}$. It is easy to show that the bounds (5.4–5.7) imply the same bounds with C replaced by D. Combining these bounds with (7.10), we obtain

$$\mathbf{g}_\Psi(e_i, e_j) = 4\pi^2 g(e_i, e_j) + O(\lambda^2), \quad \mathbf{g}_\Psi(e_0, e_\mu) = 8\pi^2 \delta_{\mu 0} + o(1). \tag{7.13}$$

From (7.6–7.9) we can then conclude that

$$e_\nu\left(\mathbf{g}_\Psi(e_0, e_\mu)\right) = O(\lambda^\alpha), \quad e_\nu\left(\mathbf{g}_\Psi(e_i, e_j)\right) = O(\lambda). \tag{7.14}$$

Now we move onto the second stage of the substitution procedure, writing $\frac{\partial}{\partial x^i} = B^j_i e_j$ (by definition $\frac{\partial}{\partial \lambda} = e_0$) and using (7.10) and (7.13–7.14). Note that the matrix B is independent of λ. Performing the substitution, we immediately obtain $\frac{\partial}{\partial \lambda}\mathbf{g}_{ij} = O(\lambda)$ (which is (7.4)), and

$$\frac{\partial}{\partial x^i}\mathbf{g}_{00} = O(\lambda^\alpha), \quad \frac{\partial}{\partial \lambda}\mathbf{g}_{0\mu} = O(\lambda^\alpha), \tag{7.15}$$

which is most of (7.5). To get the rest of (7.5) we have to bootstrap a little, because the "$o(1)$" term in (7.13) is not yet strong enough. First, note that since $\frac{\partial}{\partial x^i}(p) = e_i(p)$, from (7.10) we immediately have $\mathbf{g}_{ij} = 4\pi^2 g_{ij} + O(\lambda^2)$

(which is (7.1)), and $\mathbf{g}_{0\mu} = 8\pi^2\delta_{\mu 0} + o(1)$. In particular, $\mathbf{g}_{0\mu} = \delta_{0\mu}$ at $\lambda = 0$, so integrating the second part of (7.15) gives (7.2), which can now be used to show $\frac{\partial}{\partial x^i}\mathbf{g}_{0j} = O(\lambda^\alpha)$. Hence we arrive at (7.5).

It remains only to derive (7.3). Since we are computing only at (p, λ) and the covariant derivatives of the e_i vanish at p, this bound follows from (7.13–7.14) by direct substitution.

Since the (extended) metric $\mathbf{g}_\Psi$ is C^1, the second fundamental form of the boundary is well-defined. As corollaries of Theorem 7.1, we obtain the following two theorems.

Theorem 7.2. *There exists $\alpha > 0$ for which the continuous extension of $\mathbf{g}_\Psi$ to $[0, \lambda_0) \times M$ is $C^{1,\alpha}$.*

PROOF. Since the metric coefficients are C^∞ on the interior, this follows from (7.1–7.5) by the same argument as in the proof of Theorem 5.2. □

Theorem 7.3. *Let h denote the second fundamental form of the submanifolds $\{\lambda = \text{constant}\}$ (relative to the metric $\mathbf{g}_\Psi$). Then*

$$|h| \leq c\lambda^\alpha.$$

In particular, h vanishes on the boundary $\{\lambda = 0\}$, so the boundary is a totally geodesic submanifold.

PROOF. Since the submanifolds $M_c := \{\lambda = c\}$ have codimension 1, we can define the second fundamental form of M_c up to sign as the the section of $\mathrm{Sym}^2 T^* M_c$ given by

$$h(X, Y) = \mathbf{g}_\Psi(\nabla_X N, Y) = -\mathbf{g}_\Psi(\nabla_X Y, N), \tag{7.16}$$

where N is a unit normal vector field. In terms of local coordinates $\{x^\mu\}$ of Theorem 7.1, one has

$$h = \Gamma^0_{ij} dx^i \otimes dx^j.$$

But if we compute in normal coordinates based at an arbitrary point p, Theorem 7.1 gives $\Gamma^0_{ij} = O(\lambda^\alpha)$. □

Finally, we discuss some implications of Theorems 7.1–7.3 for the exponential map on $(\overline{\mathcal{M}_{\lambda_0}}, \mathbf{g})$. To simplify the discussion, we first extend the metric $\mathbf{g}_\Psi$ to $\mathcal{M}_{\mathrm{dbl}} := (-\lambda_0, \lambda_0) \times M$ (the double of $\overline{\mathcal{M}_{\lambda_0}}$) by reflection across the boundary. Replacing the manifold-with-boundary by one without a boundary allows us to quote without change various theorems having open sets in their hypotheses. Note that curvature is defined only on the complement $\mathcal{M}'$ of $\{\lambda = 0\}$ in $\mathcal{M}_{\mathrm{dbl}}$.

Since we don't know that the Christoffel symbols in Theorem 7.1 are uniformly Lipschitz on $\mathcal{M}_{\rm dbl}$ we cannot appeal directly to the fundamental theorem of ODE's to establish existence and uniqueness of geodesics with given initial conditions. However, Theorem V.8.1 of [Har2] implies that (in a given coordinate patch) if the Christoffel symbols of a C^1 metric $\tilde{g}$ satisfy a certain "L-Lipschitz" condition, then existence and uniqueness do hold for solutions to the geodesic equation (on an open set in initial-condition space), and moreover the dependence on initial conditions is locally uniformly Lipschitz. Hartman's L-Lipschitz condition is satisfied if there exists a sequence of C^1 matrix-valued 1-forms $\{\Gamma^\alpha_{\beta\mu}(n)dx^\mu\}$ converging uniformly to the Christoffel connection forms of $\tilde{g}$, such that the components of the 2-forms $d\Gamma(n)$ are uniformly bounded; equivalently, if the curvature 2-forms $d\Gamma(n)+\Gamma(n)\bigwedge\Gamma(n)$ are uniformly bounded (see Hartman's Exercise V.6.2 on p. 106 and its solution on p. 563). For $\tilde{g}=\mathbf{g}_\Psi$ we can obtain such a sequence by convolving the Christoffel symbols $\Gamma^\alpha_{\beta\mu}$ of $\mathbf{g}_\Psi$ with a sequence of mollifiers δ_n approaching a δ-function. To see that the curvature bound is satisfied, first note that since the $\Gamma^\alpha_{\beta\mu}$ are continuous, and the curvature of $\mathbf{g}_\Psi$ is smooth and bounded on $\mathcal{M}'_{\rm dbl}$ (Theorem 1.1), the 2-form $d\Gamma$ is smooth and bounded on $\mathcal{M}'_{\rm dbl}$, hence integrable on $\mathcal{M}_{\rm dbl}$. Hence $d(\delta_n*\Gamma)=\delta_n*d\Gamma$ (where $\delta_n*(f_\mu dx^u):=(\delta_n*f_\mu)dx^\mu$, etc.), and therefore $d\Gamma(n)$ is uniformly bounded. Using Theorem 7.1 to give a uniform bound on $\Gamma(n)$, we obtain the uniform curvature bound.

Thus the exponential map on $\mathcal{M}'_{\rm dbl}$ exists and is unique. Since $\{\lambda=0\}$ is totally geodesic, geodesics emanating from $\{\lambda=0\}$ with initial velocity in the half-space $\{\dot{\lambda}\geq 0\}$ stay in our original un-doubled manifold $\overline{\mathcal{M}_{\lambda_0}}$. Let $T_+(\overline{\mathcal{M}_{\lambda_0}})=T\mathcal{M}_{\lambda_0}\bigcup\{\text{the } \{\dot{\lambda}\geq 0\} \text{ half-spaces in } T\overline{\mathcal{M}_{\lambda_0}}|_{\partial\mathcal{M}}\}$, topologized as a subset of $T(\overline{\mathcal{M}_{\lambda_0}})$. Then we have proven

Theorem 7.4. *The exponential map is well-defined on some relatively open neighborhood of the zero-section of $T_+(\overline{\mathcal{M}_{\lambda_0}})$, and is locally uniformly Lipschitz. In particular the normal exponential map from the boundary is well-defined on some relatively open neighborhood of the zero-section and is locally uniformly Lipschitz.*

We remark that if the curvature of $\mathcal{M}_{\lambda_0}$ were known to extend continuously to $\overline{\mathcal{M}_{\lambda_0}}$, then Theorem V.6.1 of [Har2] would imply that the dependence of geodesics on initial data is C^1, and hence the normal exponential map would be a local diffeomorphism. Even without knowing continuity of the curvature, it may be possible to deduce greater regularity of the normal exponential map by other means—there is more geometry in our setup than was exploited in the ODE proof above—but we will not pursue that here.

8. Appendix: Variation of Holonomy and Families of Normal Coordinate Systems

We begin this section with a discussion of the parameter space in which variation of normal coordinates is best viewed. In order to perform the variation, we make use of a very general formula for variation of holonomy, which we provide in §8.2. In §8.3, we apply this to produce our desired formula for variation of coordinate functions. For the applications in §6 we need the formula to be differentiable and have certain bounds on its derivatives, which necessitates our providing more detail in §8.1 and §8.3 than would otherwise be needed—indeed, this is the hard part.

8.1. **The parameter space for normal coordinate systems.** Let (M, g) be a compact Riemannian manifold of dimension m. The set of normal coordinate systems on M is parametrized by the orthonormal frame bundle $F(M)$. Given a point $p \in M$ and a frame $\mathbf{e}_p$ at p, asking how the normal coordinates of a fixed point q (near p) change as we vary p only makes sense if we prescribe a way for the frame $\mathbf{e}_p$ to change with p as well.

It's useful, therefore, to introduce a space N which incorporates both the parametrizing frames $\mathbf{e}_p$ and the points q at which the normal coordinates are evaluated. Let $W \subset M \times M$ be a closed neighborhood of the diagonal with the property that for all $(p, q) \in W$, each of p, q is contained in a normal-coordinate neighborhood of the other. Define N to be pullback of the frame bundle $F(M)$ to W by projection onto the first factor from W to M, and N^* to be the portion of N lying above the complement of the diagonal. We use the notation $(\mathbf{e}_p, q)$ for points of N, where $(p, q) \in W$ and $\mathbf{e}_p$ is a frame at p. By a "normal neighborhood" of $p \in M$ we will mean an open set U containing p for which $U \times U \subset W$.

N comes equipped with a smooth map to $\mathbf{R}^m$ sending $(\mathbf{e}_p, q)$ to the m-tuple of coordinates of q in the normal coordinate system $\{x^i_{\mathbf{e}_p}\}$ defined by $\mathbf{e}_p$. We denote the component functions of this map by $\tilde{x}^i : N \to \mathbf{R}$. Thus, normal coordinates become global functions on N.

Let r denote both the distance function on W and the pullback of the distance function to N. We are often interested in the order of vanishing of functions on N or N^* as $r \to 0$. We will always use the notation "$O(r^n)$" in a *uniform* sense—i.e. we say $f : N \to \mathbf{R}$ is $O(r^n)$ iff there is a constant c for which $f(\mathbf{e}_p, q) \leq cr(p, q)^n$ everywhere in N (or N^*, if $n \leq 0$). For some expressions involving normal coordinates (e.g. those arising in the expansion of the metric) we often come across bounds of the form

$$|f(\mathbf{e}_p, q)| \leq c(\mathbf{e}_p) r_p(q)^n, \tag{8.1}$$

where r_p denotes distance to p. In such instances f is often $O(r^n)$ in the sense above, using compactness of N and the construction of the bound (8.1). When this is the case we will simply assert that f is $O(r^n)$ without explicit proof of uniformity.

There are two types of vector fields on N that will be of concern to us. First, using the natural isomorphism $T_{(\mathbf{e}_p,q)}N \cong T_{\mathbf{e}_p}(F(M)) \times T_q M$, we define smooth global vector fields $X_1, \ldots, X_m$ on N by

$$X_i(\mathbf{e}_p, q) \quad = \quad (0, \frac{\partial}{\partial x^i}(q)) \in T_{\mathbf{e}_p}(F(M)) \times T_q M,$$

where $\frac{\partial}{\partial x^i}$ are the coordinate vector fields near p in the normal coordinate system $\{x^i = x^i_{\mathbf{e}_p}\}$. The $\{X_i\}$ enter when one wants to differentiate an expression in a temporarily fixed normal coordinate system, then examine how the answer depends on parameters in N.

The second type of vector field (actually a vector field with a restricted domain) is the type associated with varying a normal coordinate system. Given any vector $V \in T_{\mathbf{e}_p} M$, for each q there is a corresponding vector $\tilde{V}_q = (V, 0) \in T_{\mathbf{e}_p}(F(M)) \times T_q M \cong T_{(\mathbf{e}_p,q)}N$; thus $\tilde{V}$ is a vector field along a submanifold $\{\mathbf{e}_p\} \times U_p \subset N$, where U_p is a normal neighborhood of p. A general variation of normal coordinates is simply the function on U_p given by $q \mapsto \{\tilde{V}(\tilde{x}^i)\Big|_{(\mathbf{e}_p,q)}\}_{i=1}^m$. For the variations we consider, we will always take V to be the horizontal lift (with respect to the Levi-Civita connection) of some $v_p \in T_p M$.

Definition 1. We say that $f : N \to \mathbf{R}$ (or $N^* \to \mathbf{R}$) is $O_{\text{strong}}(r^n)$ if f is $O(r^n)$ and if for any multi-index $(i_1, \ldots, i_k)$ the derivatives $X_{i_1} \ldots X_{i_k} f$ are $O(r^{n-k})$ (equivalently, if the k^{th} covariant derivative of f with respect to the second variable in N is $O(r^{n-k})$). Similarly, if f has auxiliary dependence on some parameter A, we say f is write $O_{\text{strong}}(Ar^n)$ if all m^{th} derivatives of the above form are uniformly $O(Ar^{n-m})$ (with constants independent of A).

For the applications in this paper, it suffices to take $m \leq 2$ in this definition.

The fact that a given function on N is $O_{\text{strong}}(r^n)$ does not by itself imply that its variation under a vector field V as above is $O(|v|r^{n-1})$ (although for the functions whose variations we compute this turns out to be the case); it is only derivatives with respect to the *second* variable (i.e. q in $(\mathbf{e}_p, q)$) that we know *a priori* to be $O(r^{n-1})$. This forces us to establish separately, in §8.3, the order of vanishing of the variations of functions $\zeta_{ij}, \hat{\psi}_{ij}$ (see Definition 2 below) known already to be $O_{\text{strong}}(r)$.

In order to prove that various quantities encountered in §8.3 have the "O_{strong}" behavior we assert, we will need the following lemma and the subsequent Corollary 8.2.

Lemma 8.1. *Given $(\mathbf{e}_p, q) \in N$, let $\mathbf{e}_q$ be the frame obtained by parallel translating $\mathbf{e}_p$ to q along the minimal geodesic, and let $\{x^i\}$, $\{y^i\}$ be the normal coordinate systems determined by $\mathbf{e}_p, \mathbf{e}_q$ respectively. Then*

$$e_i(p) = \left[\delta_{ij} + \frac{1}{6}R_{ikjl}(p)x^k(q)x^l(q) + O_{\text{strong}}(r^3)\right]\frac{\partial}{\partial y^j}(p), \tag{8.2}$$

where $\{R_{ikjl}(p)\}$ are the components of the Riemann tensor at p in the frame $\mathbf{e}_p$.

This lemma is is more subtle than it first appears. N admits an involution τ defined by $\tau(\mathbf{e}_p, q) = (\mathbf{e}_q, p)$, where $\mathbf{e}_q$ is the radially parallel translate of $\mathbf{e}_p$ as in the lemma. The fact that a function $f : N \to \mathbf{R}$ is $O_{\text{strong}}(r^n)$ does not by itself imply that $\tau^* f$ is $O_{\text{strong}}(r^n)$. (See the comment after Definition 1; the situation here is similar.) This observation applies to the remainder term in (8.2). Had we started with a frame at q rather than at p, and were regarding the bracketed expression in (8.2) as a function of coordinates $(\mathbf{e}_q, p)$ on N rather than of $(\mathbf{e}_p, q)$ the "O_{strong}" assertion would be easier to prove. Proving (8.2) as it stands is more delicate because the coordinate system defining $\{\frac{\partial}{\partial y^i}\}$ itself depends on q.

Proof of Lemma 8.1: Parametrize the minimal geodesic γ from q to p proportionally to arclength, with $\gamma(0) = q$, $\gamma(1) = p$; in the coordinates $\{y^i\}$, we have $\gamma(t) = ty(p)$. Let $\mathbf{e}(t)$ be the frame at $\gamma(t)$ obtained by parallel translating $\mathbf{e}_q$ along γ, and write $e_i(ty) = f^j{}_i(t)\frac{\partial}{\partial y^j}$; note that $f^j{}_i(0) = \delta_{ij}$. Using $\nabla_{\gamma'} e_i = 0$ we obtain the equation

$$f' + Af = 0, \tag{8.3}$$

where

$$A(t)^i{}_j = \langle dy^i, \nabla_{y^k(p)\partial_k}\partial_j\rangle\big|_{ty(p)} = y^k(p)\Gamma^i{}_{jk}(\mathbf{e}_q; ty(p)). \tag{8.4}$$

Here $\Gamma^i{}_{jk}(\mathbf{e}_q; ty(p))$ are the Christoffel symbols with respect to the frame $\mathbf{e}_q$, evaluated at $\gamma(t)$. Since $\Gamma^i{}_{jk}(\mathbf{e}_q; 0) = 0$,

$$\begin{aligned} A(t)^i{}_j &= y^k(p)\int_0^1 \frac{\partial}{\partial t_1}\Gamma^i{}_{jk}(\mathbf{e}_q; t_1 ty(p))dt_1 \\ &= ty^k(p)y^l(p)\int_0^1 (\frac{\partial}{\partial y^l}\Gamma^i{}_{jk})(\mathbf{e}_q; t_1 ty(p))dt_1. \end{aligned}$$

The last integral above is determined completely (and smoothly) by $(\mathbf{e}_p, q)$ and t, so we will write it as ${B_{kl}}^i{}_j(\mathbf{e}_p, q, t)$; we regard this as a matrix-valued function B_{kl} with matrix indices i, j. Since $y^k(p) = -x^k(q)$, we can rewrite (8.3) as

$$f' + tx^k(q)x^l(q)B_{kl}(\mathbf{e}_p, q, t)f = 0. \tag{8.5}$$

Because the B_{kl} are smooth on $N \times [0,1]$ and $x^k(q)x^l(q)$ is $O_{\text{strong}}(r^2)$, the solution to the linear equation (8.3) with initial condition $f(0) = I$ is $I + O_{\text{strong}}(t^2r^2)$, as one can see, for example, by expressing the solution as a path-ordered exponential.

We can take the preceding argument one order further, writing ${B_{kl}}^i{}_j(\mathbf{e}_p, q, t) =$

$$\int_0^1 \left((\frac{\partial}{\partial y^l}{\Gamma^i}_{jk})(\mathbf{e}_q; 0) + t_1 t y^n \int_0^1 (\frac{\partial^2}{\partial y^l \partial y^n}{\Gamma^i}_{jk})(\mathbf{e}_q; t_2 t_1 t y(p)) dt_2 \right) dt_1.$$

It is well-known that $(\partial_l {\Gamma^i}_{jk})(\mathbf{e}_q; 0) = -\frac{1}{3}(R_{ijkl}(q) + R_{ikjl}(q))$ (components of the Riemann tensor being taken relative to $\mathbf{e}_q$). Since $x^k x^l R_{ijkl} = 0$, we can therefore rewrite (8.5) in the form

$$f' + (-\frac{1}{3}tx^k(q)x^l(q)R_{\cdot k \cdot l} - x^k(q)x^l(q)x^m(q)B_{klm}(\mathbf{e}_p, q, t))f = 0, \tag{8.6}$$

where now B_{klm} is smooth in all parameters. From (8.6) we obtain the solution

$$f(t)^i{}_j = \delta_{ij} + \frac{1}{6}t^2 x^k(q)x^l(q)R_{ikjl}(q) + O_{\text{strong}}(r^3t^3).$$

Setting $t = 1$ and noting that $R_{ikjl}(q) = R_{ikjl}(p) + O_{\text{strong}}(r)$, (8.2) follows.

An immediate corollary of Lemma 8.1 is the following fact concerning the Jacobian of a normal-coordinate change.

Corollary 8.2. *Let $p, q \in M$ with each point contained in a normal-coordinate neighborhood of the other. Let $\{e_i(p)\}$ be an orthonormal basis of T_pM and let $\{e_i(q)\}$ be the basis obtained by parallel translating $\{e_i(p)\}$ along the minimal geodesic from p to q. Let $\{x_p^i\}, \{x_q^i\}$ be the corresponding normal coordinate systems centered at p, q respectively and let $r = \text{dist}(p, q)$. Then*

$$\frac{\partial x_q^j}{\partial x_p^i}(p) = \delta_{ij} + O_{\text{strong}}(r^2). \tag{8.7}$$

PROOF. Simultaneously, we have $e_i(p) = \frac{\partial}{\partial x_p^i}(p) = \frac{\partial x_q^j}{\partial x_p^i}(p)\frac{\partial}{\partial x_q^j}(p)$ and (from (8.2)) $e_i(p) = (\delta_{ij} + O_{\text{strong}}(r^2))\frac{\partial}{\partial x_q^j}(p)$. □

In addition to the global coordinate functions $\{\tilde{x}^i\}$ on N, there are several other global functions on N used earlier in this paper.

Definition 2. (i) The m^2 functions $g^{ij} : N \to \mathbf{R}$ are defined by $g^{ij}(\mathbf{e}_p, q) = g(dx^i, dx^j)\big|_q$, where $\{x^i = x^i_{\mathbf{e}_p}\}$ are the normal coordinates near p determined by $\mathbf{e}_p$. (ii) The m^2 functions $\zeta_{ij} : N \to \mathbf{R}$ are defined by $\zeta_{ij}(\mathbf{e}_p, p) = 0$ and

$$\zeta_{ij}(\mathbf{e}_p, q) = \frac{\partial^2 r_q}{\partial x^i \partial x^j}\bigg|_p - \frac{\partial^2 r_p}{\partial x^i \partial x^j}\bigg|_q = \frac{\partial^2 r_q}{\partial x^i \partial x^j}\bigg|_p - \left(\frac{\delta_{ij}}{r} - \frac{x^i(q)x^j(q)}{r^3}\right) \tag{8.8}$$

for $p \neq q$; here r_q denotes distance to q. Alternatively, if we define normal coordinates $\{y^i\}$ as in Lemma 8.1, then $\zeta_{ij} = \frac{\partial^2 r_q}{\partial x^i \partial x^j}\Big|_p - \frac{\partial^2 r_q}{\partial y^i \partial y^j}\Big|_p$. (iii) The m^2 functions $\hat{\psi}_{ij} : N \to \mathbf{R}$ are defined by $\hat{\psi}_{ij}(\mathbf{e}_p, p) = 0$ and $\hat{\psi}_{ij}(\mathbf{e}_p, q) = -\zeta_{ij} + r^{-1}(g^{ij} - \delta_{ij})$ for $p \neq q$.

Lemma 8.3. *The functions ζ_{ij} and $\hat{\psi}_{ij}$ are $O(r)$.*

PROOF. Since $g^{ij} = \delta_{ij} + O(r^2)$, we need only show $\zeta_{ij} = O(r)$. This was proven in [GP2, Lemma 4.2] using Jacobi fields, but we give a different proof here to obtain an expression for ζ_{ij} that allows us later to compute its variation.

Let $(\mathbf{e}_p, q) \in N$, let $\{x^i\}$ be the normal coordinates determined by $\mathbf{e}_p$ and let $\mathbf{e}_q, \{y^i\}$ be the frame and normal coordinates defined in Lemma 8.1. When we are not differentiating, we will write $r = r_q$ and $y^i = y^i(p)$.

First note that

$$\frac{\partial r_q}{\partial x^i}(p) = -\frac{x^i(q)}{r}. \tag{8.9}$$

This can be derived from the formula for first variation of arclength; see [GP2, Lemma 4.2].

Next, we have

$$\nabla dr_q|_p = \nabla(r_q^{-1} y^i dy^i)\big|_p = (\frac{\delta_{ij}}{r} - \frac{y^i y^j}{r^3}) dy^i \otimes dy^j + r^{-1} y^i \nabla dy^i.$$

On the other hand, since $\nabla dx^i|_p = 0$,

$$\nabla dr_q = \frac{\partial^2 r_q}{\partial x^i \partial x^j} dx^i \otimes dx^j$$

Since $y^i(p) = -x^i(q)$, comparing the last two equations we find

$$\begin{aligned} \zeta_{ij}(\mathbf{e}_p, q) dx^i \otimes dx^j\big|_p &= (\frac{\delta_{ij}}{r} - \frac{x^i(q)x^j(q)}{r^3})\,(dy^i \otimes dy^j - dx^i \otimes dx^j)\big|_p \\ &\quad - r^{-1} x^i(q) \nabla dy^i\big|_p . \end{aligned} \tag{8.10}$$

From Corollary 8.2, $(dy^i \otimes dy^j - dx^i \otimes dx^j)|_p = O(r^2)$. Since $\nabla dy^i|_p = O(r^2)$ as well, the assertion of the lemma follows from (8.10). $\square$

With a little more work, one can show that ζ_{ij} is $O_{\text{strong}}(r)$ (and hence so is $\hat{\psi}_{ij}$); cf. the proof of Lemma 8.1. We omit the argument since the stronger statement will not be used.

8.2. **Variation of holonomy in general vector bundles.** Our derivation in §8.3 of the formula for variation of normal coordinates will make use of a more general formula for variation of holonomy.

Let $E \to M$ be a vector bundle with connection ∇ and curvature K. We first consider the variation in the parallel transport map along a 1-parameter family of curves with fixed endpoints. Later we will specialize to loops. Below, t is always used for the parameter along a curve, and s for the parameter of variation. We will use "prime" to denote d/dt and "dot" to denote d/ds.

Proposition 8.4. *Let $q, p \in M$ and let $\gamma_0 : [0,1] \to M$ be a smooth curve with $\gamma_0(0) = q$, $\gamma_0(1) = p$. Let $\gamma : [0,\epsilon) \to M$ be a variation of γ with fixed endpoints, and write $\gamma_s(t) = \gamma(s,t)$. Let $X(t) = X_0(t) = \gamma_0'(t)$, and (abusing notation) let $V(t) = V_0(t)$ be the variation vector field $\frac{\partial\gamma(s,t)}{\partial s}|_{s=0}$ along γ. Let $P_s \in Hom(E_q, E_p)$ be the parallel transport map along γ_s. Then*

$$\frac{dP_s}{ds}|_{s=0} = -P_{\gamma(0)\to\gamma(1)} \circ \int_0^1 P_{\gamma(\tau)\to\gamma(0)} \circ K(V(\tau), X(\tau)) \circ P_{\gamma(0)\to\gamma(\tau)} d\tau, \tag{8.11}$$

where $P_{\gamma(a)\to\gamma(b)}$ denotes parallel transport along γ from $\gamma(a)$ to $\gamma(b)$. Equivalently, we can write this as

$$\dot{P}(0) = \int_0^1 P_{\gamma(\tau)\to\gamma(1)} \circ K(X(\tau), V(\tau)) \circ P_{\gamma(0)\to\gamma(\tau)} d\tau. \tag{8.12}$$

(See also equation (8.18) below, for a local-frame version of this formula.)

PROOF. First assume that the image of γ_0 lies in an open set $U \subset M$ over which E is trivial, and let $\{e_\alpha\}$ be a framing of $E|_U$. Since we are interested in a derivative at $s = 0$, we may assume ϵ is sufficiently small that the image of γ lies in U.

Let ω be the (matrix-valued) connection form over U with respect to $\{e_\alpha\}$; thus $\nabla_Y e_\alpha = e_\beta \omega^\beta{}_\alpha(Y)$. For each α let $f_\alpha(s,t)$ be the parallel translate of $e_\alpha(q)$ along γ_s from q to $\gamma_s(t)$. Then there is a unique matrix-valued function h on $[0,\epsilon) \times [0,1]$ such that

$$f_\alpha(s,t) = e_\beta(\gamma_s(t)) h^\beta{}_\alpha(s,t) = P_{\gamma_s(0)\to\gamma_s(t)}(e_\alpha(q)); \tag{8.13}$$

note that $h(s,0) = Id$ for all s. The parallelism of f_α along each γ_s implies that for each s, $h_s := h(s,\cdot)$ is the fundamental solution of the equation

$$h_s'(t) + A_s(t)h_s(t) = 0, \tag{8.14}$$

where $A_s(t) = A(s,t)$ is the matrix-valued function

$$A_s = \iota_{\partial/\partial t}\gamma^*\omega. \tag{8.15}$$

Differentiating (8.14) with respect to s at $s = 0$, we obtain $(\dot{h})'(t) + A_0(t)\dot{h}(t) = -\dot{A}(t)h_0(t)$. Note that $\dot{h}(0) = 0$ since $h_s(0) \equiv Id$. Since h_0 is the fundamental solution of (8.14) for $s = 0$, this implies that

$$\dot{h}(t) = -h_0(t)\int_0^t h_0^{-1}(\tau)\dot{A}(\tau)h_0(\tau)d\tau. \tag{8.16}$$

We next compute $\dot{A}(\tau)$ to plug it into (8.16). From (8.15) we have

$$\begin{aligned}\dot{A}_s(t) = \mathcal{L}_{\partial/\partial s}(\iota_{\partial/\partial t}\gamma^*\omega) &= \iota_{\partial/\partial t}\mathcal{L}_{\partial/\partial s}\gamma^*\omega \\ &= \iota_{\partial/\partial t}(\iota_{\partial/\partial s}d\gamma^*\omega + d\iota_{\partial/\partial s}\gamma^*\omega) \\ &= d\omega(V_s(t), X_s(t)) + \frac{\partial}{\partial t}\langle\omega, V_s(t)\rangle\end{aligned}$$

where $V_s(t) = \gamma_{*(s,t)}\frac{\partial}{\partial s}$ and $X_s(t) = \gamma_{*(s,t)}\frac{\partial}{\partial t}$. Plugging into (8.16), we obtain

$$\begin{aligned}\dot{h}(t) = -h_0(t) \quad &\cdot \quad \{\int_0^t h_0^{-1}(\tau)d\omega(V(\tau), X(\tau))h_0(\tau)d\tau \\ &+ \quad \int_0^t h_0^{-1}[\frac{\partial}{\partial\tau}\langle\omega(\gamma_0(\tau)), V(\tau)\rangle]h_0(\tau)d\tau\}.\end{aligned} \tag{8.17}$$

In the second integral, integrate by parts, and in the boundary term use the fact that $V(0) = 0$. Using (8.14-8.15), we also have $h_0'(\tau) = -\langle\omega, X(\tau)\rangle h_0(\tau)$ and $d/d\tau(h_0^{-1}(\tau) = h_0^{-1}(\tau)\langle\omega, X(\tau)\rangle$. This puts the second integral in (8.17) into the form

$$h_0^{-1}(t)\langle\omega(\gamma_0(t)), V(t)\rangle h_0(t) + \int_0^t h_0^{-1}(\tau)\omega\bigwedge\omega(V(\tau), X(\tau))h_0(\tau)d\tau.$$

Since $d\omega + \omega\wedge\omega$ is the matrix $\hat{K}$ of K in the frame $\{e_\alpha\}$, we therefore obtain

$$\dot{h}(t) = -\langle\omega(\gamma_0(t)), V(t)\rangle h_0(t) - h_0(t)\int_0^t h_0^{-1}(\tau)\hat{K}(V(\tau), X(\tau)h_0(\tau)d\tau.$$

In particular this applies when $t = 1$. Since $V(1) = 0$, we have

$$\dot{h}(1) = -h_0(1)\int_0^1 h_0^{-1}(\tau)\hat{K}(V(\tau), X(\tau)h_0(\tau)d\tau. \tag{8.18}$$

To identify in invariant form the operator appearing here, consider the operator $P : E_q \to E_p$ given by

$$P(v) = -P_{\gamma_0(0)\to\gamma_0(1)} \int_0^1 P_{\gamma_0(\tau)\to g(0)} \circ K(V(\tau), X(\tau))d\tau.$$

A straightforward calculation using (8.13) and the fact that $K(\cdot,\cdot)((e_\alpha(z)) = e_\mu(z)\hat{K}^\mu{}_\alpha(\cdot,\cdot)$ shows that

$$\begin{aligned} P(e_\alpha(q)) &= -e_\beta(p)(h_0)^\beta{}_\nu(1) \int_0^1 (h_0^{-1})^\nu_\mu(\tau)\hat{K}^\mu{}_\sigma(V(\tau), X(\tau))(h_0)^\sigma{}_\alpha(\tau)d\tau \\ &= e_\beta(p)\dot{h}(1)^\beta{}_\alpha \end{aligned}$$

by (8.18). On the other hand, $P_s(e_\alpha(p)) = e_\beta(p)h(s,1)^\beta{}_\alpha$, implying

$$\dot{P}|_{s=0}(e_\alpha(p)) = e_\beta(p)\dot{h}(1)^\beta{}_\alpha = P(e_\alpha(q)).$$

By linearity, we obtain (8.11).

To remove our initial assumption that the image of γ_0 lay in a trivializing neighborhood for E, simply use the variation γ to pull back E and ∇ to the rectangle $[0,\epsilon)\times[0,1]$. The pulled-back bundle is trivial, so the previous argument applies. Unwinding the pullbacks in the resulting formula, we again arrive at (8.11). □

One class of variations will be of particular concern to us. Suppose $s \mapsto \alpha_s$ is a smooth variation of a curve α_0 from q to p, with $\alpha_s(0) \equiv q$ but $\alpha_s(1)$ variable. Suppose further that $s \mapsto \beta_s$ is a smooth variation of the constant curve $\beta_0 \equiv p$, such that for all s we have $\beta_s(0) = p$, $\beta_s(1) = \alpha_s(1)$ (see Figure 1). For each s we define the curve γ_s by traveling along α_0 from q to p, then from p to $\alpha_s(1)$ along β_s, then from $\alpha_s(1)$ back to q along the inverse of α_s. (Note that γ_0 is simply α followed by α^{-1} from q to p and back again.)

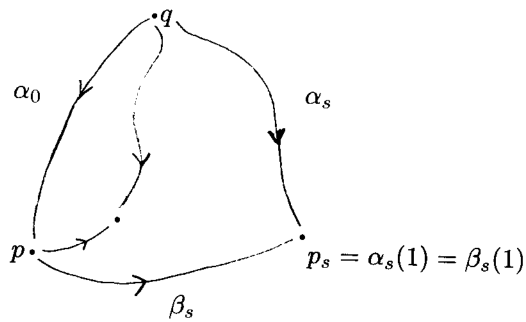

Figure 1

In this situation we have the following.

Corollary 8.5. *Let the variations α, β, γ be as just described. Let $Y(t) = \alpha_0'(t)$ be the tangent vector field along α_0, and let $W(t) = \dot{\alpha}(t)|_{s=0}$ be the variation vector field of α along α_0. Let $P_s \in End(E_q)$ be the holonomy around the loop γ_s. Then*

$$\dot{P}(0) = \int_0^1 P_{\alpha(\tau)\to q} \circ K(Y(\tau), W(\tau)) \circ P_{q\to\alpha(\tau)} d\tau, \tag{8.19}$$

where the parallel translations are taken along α.

PROOF. We can parametrize the variation γ_s by

$$\gamma_s(t) = \begin{cases} \alpha'(t), & 0 \le t < 1, \\ \beta_s(t-1), & 1 \le t < 2, \\ \alpha_s(3-t), & 2 \le t \le 3. \end{cases}$$

Let $X(t), V(t)$ be as in Proposition 8.4, $0 \le t \le 3$. Note that $V(t) \equiv 0$ for $0 \le t \le 1$ and that, since β_0 is a constant curve, $X(t) \equiv 0$ for $1 \le t \le 2$. For $2 \le t \le 3$ we have $X(t) = -Y(3-t)$ and $V(t) = W(3-t)$, so the result follows from (8.12). $\square$

Remark. In particular, this corollary applies to both of the following situations. (a) If $\beta(s) = \alpha_s(1)$ and $\beta_s(t) = \beta(st)$, so that the images of the β_s are progressively larger portions of the curve β_1. (b) If M is Riemannian, $\alpha_s(1)$ lies in a normal neighborhood of p, and β_s is the unique minimal geodesic from p to $\alpha_s(1)$.

8.3. **Application: variation of normal coordinate systems.** We are now ready to derive our basic variation formula. Again s will denote the variation parameter throughout this section.

Let $p \in M$, let U be a normal neighborhood of p, and let $\mathbf{e}^0 = (e_1^0(p), \ldots, e_n^0(p))$ be an orthonormal basis of T_pM. Let $v \in T_pM$, and let $\beta : [0, \epsilon] \to U$ be a curve with $\beta(0) = p$ and $\beta'(0) = v$. There are two obvious ways to extend $\mathbf{e}$ to the curve β, defining $e_i^s(\beta(s)) \in T_{\beta(s)}M$ to be the parallel translate of $e_i^0(p)$ from p to $\beta(s)$ along either β or the unique minimal geodesic from p to $\beta(s)$.

Below, we will write $p_s = \beta(s)$ and will always take $\{e_i^s(p_s)\}$ to be defined as in one of these two ways. It is irrelevant which way is used since, in either case, the tangent vector at $s = 0$ to the curve $s \mapsto \mathbf{e}^s$ in $F(M)$ is the Levi-Civita horizontal lift V of v, so that the vectors $V_q = \{\dot{\mathbf{e}}, 0\} \in T_{(\mathbf{e}_p,q)}N\}$ form a variation vector field $\tilde{V}$ as discussed just before Definition 1. Therefore if we let $\{x_s^i\}$ denote the normal coordinate system centered at p_s determined by the frame $\mathbf{e}^s = (e_1^s(p_s), \ldots, e_n^s(p_s))$, then

$$\dot{x}^i(q) := \frac{d}{ds} x_s^i(q)\bigg|_{s=0} = \tilde{V}(\tilde{x}^i)\bigg|_{(\mathbf{e}_p,q)}.$$

We will prove the following:

Proposition 8.6. *In the situation above, with q fixed and with the coordinate center-point p varying along a curve with initial tangent vector v, we have*

$$\dot{x}^i(q) = -v^i - \frac{2}{3} x^k x^l v^j R_{ikjl}(p) + O_{\text{strong}}(r^3|v|) \tag{8.20}$$

where $\{R_{ikjl}(p)\}, \{v^i\}$ are the components of the Riemann tensor and v in the frame $\{e_i^0(p)\}$.

PROOF. First note that $x_0^i = x^i$, and that for all $s \in [0, \epsilon], q \in U$,

$$q = \exp_p(x^i(q)e_i) = \exp_{\beta(s)}(x_s^i e_i^s) = \exp_{p_s}(x_s^i e_i^s).$$

Since any choice of β with $\dot{\beta}(0) = v$ will produce the same variation vector field, we are free to choose β conveniently. It simplifies matters if we choose

$$\beta(s) = \exp_q(w_0 + sw_1),$$

where $w_0 = -x^i(q)e_i^0(q)$ and $w_1 = (\exp_{q*}|_{w_0})^{-1}(v)$. Let α_s be the minimal geodesic from $q = \alpha_s(0)$ to $p_s = \alpha_s(1)$, and define $e_i^s(\alpha_s(t)) \in T_{\alpha_s(t)}M$ to be the parallel translate of $e_i^s(p_s)$ along α_s^{-1}. In particular this gives us a 1-parameter family of frames $\{e_i^s(q)\}$ at q, and

$$\exp_q(-x_s^i(q)e_i^s(q)) = \beta(s) = p_s \quad \forall s.$$

Differentiating at $s=0$, we obtain

$$\exp_{q*}\big|_{w_0}\left(-\dot{x}^i(q)e_i^0(q) - x^i(q)(e_i^s(q))^{\cdot}\right) = v.$$

Thus

$$\dot{x}^i(q)e_i^0(q) = -x^i(q)(e_i^s(q))^{\cdot} - w_1. \tag{8.21}$$

We will use (8.21) to compute $\dot{x}^i(q)$. First we compute the term $(e_i(q))^{\cdot}$ in (8.21) using the fact that $e_i^s(q)$ can be obtained by parallel translating $e_i^0(q)$ around the following loop:

(i) Go from q to p (in time 1) along the minimal geodesic α_0.

(ii) Then go from p to $p_s = \beta(s)$ (in time 1) along either β or the unique minimal geodesic, depending on which way has been chosen to define $\mathbf{e}_i^s$.

(iii) Then go from $\beta(s)$ back to q (in time 1) along the geodesic α_s^{-1}.

The composite curve gives us a variation of the type described in either part (a) or (b) of the remark following Corollary 8.5, depending on which way $\mathbf{e}_i^s$ has been defined. Hence the corollary applies, and the variation in holonomy around the initial loop is given by

$$\dot{P}(0) = \int_0^1 P_{\alpha_0(t)\to q} \circ R(\alpha_0'(t), \frac{\partial \alpha_s}{\partial s}(t)|_{s=0}) \circ P_{q\to\alpha_0(t)}dt,$$

where R is the Riemannian curvature 2-form and the parallel translations are taken along α_0. By our choice of β, we have $\alpha_s(t) = \exp_q(t(w_0+sw_1))$, and hence

$$\frac{\partial \alpha_s}{\partial s}(t)\bigg|_{s=0} = \frac{d}{ds}\exp_q\left(t(w_0+sw_1)\right)\bigg|_{s=0} \tag{8.22}$$

is precisely the Jacobi field $J(t)$ along α_0 with $J(0)=0,\ J(1)=v$. Thus

(8.23)
$$(e_i(q))^{\cdot} = \dot{P}(0)\left(e_i^0(q)\right) = \int_0^1 P_{\alpha_0(t)\to q}\circ R(\alpha_0'(t), J(t))\circ P_{q\to\alpha_0(t)}(e_i^0(q))dt.$$

Since the frame $\{e_i^0\}$ is parallel along α_0, (8.23) simplifies to

$$(e_i(q))^{\cdot} = \left[\int_0^1 \hat{R}^j{}_i(\alpha_0'(t), J(t))dt\right] e_j^0(q) \tag{8.24}$$

where $\hat{R}^j{}_i$ is the matrix of R in the frame $\{e_i^0(\alpha_0(t))\}$.

We want to determine the leading-order behavior of this integral and estimate the remainder. Since α_0 is a radial geodesic from q to p, we have

$$\alpha_0'(t) = -x^i(q) \left.\frac{\partial}{\partial x^i}\right|_{\alpha_0(t)} = -x^i(q)e_i^0(\alpha_0(t)). \tag{8.25}$$

Also, from (8.22), using (8.2) twice we find

(8.26)

$$J(t) = t \cdot \exp_{q*}|_{tw_0} \circ (\exp_{q*}|_{w_0})^{-1}(v) = t\left(v^l e_l^0(\alpha_0(t)) + O_{\text{strong}}((1-t)r^2|v|)\right).$$

Inserting (8.25-8.26) into (8.24), and using the fact that $\hat{R}^j{}_i(e_k^0(\alpha_0(t)), e_l^0(\alpha_0(t))) = R_{jikl}(p) + O_{\text{strong}}((1-t)r)$, we find

$$\int_0^1 \hat{R}^j{}_i(\alpha_0'(t), J(t))dt = -\frac{1}{2}x^k(q)R_{jikl}(p)v^l + O_{\text{strong}}(r^2|v|). \tag{8.27}$$

Hence

$$x^i(q)(e_i(q))^{\cdot} = \left[\frac{1}{2}x^j(q)x^k(q)R_{jikl}(p)v^l + O_{\text{strong}}(r^3|v|)\right] e_i^0(q). \tag{8.28}$$

This gives us the first term on the right-hand side of (8.21). To handle the second term, namely

$$w_1 = (\exp_{q*}|_{w_0})^{-1}(v) := u^j e_j^0(q),$$

we use Lemma 8.1. Let $\{y^i\}$ be the normal coordinates based at q determined by the frame $\{e_i^0(q)\}$. Then

$$v = v^i e_i(p) = \exp_{q*}|_{w_0}(u^j e_j^0(q)) = u^j \left.\frac{\partial}{\partial y^j}\right|_p .$$

By Lemma 8.1, we therefore have $u^j = \left[\delta_{ij} + \frac{1}{6}R_{ikjl}(p)y^k y^l + O_{\text{strong}}(r^3)\right] v^i$. But $y^k = -x^k(q)$, yielding

$$w_1 = \left[\delta_{ij} + \frac{1}{6}R_{ikjl}(p)x^k x^l + O_{\text{strong}}(r^3)\right] v^j e_i^0(q). \tag{8.29}$$

Combining (8.21), (8.28), and (8.29), we obtain (8.20). □

Next we consider the variations of several other functions on N.

Proposition 8.7. *In the setting of Proposition 8.6 we have the following.*

(a) $\dot{r} = -\frac{v^i x^i}{r}$ *(for $r \neq 0$).*

(b) $(g^{ij})^{\cdot} = O(r|v|)$.

(c) $\dot{\zeta}_{ij}$ *and* $(\hat{\psi}_{ij})^{\cdot}$ *are* $O(|v|)$.

PROOF. Notation will be as in the proof of Lemma 8.3 and Proposition 8.6. Remember that the point q is fixed throughout all variations; it is only p and $\mathbf{e}_p$ that change.

(a) Since $r = r_q$ depends only on p, not on $\mathbf{e}_p$, we have $\dot{r} = v^i \frac{\partial r}{\partial x^i}$. Now use (8.9).

(b) By definition, $g^{ij}(\mathbf{e}_{p_s}, q) = g(dx_s^i, dx_s^j)\big|_q$; hence $(g^{ij})^{\cdot} = g(d\dot{x}^i, dx^j) + g(dx^i, d\dot{x}^j)$. But from Proposition 8.6, $d\dot{x}^i = O(r|v|)$.

(c) In (8.10), replace p by p_s, $\{x^i\}$ by $\{x_s^i\}$, and $\{y^i\}$ by $\{y_s^i\}$ (the normal coordinate system determined by the frame $\mathbf{e}_i^s(q)$ constructed in the proof of Proposition 8.6). We then have an equation between tensor fields along the curve β, which we will covariantly differentiate at p. Below, when we write dx_s^i, dy_s^i, and ∇dy_s^i, we mean the fields along β whose value at s is $dx_s^i(\beta(s))$, $dy_s^i(\beta(s))$, and $(\nabla dy_s^i)(\beta(s))$ respectively.

Note that $\{dx_s^i\}$ is the coframe dual to $\mathbf{e}_i^s(p_s)$, which by construction is covariantly constant along β at p. Hence $\nabla_v(dx_s^i \otimes dx_s^j) = 0$, so, writing $y^i = y_0^i$,

$$\begin{aligned} \dot{\zeta}_{ij} dx^i \otimes dx^j\Big|_p &= \left(\frac{\delta_{ij}}{r_s} - \frac{x_s^i(q)x_s^j(q)}{r_s^3}\right)^{\cdot}(dy^i \otimes dy^j - dx^i \otimes dx^j)\Big|_p \\ &\quad + \left(\frac{\delta_{ij}}{r} - \frac{x^i(q)x^j(q)}{r^3}\right)(\nabla_v(dy^i \otimes dy^j)) \\ &\quad - (r_s^{-1}x_s^i(q))^{\cdot}\nabla dy^i\big|_p - r^{-1}x^i(q)\,\nabla_v(\nabla dy_s^i)\big|_p . \end{aligned} \tag{8.30}$$

From part (a) and (8.20) we compute $(\delta_{ij}/r_s - x_s^i(q)x_s^j(q)/r_s^3)^{\cdot} = O(|v|r^{-2})$ and $(r_s^{-1}x_s^i(q))^{\cdot} = O(|v|r^{-1})$. To compute the covariant derivative of the tensor fields involving dy_s^i, first note that as s varies, $\mathbf{e}_i^s(q)$ changes by a holonomy matrix $h(s)$ as discussed earlier. For all u near p and q, we have $y_s^i(u)\mathbf{e}_i^s(q) =$ constant, and hence $y_s^i(u) = C^i{}_j(s)y_0^i(u)$ where $C = h^{-1}$. Thus $dy_s^i|_{\beta(s)} = C^i{}_j(s)dy_0^j|_{\beta(s)}$ and $\nabla dy_s^i = C^i{}_j(s)\nabla dy_0^i|_{\beta(s)}$, implying

$$\nabla_v(dy_s^i) = \dot{C}^i{}_j dy_0^j(p) + \nabla_v dy_0^i$$

and

$$\nabla_v((\nabla dy_s^i)|_{\beta(s)}) = \dot{C}^i{}_j(\nabla_v dy_0^j)(p) + \nabla_v \nabla dy_0^i.$$

From (8.24) and (8.27), $\dot{h} = O(|v|r)$, and hence $\dot{C} = O(|v|r)$, so the preceding implies $\nabla_v(dy_s^i \otimes dy_s^j - dx_s^i \otimes dx_s^j) = O(|v|r)$ and $\nabla_v((\nabla dy_s^i)|_{\beta(s)}) = O(|v|)$. Combining all of the bounds above with those used in Lemma 8.3, and inserting the result into (8.30), we find $\dot{\zeta}_{ij} = O(|v|)$.

Finally, we have $(\dot{\hat{\psi}}_{ij})^{\cdot} = -\dot{\zeta}_{ij} + (r^{-1})^{\cdot}(g^{ij} - \delta_{ij}) + r^{-1}(g^{ij})^{\cdot}$. From parts (a) and (b) above, the last two terms are $O(|v|)$, so $(\hat{\psi}_{ij})^{\cdot} = O(|v|)$ as well. $\square$

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