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ON THE PARAHOLOMORPHIC SECTIONAL CURVATURE OF ALMOST PARA-HERMITIAN MANIFOLDS

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ABSTRACT. The paraholomorphic sectional curvature of almost para-Hermitian manifolds is investigated. After discussing those manifolds with pointwise constant paraholomorphic sectional curvature, main attention is devoted to those being isotropic. Also, some boundedness conditions for the paraholomorphic sectional curvature are studied.

1. INTRODUCTION

An *almost para-Hermitian manifold* is a differentiable almost symplectic manifold (M, Ω) which tangent bundle splits into a Whitney sum of Lagrangian subbundles, $TM = L \oplus L'$. Induced by this decomposition, there exist an almost paracomplex structure, J, and a semi-Riemannian metric, g, such that $g(JX, Y) + g(X, JY) = 0$ for all X, Y vector fields on M. This fact motivates the study of such manifolds attending to their semi-Riemannian structure and, through this paper, by an almost para-Hermitian manifold we will mean the triple (M, g, J). A special case occurs when the 2-form Ω is closed and the subbundles L and L' are involutive. Then the manifold is called *para-Kähler* and it is equivalently described by $\nabla J = 0$, ∇ being the Levi Civita connection of g. Para-Kähler manifolds present important properties and they are the most well known class of almost para-Hermitian manifolds. (See [G-MM] for a classification of almost para-Hermitian manifolds and [Cr-F-G] for a survey and further references).

Since the curvature represents the simplest and most widely studied invariant of (semi)-Riemannian manifolds, it is of interest to investigate the curvature

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properties of almost para-Hermitian manifolds. The sectional curvature of para-Kähler manifolds has been recently studied by Gadea and Montesinos in [G-MA1], where the notion of paraholomorphic sectional curvature was introduced. Since the study of the sectional curvature of indefinite metrics presents some significant differences with respect to Riemannian metrics (see [B-C-GR-H], [Gr-N], [Ku], [Kup], [N], [T] and the references therein), the main aim of this paper is to investigate further two basic problems: the restriction of the curvature tensor to degenerate sections and some boundedness conditions on the paraholomorphic sectional curvature.

Motivated by the examples at the end of this paper, the study of the properties we are interested in, must be considered in full generality, i.e., for general almost para-Hermitian manifolds. In section 3 we focus on the study of the spaces of pointwise constant paraholomorphic sectional curvature. It is obtained the expression of the curvature of those spaces, generalizing that of paracomplex space-forms and a criterium for the constancy of the paraholomorphic sectional curvature is derived (cf. Theorem 3.1). A detailed examination of the results in Theorem 3.1 suggests to study a broader class of almost para-Hermitian manifolds: those with vanishing curvature on paraholomorphic degenerate planes. This class generalizes that of almost para-Hermitian manifolds of pointwise constant paraholomorphic sectional curvature. In § 4 it is obtained the expression of the curvature of such manifolds and, using that formula, a local decomposition theorem is stated. Boundedness conditions on the paraholomorphic sectional curvature are investigated in §5 showing that it is bounded if and only if is constant at any point. Several examples motivating this study as well as showing the necessity of some assumptions made through the paper are exhibited in the last section.

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2. Preliminaries

In this section we will collect some basic material we will need further on.

2.1. **Sectional curvature of indefinite metrics.** Let (M, g) be a semi–Riemannian manifold, ∇ the Levi Civita connection and R the curvature tensor. For any non-degenerate plane $\pi =< \{= X, Y\} >$, the sectional curvature, K, is defined by

$$K(\pi) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \tag{2.1}$$

where $R(X,Y,Z,W) = g(R(X,Y)Z,W)$. It must be noted that, in opposition to Riemannian metrics, the sectional curvature is not defined on the whole Grassmannian of two–planes, since (2.1) does not make sense for degenerate planes (i.e., those with $g(X,X)g(Y,Y) - g(X,Y)^2 = 0$). Therefore, the problem of extending the definition of K to degenerate planes is of special interest. Such a problem involves the study of the restriction of the curvature tensor R to degenerate planes, and it is clear that $R(X,Y,X,Y) = 0$ for all degenerate planes $\{X,Y\}$ is a necessary condition for the desired extension. Thorpe, [T], and Dajczer and Nomizu, [D-N], showed the equivalence of that condition to constant sectional curvature.

Generalizing the properties of the curvature of a semi–Riemannian manifold, a *curvaturelike function* is defined to be a quatrilinear map, F, on a vector space V satisfying

$$
\begin{aligned}
&F(X,Y,Z,W) = -F(Y,X,Z,W) = -F(X,Y,W,Z),\\
&F(X,Y,Z,W) = F(Z,W,X,Y),\\
&F(X,Y,Z,W) + F(Y,Z,X,W) + F(Z,X,Y,W) = 0,
\end{aligned} \tag{2.2}
$$

for all $X,Y,Z,W \in V$. If $<,>$ is an inner product on V, the associated *curvaturelike tensor*, $\tilde{F}$, is defined by $<\tilde{F}(X,Y)Z,W> = F(X,Y,Z,W)$.

We close this subsection with two classical examples showing the usefulness of curvaturelike functions in the determination of the curvature of some semi–Riemannian manifolds.

The simplest curvaturelike function on a semi–Riemannian manifold (M,g) is F_0 defined by $F_0(X,Y,Z,W) = g(X,Z)g(Y,W) - g(Y,Z)g(X,W)$. Moreover, it is well–known that a semi–Riemannian manifold is of constant curvature if and only if the curvature tensor is a scalar multiple of F_0 [Gr-N, Lemma.2]. More generally, if ρ denotes the Ricci tensor of a semi–Riemannian manifold (M,g) and define

$$
\begin{aligned}
F_1(X,Y,Z,W) \;=\;& g(X,Z)\rho(Y,W) - g(Y,Z)\rho(X,W)\\
&+\rho(X,Z)g(Y,W) - \rho(Y,Z)g(X,W),
\end{aligned}
$$

it follows that F_1 is a curvaturelike function and moreover, a semi–Riemannian manifold is locally conformally flat if and only if $R - F_1$ is a multiple of F_0 at each point of M [GR-Kup, Th.3.5].

2.2. **Almost para-Hermitian manifolds.** A symplectic manifold is said to be para-Kähler if it is locally diffeomorphic to a product of two Lagrangian submanifolds, [L]. Generalizing that property, an *=20almost para-Hermitian manifold* is an almost symplectic manifold (M, Ω) which tangent bundle splits into a Whitney sum of Lagrangian subbundles, $TM = L \oplus L'$.

Equivalently to that decomposition, there exists a $(1,1)$–tensor field on M, $J = \pi_L - \pi_{L'}$, π_L (resp., $\pi_{L'}$) being the projection on L (resp., on L') such that $J^2 = Id$. Moreover, since L and L' are Lagrangian, the almost paracomplex structure J (also called almost product structure) satisfies $\Omega(JX, JY) = -\Omega(X, Y)$. Therefore, the $(0,2)$–tensor field $g(X,Y) = \Omega(X, JY)$ defines a semi–Riemannian metric on M of signature (n,n) such that

$$g(JX, JY) = -g(X, Y).$$

From now on, by an almost para-Hermitian manifold we will mean a triple (M, g, J) where g and J are a semi–Riemannian metric and an almost paracomplex structure as above. Also, note that the para-Kähler condition is now expressed by $\nabla J = 0$.

Induced by the almost paracomplex structure J, a plane π is said to be *paraholomorphic* if it is invariant by J ($J\pi \subset \pi$). The *paraholomorphic sectional curvature*, H, is defined to be the restriction of the sectional curvature to nondegenerate paraholomorphic planes, i.e.,

$$H(\pi) = -\frac{R(X, JX, X, JX)}{g(X,X)^2}. \tag{2.3}$$

We refer to [Cr-F-G], [G-MA1], [GR-H-VL] and the references therein for more information about the study of the paraholomorphic sectional curvature of para-Kähler manifolds.

A further observation is needed for the purposes of this paper. Note that the paraholomorphic sectional curvature is only defined on nondegenerate paraholomorphic planes. It is immediate to recognize from (2.3) that a necessary condition to extend the definition of H to degenerate planes is

$$R(U, JU, U, JU) = 0, \qquad \text{for all null vectors } U. \tag{2.4}$$

We will call *isotropic* to those almost para-Hermitian manifolds satisfying (2.4). Note that isotropy is a strictly weaker condition than pointwise constant paraholomorphic sectional curvature (see the examples 6.4 and 6.6 in the last section). Also note that (2.4) is a conformally invariant property of almost para-Hermitian manifolds (cf. Example 6.5).

3. Almost para-Hermitian manifolds of pointwise constant paraholomorphic sectional curvature

The purpose of this section is twofold, firstly to derive a criterium for the constancy of the paraholomorphic sectional curvature and secondly, to obtain the expression of the curvature of a space of pointwise constant paraholomorphic sectional curvature.

Theorem 3.1. *Let (M, g, J) be an almost para-Hermitian manifold. The paraholomorphic sectional curvature is pointwise constant on M if and only if one of the following equivalent conditions holds*

(*i*) $R(X, JX)JX + JR(X, JX)X \sim X$ *for all spacelike vectors* X,

(*ii*) $R(Y, JY)JY + JR(Y, JY)Y \sim Y$ *for all timelike vectors* Y,

(*iii*) $R(U, JU)JU + JR(U, JU)U = 0$ *for all null vectors* U,

where $\sim$ means "is proportional to".

Proof. We introduce the following functions G and L defined by

$$G(X,Y) = 2R(X, JX, X, JY) + 2R(X, JX, Y, JX), \tag{3.1}$$

$$\begin{aligned} L(X,Y) &= 2R(X, JX, Y, JY) + 2R(X, JY, Y, JX) \\ &\quad + R(X, JY, X, JY) + R(Y, JX, Y, JX). \end{aligned} \tag{3.2}$$

Using these two functions we have the expression

$$\begin{aligned} R(\lambda X + \mu Y, J(\lambda X + \mu Y), \lambda X + \mu Y, J(\lambda X + \mu Y)) &= \lambda^3\mu G(X,Y) \\ &\quad + \lambda\mu^3 G(Y,X) + \lambda^2\mu^2 L(X,Y) \\ &\quad + \lambda^4 R(X, JX, X, JX) + \mu^4 R(Y, JY, Y, JY). \end{aligned} \tag{3.3}$$

We will firstly show the necessity of (i). Let X be a spacelike vector, take $Y \in < X >^{\perp}$ and consider real numbers λ, μ such that $Z = \lambda X + \mu Y$ is a non null vector. If the paraholomorphic sectional curvature is pointwise constant, say c, then $R(Z, JZ, Z, JZ) = -cg(Z, Z)^2$, and from (3.3) we obtain

$$\begin{aligned} -c(\lambda^2\varepsilon_X + \mu^2\varepsilon_Y)^2 &= \lambda^4 R(X, JX, X, JX) + \mu^4 R(Y, JY, Y, JY) \\ &\quad + \lambda^3\mu G(X,Y) + \lambda\mu^3 G(Y,X) + \lambda^2\mu^2 L(X,Y), \end{aligned}$$

where $\varepsilon_X = g(X, X)$. Hence

$$\lambda^3\mu G(X,Y) + \lambda^2\mu^2(L(X,Y) + 2c\varepsilon_X\varepsilon_Y) + \lambda\mu^3 G(Y,X) = 0,$$

which implies that $G(X,Y)=0$. Thus, from (3.1), it follows that $g(R(X,JX)X + JR(X,JX)JX, JY) = 0$, in showing that (i) is necessary.

Conversely, suppose that (i) holds and show that the paraholomorphic sectional curvature is pointwise constant. Let X and Y be orthonormal vectors with $g(X,X)=1=-g(Y,Y)$. Take= λ and μ real numbers with $\lambda^2>\mu^2$ and consider the vectors $Z=\lambda X+\mu Y$ and $W=\mu X+\lambda Y$. Since Z and W are orthogonal, it follows from (i) that $g(R(Z,JZ)Z + JR(Z,JZ)JZ, JW) = 0$. Equivalently

$$\begin{aligned}R(\lambda X+\mu Y, J(\lambda X+\mu Y), \lambda X+\mu Y, J(\mu X+\lambda Y))\\ -R(\lambda X+\mu Y, J(\lambda X+\mu Y), J(\lambda X+\mu Y), \mu X+\lambda Y)=0.\end{aligned}$$

Linearizing the expression above and considering the coefficients corresponding to $\lambda^3\mu$ and $\lambda\mu^3$, it follows that $R(X,JX,X,JX)=R(Y,JY,Y,JY)$. This shows that $H(X)=H(Y)$ for all orthonormal vectors X and Y with $g(X,X)=1=-g(Y,Y)$.

To show the pointwise constancy of the paraholomorphic sectional curvature, we proceed as follows. Let $\pi_X = <\{X,JX\}>$ and $\pi_Y = <\{Y,J=Y\}>$ be nondegenerate paraholomorphic planes and assume that X and Y are unit spacelike vectors. If $<\{X,Y\}>$ is nondegenerate, let $Z\in <\{X,Y\}>^\perp$ be a unit timelike vector and consider the nondegenerate paraholomorphic plane $\pi = <\{Z,JZ\}>$. Since $X\perp Z$ and $Y\perp Z$, it follows that $H(\pi_X)=H(\pi)$ and $H(\pi_Y)=H(\pi)$, and thus $H(\pi_X)=H(\pi_Y)$. Next, suppose that $<\{X,Y\}>$ is degenerate. Take a unit spacelike $Z\in <Y>^\perp$ and consider the sequence $\{X_n\}_{n\in\mathbb{N}}$ of unit spacelike vectors given by

$$\begin{aligned}X_n&=\left[1+\frac{1}{n^2g(X,Z)^2}+\frac{2}{n}\right]^{-1/2}\left(X+\frac{1}{ng(X,Z)}Z\right), &&\text{if} g(X,Z)\neq 0,\\ X_n&=\left[1+\frac{1}{n^2}\right]^{-1/2}\left(X+\frac{1}{n}Z\right), &&\text{if} g(X,Z)=0.\end{aligned}$$

Now, it follows that $<\{X_n,Y\}>$ is nondegenerate for all $n\in\mathbb{N}$. Therefore, $H(\pi_Y)=H(\pi_{X_n})$ for all $n\in\mathbb{N}$, and the result is obtained passing to the limit.

The equivalence between (ii) and the pointwise constancy of H is obtained in a similar way. To finish the proof, we will analyze the condition (iii). If the paraholomorphic sectional curvature is pointwise constant, say c, (i) and (ii) are satisfied, and then

$$R(X,JX)X+JR(X,JX)JX=2cg(X,X)JX$$

for all nonnull vectors $X \in T_m M$. Now, if U is a null vector, considering a sequence of non null vectors approximating U and passing to the limit the necessity of (iii) is obtained.

Conversely, if (iii) holds, take X and Y orthogonal unit vectors such that $g(X, X) = -g(Y, Y)$. Since $X \pm Y$ are null vectors, it follows from (iii) that

$$\begin{aligned} 0 &= R(X+Y, J(X+Y))(X+Y) + JR(X+Y, J(X+Y))J(X+Y) \\ &= R(X-Y, J(X-Y))(X-Y) + JR(X-Y, J(X-Y))J(X-Y), \end{aligned}$$

and then,

$$\begin{aligned} 0 &= R(X+Y, J(X+Y), X+Y, J(X-Y)) \\ &\quad + R(X+Y, J(X+Y), X-Y, J(X+Y)) \\ &= R(X-Y, J(X-Y), X-Y, J(X+Y)) \\ &\quad + R(X-Y, J(X-Y), X+Y, J(X-Y)). \end{aligned}$$

After some calculations, it follows that $R(X, JX, X, JX) = R(Y, JY, Y, JY)$, and hence $H(X) = H(Y)$ as in the case (i). □

Remark. The result of previous theorem remains valid for any curvaturelike function on a para-Hermitian vector space. (Note that only properties (2.2) of the curvature tensor have been used in the proof above). This fact will be of importance in the determination of the curvature of isotropic almost para-Hermitian manifolds (cf. Theorem 4.4).

Note that the result of Theorem 3.1 reduces to [GR-H-VL, Lemma.3] for the case of para-Kähler manifolds. Also, a geometric interpretation of conditions (i) and (ii) can be stated as follows. We will say that an almost para-Hermitian manifold satisfies the *axiom of paraholomorphic spheres* (resp., *paraholomorphic planes*) at a point $m \in M$ if, for any 2–dimensional nondegenerate paraholomorphic subspace V of $T_m M$, there exists a totally umbilical paraholomorphic submanifold, S, with parallel mean curvature (resp., totally geodesic) passing through m with tangent space $T_m S = V$. Now, as a direct application of previous theorem, and proceeding as in [Gr-N, Th.3, 4], we have

Proposition 3.2. *Let (M, g, J) be an almost para-Hermitian manifold. If M satisfies the axiom of paraholomorphic spheres (resp., paraholomorphic planes) at a point $m \in M$, then the paraholomorphic sectional curvature is constant at m.*

The constancy of the paraholomorphic sectional curvature was previously studied in [G-MA1] and [GR-H-VL] in the framework of para-Kähler manifolds (see also [Cr-F-G]). The special significance of para-Kähler manifolds comes from the fact that the paracomplex structure is parallel, and hence, the curvature tensor satisfies

$$R(X,Y,JZ,JW) = R(JX,JY,Z,W) = -R(X,Y,Z,W). \tag{3.4}$$

Although previous identity is not valid in general for the curvature tensor of an almost para-Hermitian manifold, it is possible to construct a curvaturelike function, R', satisfying (3.4) as follows

$$\begin{aligned} R'(X,Y,Z,W) = {} & 3R(X,Y,Z,W) - 3R(X,Y,JZ,JW) - 3R(JX,JY,Z,W) \\ & +3R(JX,JY,JZ,JW) - R(X,Z,JY,JW) - R(JX,JZ,Y,W) \\ & +R(X,W,JY,JZ) + R(JX,JW,Y,Z) - R(X,JW,JY,Z) \\ & -R(JX,W,Y,JZ) + R(X,JZ,JY,W) + R(JX,Z,Y,JW). \end{aligned}$$

The significance of R' is pointed out in the following

Theorem 3.3. *Let (M,g,J) be an almost para-Hermitian manifold. Then, the paraholomorphic sectional curvature is constant c at a point $m \in M$ if and only if*

$$R' = 4cR_0$$

at the point m, where R_0 is the curvaturelike function

$$\begin{aligned} R_0(X,Y,Z,W) \quad = \quad & g(X,Z)g(Y,W) - g(Y,Z)g(X,W) + g(X,JZ)g(JY,W) \\ & -g(Y,JZ)g(JX,W) + 2g(X,JY)g(JZ,W). \end{aligned}$$

PROOF. It is a straightforward calculation to show that R' satisfies (2.2) and (3.4). Moreover, since $R'(X,JX,X,JX) = 16R(X,JX,X,JX)$ for all vectors X, it follows that the paraholomorphic sectional curvature of R' is constant at the point m and 16 times that of R. Now, the proof follows as in [G-MA1]. (Note that only the properties of a curvaturelike function and (3.4) are used in the proof of [G-MA1, Cor.3.5]). □

At this point, two remarks are needed on the result of previous theorem.

1. The paraholomorphic sectional curvature determines the curvature of para-Kähler manifolds. This result is, however, no longer valid for general almost para-Hermitian manifolds even if H vanishes (see Example 6.2).

2. Gadea and Montesinos showed in [G-MA1] the existence of a Schur lemma for the constancy of the paraholomorphic sectional curvature of para-Kähler manifolds. Although that result can be generalized to the broader class of Nearly para-Kähler manifolds, it is not valid in general. In fact, following the ideas of [Gra-V, Sect.5], it is not difficult to construct para-Hermitian structures of pointwise constant paraholomorphic sectional curvature on $\mathbb{R}^{2n}$ such that H is not globally constant.

An equivalent condition to the validity of the Schur lemma is obtained as follows. First we introduce two $(0,2)$–tensor fields derived from the curvature. One is the usual Ricci tensor,

$$\rho(X,Y) = \operatorname{trace}\{Z \mapsto R(X,Z)Y\} \tag{3.5}$$

and the second is the $*$–Ricci tensor,

$$\rho^*(X,Y) = \operatorname{trace}\{Z \mapsto -JR(X,JY)Z\}. \tag{3.6}$$

Now, if τ and τ^* denote the scalar curvature and the $*$–scalar curvature respectively, the following holds

Proposition 3.4. *Let (M,g,J) be an almost para-Hermitian manifold of pointwise constant paraholomorphic sectional curvature. Then H is constant on M if and only if the scalar curvature $\tau - 3\tau^*$ is constant on M.*

PROOF. Since at any point $R' = c(m)R_0$, it must be $\tau' = c(m)\tau_0$, where τ' and τ_0 denote the scalar curvature of the curvaturelike functions R' and R_0 respectively. After some calculations it follows that $\tau' = 4(\tau - 3\tau^*)$ and $\tau_0 = 4n(n+1)$ from where the result follows. □

4. Isotropic almost para-Hermitian manifolds

We recall that an almost para-Hermitian manifold $(M,\ g,\ J)$ is said to be *isotropic* if the restriction of the curvature tensor to degenerate paraholomorphic planes vanishes identically.

Now, if (2.4) holds then $g(R(U,JU)JU + JR(U,JU)U,U) = 0$, which shows that $R(U,JU)JU + JR(U,JU)U \in \langle U \rangle^{\perp}$. Next, let V be a null vector in $\langle U \rangle^{\perp}$ and consider the null vectors $U + \lambda V$, $\lambda \in \mathbb{R}$. Since M is isotropic, $R(U+\lambda V, J(U+\lambda V), U+\lambda V, J(U+\lambda V)) = 0$. Linearizing this expression, and considering the coefficient of λ, one gets $g(R(U,JU)JU + JR(U,JU)U,V) = 0$. Hence, it follows that $R(U,JU)JU + JR(U,JU)U$ lies in the direction of $\langle U \rangle$ and thus

$$R(U,JU)JU + JR(U,JU)U = c_U\, U. \tag{4.1}$$

Therefore, it is immediate to recognize that (4.1) is an equivalent characterization of (2.4).

Remark. It is clear from previous argument and (*iii*) in Theorem 3.1, that isotropy is a generalization of the pointwise constancy of the paraholomorphic sectional curvature.

The main purpose of this section is to obtain the expression of the curvature of isotropic almost para-Hermitian manifolds, which must generalize the result of Theorem 3.3 according to previous remark. A key observation to do that is the possibility of constructing "isotropic curvaturelike functions" on the basis of the examples exhibited in the Preliminaries. Let φ be a symmetric $(0,2)$–tensor field on M such that $\varphi(JX, JY) = -\varphi(X, Y)$ and define

$$\begin{aligned}\mathbf{F}_{\varphi}(X,Y,Z,W) \;=\;& g(X,Z)\varphi(Y,W) - g(Y,Z)\varphi(X,W) + g(X,JZ)\varphi(JY,W)\\ & -g(Y,JZ)\varphi(JX,W) + 2g(X,JY)\varphi(JZ,W)\\ +\;& \varphi(X,Z)g(Y,W) - \varphi(Y,Z)g(X,W) + \varphi(X,JZ)g(JY,W)\\ & -\varphi(Y,JZ)g(JX,W) + 2\varphi(X,JY)g(JZ,W).\end{aligned}$$

A straightforward calculation shows that $\mathbf{F}_{\varphi}$ is a curvaturelike function, it satisfies (3.4) and moreover, it is isotropic (i.e., its restriction=20to degenerate paraholomorphic planes vanishes identically). Hence, for each null vector U there exists c_U^{φ} such that the associated curvaturelike tensor $\tilde{\mathbf{F}}_{\varphi}$ satisfies $\tilde{\mathbf{F}}_{\varphi}(U, JU)JU + J\tilde{\mathbf{F}}_{\varphi}(U, JU)U = c_U^{\varphi}\, U$.

Therefore, if (M, g, J) is an isotropic almost para-Hermitian manifold, to obtain the expression of its curvature, it is enough to find a symmetric $(0,2)$–tensor field, φ, as above in such a way that c_U^{φ} coincides with c_U for all null vectors U. To obtain the desired φ, we compute the values of the Ricci and *–Ricci tensors on null vectors. We begin with the following

Lemma 4.1. *Let (M, g, J) be an isotropic almost para-Hermitian manifold and Z a nonnull vector. For each null vector U in $< \{Z, JZ\} >^{\perp}$, it holds*

$$\begin{aligned}2\varepsilon_Z c_U \;=\;& R(U,Z,U,Z) - R(U,JZ,U,JZ) - R(JU,Z,JU,Z)\\ & +R(JU,JZ,JU,JZ) - 6R(U,JU,Z,JZ).\end{aligned} \tag{4.2}$$

Proof. Let V be a null vector in $< \{Z, JZ\} >^{\perp}$ with $g(U, V) = -\frac{1}{2}$. Then $W_t = \frac{1}{\sqrt{t}}(U + t\varepsilon_Z V)$ is a nonnull vector for each $t > 0$ such that $g(W_t, W_t) = -g(Z, Z)$ and $g(W_t, Z) = 0$. Since $Z \pm W_t$ is a null vector, it follows that

$$R(Z \pm W_t, J(Z \pm W_t), Z \pm W_t, J(Z \pm W_t)) = 0.$$

Linearizing previous expression, after some calculations, it follows that

$$R(Z,JZ,Z,JZ)+R(W_t,JW_t,W_t,JW_t)=-2R(Z,JZ,W_t,JW_t)$$
$$-2R(Z,JW_t,W_t,JZ)-R(Z,JW_t,Z,JW_t)-R(W_t,JZ,W_t,JZ).$$

Linearizing this expression, taking into consideration that the manifold is isotropic and multiplying both sides by t, we obtain

$$0=tR(Z,JZ,Z,JZ)+\varepsilon_Z G(U,V)+t^2\varepsilon_Z G(V,U)+tL(U,V)$$
$$+2R(Z,JZ,U+t\varepsilon_Z V,JU+t\varepsilon_Z JV)+2R(Z,JU+t\varepsilon_Z JV,U+t\varepsilon_Z V,JZ)$$
$$+R(Z,JU+t\varepsilon_Z JV,Z,JU+t\varepsilon_Z JV)+R(U+t\varepsilon_Z V,JZ,U+t\varepsilon_Z V,JZ).$$

Now, taking limits when $t\to 0$,

$$\begin{aligned} 0 &= R(U,JZ,U,JZ)+R(JU,Z,JU,Z) \\ &\quad +2R(U,JU,Z,JZ)+2R(Z,JU,U,JZ)+\varepsilon_Z G(U,V). \end{aligned}$$

Putting JZ instead of Z in the above expression and subtracting we have

$$\begin{aligned} 0 &= 2\varepsilon_Z G(U,V)+4R(U,JU,Z,JZ)+2R(Z,JU,U,JZ) \\ &\quad +2R(U,Z,JU,JZ)+R(U,JZ,U,JZ)+R(JU,Z,JU,Z) \\ &\quad -R(U,Z,U,Z)-R(JU,JZ,JU,JZ). \end{aligned}$$

Finally, since $G(U,V)=c_U$, (4.2) is obtained using the first Bianchi identity. □

Lemma 4.2. *Let (M,g,J) be an almost para-Hermitian manifold and $U\in T_mM$ a null vector such that $U\neq \pm JU$. Then, one of the following holds,*

(i) *There exist orthonormal vectors $X,Y\in T_mM$ with $<\{X,JX\}>\perp<\{Y,JY\}>$ such that $U=k(X+Y)$, for some real k, or*

(ii) *There exists a sequence $\{U_n\}_{n\in\mathbb{N}}$ of null vectors satisfying (i) such that $U=\lim_{n\to\infty}U_n$.*

PROOF. Let U be a null vector, take X, Y orthogonal vectors such that $U=(X+Y)$ and consider the subspace $V=<\{X,JX,Y,JY\}>$.

If V is nondegenerate, define $W_1=<\{X,JX\}>$ and W_2 its orthogonal complement in V. Then $V=W_1\oplus W_2$ and there exists an orthogonal decomposition $U=U_1+U_2$, where U_i is the component of U in W_i, $i=1,2$. If U_1 is nonnull, then U_2 is so and, after normalizing both vectors, the desired decomposition is obtained showing (i).

If U_1 is a null vector (note that both W_1 and W_2 have induced Lorentzian metric) we proceed as follows. If $\dim V=2$, it must be $V=<\{X,JX\}>=$

$< \{Y, JY\} >$. Then $U = (X \pm JX)$ and thus $JU = \pm U$, which is a contradiction. Also, since V is nondegenerate, dimension of V cannot be 3. Hence, assume that $\dim V = 4$, and choose null vectors $V_i \in W_i$, $i = 1, 2$, such that $g(U_1, V_1) = 1$ and $g(U_2, V_2) = -1$. Now, $U_n = U_1 + U_2 + \frac{1}{n}(V_1 + V_2)$ is a sequence= of null vectors approximating U. Furthermore, note that each U_n is of the form $U_n = k_n(X_n + Y_n)$ with X_n, Y_n spanning orthogonal paraholomorphic planes for all $n \in \mathbb{N}$, where

$$X_n = (\frac{n}{2})^{1/2}(U_1 + \frac{1}{n}V_1), \quad Y_n = (\frac{n}{2})^{1/2}(U_2 + \frac{1}{n}V_2), \qquad k_n = (\frac{2}{n})^{1/2}.$$

To finish the proof, let consider the case of V being degenerate. Since $g(X, X) = 1 = g(JY, JY)$, there exists a sequence $\{X_n\}_{n\in\mathbb{N}}$ of spacelike unit vectors approximating X such that the subspace $V_n = < \{X_n, JX_n, Y, JY\} >$ is nondegenerate for all $n \in \mathbb{N}$ (see the proof of Theorem 3.1). Now, it is easy to construct a sequence of null vectors $\tilde{U}_n$ approximating the null vector U such that each $\tilde{U}_n$ lies in V_n for all $n \in \mathbb{N}$. Since V_n is nondegenerate for all $n \in \mathbb{N}$, the desired sequence of null vectors approximating U is obtained. □

Now, we state the following result which is a key observation for the study of isotropic almost para-Hermitian manifolds.

Lemma 4.3. *Let (M^{2n}, g, J) be an isotropic almost para-Hermitian manifold. Then,*

$$\rho(U,U) - \rho(JU,JU) - 6\rho^*(U,U) = (2n+4)c_U \tag{4.3}$$

for each null vector U on M.

PROOF. We will prove (4.3) in two steps corresponding to cases (i) and (ii) in Lemma 4.2. Let firstly consider those null vectors U admitting an expression $U = k(X + Y)$ for some $k \in \mathbb{R}$, where X and Y are spacelike and timelike unit vectors such that $< \{X, JX\} > \perp < \{Y, JY\} >$.

Now, if U is as above, take $\{Z_1, \ldots, Z_{n-2}, JZ_1, \ldots, JZ_{n-2}\}$ in such a way that $\{X,\ JX,\ Y,\ JY,\ Z_1,\ \ldots,\ Z_{n-2},\ JZ_1,\ \ldots,\ JZ_{n-2}\}$ be a local orthonormal frame.

Then one has

(4.4)

$$
\begin{aligned}
&\rho(U,U) - \rho(JU,JU) \\
&\quad = R(U,X,U,X) - R(U,Y,U,Y) - R(JU,X,JU,X) \\
&\quad +R(JU,Y,JU,Y) - R(U,JX,U,JX) + R(U,JY,U,JY) \\
&\quad +R(JU,JX,JU,JX) - R(JU,JY,JU,JY) + \sum_{i=1}^{n-2} \varepsilon_{Z_i}\{R(U,Z_i,U,Z_i) = \\
&\quad +R(JU,JZ_i,JU,JZ_i) - R(U,JZ_i,U,JZ_i) - R(JU,Z_i,JU,Z_i)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\rho^*(U,U) &= R(U,JU,X,JX) - R(U,JU,Y,JY) \\
&\quad + \sum_{i=1}^{n-2} \varepsilon_{Z_i} R(U,JU,Z_i,JZ_i).
\end{aligned}
\tag{4.5}
$$

Now, since $U = k(X+Y)$, a straightforward calculation shows that

$$
\begin{aligned}
0 &= R(U,X,U,X) - R(U,Y,U,Y) \\
&= R(JU,X,JU,X) - R(JU,Y,JU,Y), \\
2c_U &= R(JU,Y,JU,Y) - R(JU,X,JU,X) \\
&\quad +R(U,JY,U,JY) - R(U,JX,U,JX), \\
c_U &= R(U,JU,Y,JY) - R(U,JU,X,JX).
\end{aligned}
$$

Hence, from (4.4) and (4.5), it follows that

(4.6)

$$
\begin{aligned}
&\rho(U,U) - \rho(JU,JU) - 6\rho^*(U,U) = 8c_U + \\
&\qquad + \sum_{i=1}^{n-2} \varepsilon_{Z_i}\{R(U,Z_i,U,Z_i) + R(JU,JZ_i,JU,JZ_i) \\
&\qquad -R(U,JZ_i,U,JZ_i) - R(JU,Z_i,JU,Z_i) - 6R(U,JU,Z_i,JZ_i)\},
\end{aligned}
$$

and the result is obtained from Lemma 4.1.

For an arbitrary null vector U, if $JU = \pm U$ the result is obvious. In the case of $JU \neq \pm U$, we can choose a suitable sequence of null vectors $\{U_n\}_{n\in\mathbb{N}}$ approximating the null vector U, with each U_n satisfying (i) in Lemma 4.2 and the result is obtained after passing to the limit. □

Previous lemma suggests the definition of the following symmetric bilinear form

$$\varphi(X,Y) := \rho(X,Y) - \rho(JX,JY) - 3\rho^*(X,Y) + 3\rho^*(JX,JY). \tag{4.7}$$

Note that $\varphi(JX,JY) = -\varphi(X,Y)$ and, moreover, $\varphi(U,U) = (2n+4)c_U$ for all null vectors U, according to (4.3).

In what follows, R_1 will denote the curvaturelike function obtained from $\mathbf{F}_\varphi$ for φ defined by (4.7). Now, we state the main result of this section

Theorem 4.4. *Let (M^{2n}, g, J) be an almost para-Hermitian manifold. Then M is isotropic if and only if*

$$R' = -\frac{\tau - 3\tau^*}{(n+1)(n+2)} R_0 + \frac{1}{n+2} R_1. \tag{4.8}$$

Proof. It is clear that if (4.8) holds, M is isotropic. Conversely, to show the necessity of condition (4.8), let us consider the curvaturelike function $F = R - \frac{1}{16(n+2)} R_1$. Since $\varphi(U,U) = (2n+4)c_U$, it follows that $\tilde{F}(U,JU)JU + J\tilde{F}(U,JU)U = 0$, and thus the associated curvaturelike function F' must be a multiple of R_0 at each point. Furthermore, since R_1 satisfies (3.4), it must be $16R_1 = R_1'$, which shows that $R' - \frac{1}{(n+2)} R_1 = CR_0$. What remains to do is to determine the function C.

We proceed in the following way. Since $R' = CR_0 + \frac{1}{(n+2)} R_1$, if τ', τ_1 and τ_0 denote the scalar curvatures of R', R_1 and R_0 respectively, it must be $\tau' = C\tau_0 + \frac{1}{n+2}\tau_1$. A straightforward calculation shows that $\tau' = 4(\tau - 3\tau^*)$, $\tau_1 = 8(n+1)(\tau - 3\tau^*)$ and $\tau_0 = 4n(n+1)$. Thus $C = -\dfrac{(\tau - 3\tau^*)}{(n+1)(n+2)}$, which finishes the proof. □

As a consequence of (4.3) and (4.8), a necessary and sufficient condition for an isotropic almost para-Hermitian manifold to be of pointwise constant paraholomorphic sectional curvature is obtained in terms of the tensor field φ defined in (4.7).

Corollary 4.5. *An isotropic almost para-Hermitian manifold has pointwise constant paraholomorphic sectional curvature if and only if the symmetric bilinear form φ is a multiple of the metric tensor at each point.*

Proof. If φ is a multiple of the metric g, then c_U vanishes identically for all null vectors as a consequence of (4.3) and hence the result follows from Theorem 3.1(*iii*). Conversely, if H is constant at each point, it must be $c_U = 0$ forall null U and hence, $\varphi(U,U) = 0$ for all null U. Now, that φ is a multiple of the metric is obtained from [N, Lemma.A]. □

Also, the following is obtained as a consequence of previous theorem.

Theorem 4.6. *Let (M, g, J) be a connected isotropic para-Kähler manifold. Assume the scalar curvature to be constant and the Ricci operator diagonalizable. Then,*

(a) *M is a space of constant paraholomorphic sectional curvature, or*
(b) *M is locally isometric to a product $M_1 \times M_2$ of para-Kähler manifolds of constant paraholomorphic sectional curvature c and $-c$ respectively.*

PROOF. If (M, g, J) is a para-Kähler manifold the Ricci tensors ρ and ρ^* satisfy $\rho = -\rho^*$ and therefore $\tau = -\tau^*$. Then, the symmetric bilinear form φ defined in (4.7) reduces to $\varphi(X,Y) = 8\rho(X,Y)$. Also, since $R' = 16R$, (4.8) takes the form

$$R = -\frac{\tau}{4(n+1)(n+2)} R_0 + \frac{1}{2(n+2)} R_1.$$

First of all, we will show that an isotropic para-Kähler manifold is locally symmetric if and only if the scalar curvature is constant. The necessity is clear. Therefore, we will show the converse. By the second Bianchi identity, we have:

$$\begin{aligned} 0 = \; & \sigma_{(X,Y,Z)} \nabla_Z \{ g(X,W)\rho(Y,T) - g(Y,W)\rho(X,T) + \rho(X,W)g(Y,T) \\ & - \rho(Y,W)g(X,T) + g(X,JW)\rho(JY,T) - g(Y,JW)\rho(JX,T) \\ & + \rho(X,JW)g(JY,T) - \rho(Y,JW)g(JX,T) \\ & + 2g(X,JY)\rho(JW,T) + 2\rho(X,JY)g(JW,T) \}, \end{aligned}$$

where $\sigma_{(X,Y,Z)}$ denotes the cyclic sum over Z, X, Y.

Let $\{e_1, \ldots, e_{2n}\}$ be an orthonormal local frame. Putting $Z = T = e_i$, multiplying by $\varepsilon_i = g(e_i, e_i)$, and taking the sum over $i = 1, \ldots, 2n$ one obtains:

$$\begin{aligned} 0 = \; & (2n+3)\{(\nabla_X \rho)(Y,Z) - (\nabla_Y \rho)(X,Z)\} - (\nabla_{JX}\rho)(Y,JZ) \\ & + (\nabla_{JY}\rho)(X,JZ) + 2(\nabla_{JZ}\rho)(X,JY) + g(X,Z)\sum_{i=1}^{2n} \varepsilon_i (\nabla_{e_i}\rho)(Y,e_i) \\ & + g(X,JZ)\sum_{i=1}^{2n} \varepsilon_i (\nabla_{e_i}\rho)(JY,e_i) + g(Y,Z)\sum_{i=1}^{2n} \varepsilon_i (\nabla_{e_i}\rho)(X,e_i) \\ & - g(Y,JZ)\sum_{i=1}^{2n} \varepsilon_i (\nabla_{e_i}\rho)(JX,e_i) + 2g(X,JY)\sum_{i=1}^{2n} \varepsilon_i (\nabla_{e_i}\rho)(JZ,e_i) \;:. \end{aligned}$$

Since scalar curvature is constant, it follows (see for example [Ne, pag.88]) that $(div\,\rho)(X) = 0$ for all vector fields X. Using this fact in the expression above we have:

$$
\begin{aligned}
0 &= (2n+3)\{(\nabla_Y\rho)(X,Z)-(\nabla_X\rho)(Y,Z)\} \\
&\quad -(\nabla_{JY}\rho)(X,JZ)+(\nabla_{JX}\rho)(Y,JZ)-2(\nabla_{JZ}\rho)(X,JY).
\end{aligned} \tag{4.9}
$$

It follows from (3.4) after some calculations that the Ricci tensor of a para-Kähler manifold satisfies $(\nabla_X\rho)(Y,JZ)+(\nabla_Y\rho)(Z,JX)+(\nabla_Z\rho)(X,JY)=0$. Then, it follows from (4.9) that $(\nabla_{JZ}\rho)(X,JY)=0$, which shows that the Ricci tensor is parallel, and therefore that the manifold is locally symmetric by (4.8).

Since M is locally symmetric and the Ricci tensor is diagonalizable, the eigenvalues of the Ricci operator are constant and the corresponding eigenspaces define parallel distributions on M. Therefore, M is locally a product of Einstein spaces. Furthermore, since the integral manifolds of those distributions are totally geodesic, they are para-Kähler manifolds of constant paraholomorphic sectional curvature. Therefore, M is locally a product $M_1(c_1)\times\cdots\times M_k(c_k)$ of para-Kähler manifolds with constant paraholomorphic sectional curvature.

Now, if the number of factors reduces to one, the case (a) is proved. To finish the proof, we will show that M is locally flat or the number of factors is exactly two, obtaining (b). Let us suppose that $M=M_1(c_1)\times M_2(c_2)\times M_3(c_3)$. There are null vectors $X=(X_1,X_2,0)$, $Y=(Y_1,0,Y_3)$ and $Z=(Z_1,Z_2,Z_3)$. Using the isotropy condition with X and Y it follows that $c_1=-c_2=-c_3$. Hence, M is locally a product $M_1(c)\times M_2(-c)\times M_3(-c)$. Once again, using the isotropy condition with Z it follows that $c=0$, which shows that M is locally flat. Analogously, it is shown that M is flat if the number of different eigenvalues of the Ricci operator is greater than two. In the case of two distinct eigenvalues, (b) is obtained proceeding as above. □

Remark. The assumption in previous theorem on the diagonalizability of the Ricci tensor cannot be removed as shown in Example 6.6.

Remark. Note that the special significance of being M para-Kähler in the theorem above comes from the fact that the curvaturelike function R' (which coincides with the curvature tensor R of M) satisfies the second Bianchi identity. This last property cannot be assumed to be satisfied by R' in general.

To close this section, we state a similar result to Theorem 4.6 for the general case of an isotropic almost para-Hermitian manifold.

Theorem 4.7. *Let (M,g,J) be a connected isotropic almost para-Hermitian manifold with parallel bilinear form φ. If the operator Q_φ associated with φ ($\varphi(X,Y)=g(Q_\varphi(X),Y)$) is diagonalizable, then*

(a) *M has constant paraholomorphic sectional curvature, or*
(b) *M is locally isometric to a product* $M = M_1(c) \times M_2(-c)$ *of two almost para-Hermitian manifolds of constant paraholomorphic sectional curvature.*

PROOF. As in the previous theorem, the eigenspaces of Q_φ define parallel distributions on M and the result is obtained in an analogous way. Furthermore, note that the paraholomorphic sectional curvature is globally constant on each factor of the decomposition since the parallelizability of Q_φ ensures the constancy of the scalar curvature $\tau - 3\tau^*$ (see Proposition 3.4). □

5. Boundedness on the paraholomorphic sectional curvature

The sectional curvature of a Riemannian manifold is a real function defined at each point $m \in M$ on the Grassmannian of two-planes, $G_2(T_mM)$. Since $G_2(T_mM)$ is compact, it follows that K is bounded at each point for positive definite metrics. However, the sectional curvature of a semi-Riemannian metric is bounded if and only if it is constant [Ku], [N] (see also [B-C-GR-H, Th.2.1], [Kup]). The aim of this section is to study some boundedness conditions on the paraholomorphic sectional curvature of almost para-Hermitian manifolds.

Theorem 5.1. *Let* (M^{2n}, g, J) *be an isotropic almost para-Hermitian manifold,* $n \geq 3$. *The paraholomorphic sectional curvature is pointwise constant if and only if the curvature of nondegenerate paraholomorphic planes in* $< \{Z, JZ\} >^\perp$ *is bounded from below or from above for all nonnull vectors* Z.

PROOF. Let X be a spacelike unit vector, and consider the paraholomorphic plane $\pi_1 = < \{X, JX\} >$. Let $\pi_2 = < \{Y, JY\} >$ be a nondegenerate paraholomorphic plane in T_mM such that $< \{X, JX\} > \perp < \{Y, JY\} >$ and assume that $g(X,X) = 1 = -g(Y,Y)$. Then $\lambda X + \mu Y$ and $\mu X + \lambda Y$ are unit spacelike and timelike vectors for $\lambda^2 - \mu^2 = 1$.

Since $n \geq 3$ and the subspace $< \{X, JX, Y, JY\} >$ is nondegenerate, there exists a nondegenerate paraholomorphic plane $< \{Z, JZ\} >$ in the orthogonal complement $< \{X, JX, Y, JY\} >^\perp$. Therefore, $\pi_3 = < \{\lambda X + \mu Y, J(\lambda X + \mu Y)\} >$ and $\pi_4 = < \{\mu X + \lambda Y, J(\mu X + \lambda Y)\} >$ are nondegenerate e paraholomorphic planes in $< \{Z, JZ\} >^\perp$. Since we assume that the paraholomorphic sectional curvature is bounded from below on paraholomorphic planes in $< \{Z, JZ\} >^\perp$, there exists N such that

$$N \leq H(\pi_3) = -R(\lambda X + \mu Y, J(\lambda X + \mu Y), \lambda X + \mu Y, J(\lambda X + \mu Y)), \tag{5.1}$$

and

$$N \leq H(\pi_4) = -R(\mu X + \lambda Y, J(\mu X + \lambda Y), \mu X + \lambda Y, J(\mu X + \lambda Y)). \tag{5.2}$$

Since $g(X,X) = 1 = -g(Y,Y)$, $U = X \pm Y$ is a null vector and= using that M is isotropic,

$$R(X \pm Y, J(X \pm Y), X \pm Y, J(X \pm Y)) = 0.$$

Linearizing previous expression, it follows that the functions G and L defined in (3.1) and (3.2) respectively satisfy $G(X,Y) = -G(Y,X)$ and $L(X,Y) = -R(X,JX,X,JX)$ $-R(Y,JY,Y,JY)$. Hence, since $\lambda^2 - \mu^2 = 1$, (5.1) and (5.2) become

$$\begin{aligned} N \leq & \lambda^2\{R(Y,JY,Y,JY) - R(X,JX,X,JX)\} - R(Y,JY,Y,JY) \\ & -\lambda\mu G(X,Y) \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} N \leq & \lambda^2\{R(X,JX,X,JX) - R(Y,JY,Y,JY)\} - R(X,JX,X,JX) \\ & +\lambda\mu G(X,Y), \end{aligned} \tag{5.4}$$

respectively, for all λ, μ with $\lambda^2 - \mu^2 = 1$.

These expressions also hold if we replace μ by $-\mu$, which easily implies that $R(X,JX,X,JX) = R(Y,JY,Y,JY)$. Thus, we conclude that $H(X) = H(Y)$ whenever $< \{X, JX\} > \perp < \{Y, JY\} >$, with $g(X,X) = 1 = -g(Y,Y)$.

Now the pointwise constancy of the paraholomorphic sectional curvature follows in a similar way as in Theorem 3.1. □

In the general case, if M is not assumed to be isotropic, we have the following

Theorem 5.2. *Let (M^{2n}, g, J) be an almost para-Hermitian manifold, with $n \geq 3$. The paraholomorphic sectional curvature is pointwise constant if and only if the curvature of nondegenerate paraholomorphic planes in $< \{Z, JZ\} >^\perp$ is bounded from below and from above for all nonnull vectors Z.*

Proof. Let U be a null vector. If $JU = \pm U$, it follows trivially that the restriction of the curvature tensor $R(U,JU,U,JU) = 0$. Next, assume U to be in the case (i) of Lemma 4.2. Since $n \geq 3$, let Z be a spacelike vector in $< \{U, JU\} >^\perp$. Since U is a null vector in $< \{Z, JZ\} >^\perp$, and the restriction of the metric g to $< \{Z, JZ\} >^\perp$ is nondegenerate with signature $(n-1, n-1)$, we can choose a null vector V in $< \{Z, JZ\} >^\perp$ with $g(U,V) = -\frac{1}{2}$. Thus, for all real numbers $t > 0$, $A_t = \dfrac{U + tV}{\sqrt{t}}$ is a unit timelike vector.

If the curvature of nondegenerate paraholomorphic planes in $< \{Z, JZ\} >^{\perp}$ is bounded by N, then for the plane $\pi_t = < \{A_t, J(A_t)\} >$ we have $|H(\pi_t)| \leq N$, and therefore

$$|R(U+tV, J(U+tV), U+tV, J(U+tV))| \leq t^2 N$$

for all $t > 0$. Taking limits when $t \to 0$, we obtain $R(U, JU, U, JU) = 0$.

Now, if U is an arbitrary null vector, proceeding as in Lemma 4.2 (ii), it is possible to approximate it by a sequence of null vectors satisfying condition (i) in the mentioned lemma. Then, it follows that $R(U, JU, U, JU) = 0$ just passing to the limit, and thus, M is isotropic. Therefore, the pointwise constancy of H follows from previous theorem. □

6. Examples

In this section we will show some examples of almost para-Hermitian manifolds. Special attention is devoted to the examination of those being isotropic and of pointwise constant paraholomorphic sectional curvature. We refer to [Cr-F-G] and the references therein for more examples of almost para-Hermitian manifolds.

Example 6.1. The paracomplex projective models $P_n(B)$.

They were introduced by Gadea and Montesinos in [G-MA1] and they are the models of para-Kähler manifolds of nonvanishing constant paraholomorphic sectional curvature. (See [Cr-F-G, sect.6] and [G-MA2]).

Example 6.2. Non-flat almost para-Kähler manifolds with vanishing paraholomorphic sectional curvature.

The tangent bundle, TM, of any Riemannian manifold (M, g) is naturally endowed with an almost para-Kähler structure $(J, \tilde{g})$ defined by Cruceanu, [Cr], as follows: $J(X^H + Y^V) = X^H - Y^V$ and $\tilde{g}(X^H + Y^V, U^H + W^V) = g(X, W)^V + g(Y, U)^V$, where X^V and X^H denote the vertical and horizontal lifts of the vector field X to the tangent bundle with respect to the metric connection of (M, g) [Y-I]. It is immediate to recognize that $\tilde{g}$ coincides with the complete lift, g^C, of g and moreover, that the 2–form Ω defined by $(J, \tilde{g})$ is closed.

The curvature of the semi–Riemannian manifold (TM, g^C) was recently studied in [CV-GR-VA]. For sake of completeness, we state the following:

Lemma 6.3. [CV-GR-VA, Lemma.2.1] *Let (M, g) be a Riemannian manifold, g^C the complete lift of g, and let $\tilde{R}_\xi$ denote the curvature tensor of g^C at a point*

$\xi \in TM$ Then

$$\begin{aligned}\tilde{R}_\xi\left(X_1^V+Y_1^H, X_2^V+Y_2^H\right)\left(X_3^V+Y_3^H\right) &= \left\{R_{\pi(\xi)}(Y_1,Y_2)Y_3\right\}^H + \left\{(\nabla_\xi R)(Y_1,Y_2)Y_3\right\}^V \\ &\quad + \left\{R_{\pi(\xi)}(X_1,Y_2)Y_3 + R_{\pi(\xi)}(Y_1,X_2)Y_3 + R_{\pi(\xi)}(Y_1,Y_2)X_3\right\}^V\end{aligned}$$

for all X_i, Y_i, $i = 1,2,3$, vector fields on M.

Now, if Z is a tangent vector to TM at a point $\xi \in TM$, decompose it into its vertical and horizontal components $Z = X^H + Y^V$. Then, it follows from previous Lemma that $\tilde{R}(Z, \tilde{J}Z, Z, \tilde{J}Z) = -4\tilde{R}(X^H, Y^V, Y^V, X^H) = 0$, which shows that the paraholomorphic sectional curvature of $(TM, g^C, \tilde{J})$ vanishes identically. (Note that, however, (TM, g^C) is not flat unless M be so. Moreover, in that case, (TM, g^C, J) is a locally flat para-Kähler manifold).

Example 6.4. Isotropic almost para-Hermitian manifolds but not of constant paraholomorphic sectional curvature.

Let (M_1, g_1, J_1) and (M_2, g_2, J_2) be two almost para-Hermitian manifolds of constant paraholomorphic sectional curvature c and $-c$ respectively. Then, the product manifold $M = M_1 \times M_2$ endowed with the metric $g((X_1, X_2), (Y_1, Y_2)) = g_1(X_1, Y_1) + g_2(X_2, Y =_2)$ and the almost paracomplex structure $J(X_1, X_2) = (J_1X_1, J_2X_2)$ is an almost para-Hermitian manifold. Now, proceeding as in [B-C-GR-H, Example.3.1], it is easy to show that M is an isotropic manifold with nonconstant paraholomorphic sectional curvature unless $c = 0$. Moreover, note that M is para-Kähler if and only if both factors are so.

Example 6.5. Locally conformal manifolds to isotropic almost para-Hermitian manifolds.

Let (M_1, g_1, J_1) and (M_2, g_2, J_2) be two locally conformal almost para-Hermitian manifolds (i.e., there exists a local diffeomorphism ϕ from M_1 to M_2 such that $\phi_* J_1 = J_2\phi_*$ and $\phi^* g_2 = e^{2\sigma} g_1$ for some real function σ defined on M_1). Then, there exists a correspondence between null vectors on M_1 and M_2, and moreover, it follows after some calculations that $R_2(\phi_*U, J_2\phi_*U, \phi_*U, J_2\phi_*U) = e^{2\sigma}\, R_1(U, J_1U, U, J_1U)$ for all null vectors U on M_1. Therefore, *isotropy is a conformally invariant property of almost para-Hermitian manifolds.*

Example 6.6. The tangent bundle of a paracomplex space form.

Let (M, g, J) be an almost para-Hermitian manifold. The horizontal lifts (with respect to the metric connection of g) of the metric, g^H, and the almost paracomplex structure, J^H, induce an almost para-Hermitian structure on TM. (Note that, since $\nabla g = 0$, then $g^H = g^C$. Also, if (M, g, J) is a para-Kähler manifold, then $J^H = J^C$ since $\nabla J = 0$ and, moreover, it follows that (M, g, J) is a para-- Kähler manifold if and only if (TM, g^H, J^H) is so). Using the results in Lemma 6.3, we have the following

Lemma 6.7. *Let (M, g, J) be an almost para-Hermitian manifold. Then, (TM, g^H, J^H) is isotropic if and only if (M, g, J) is of pointwise constant paraholomorphic sectional curvature and $g((\nabla_\xi R)(X, JX)X, JX) = 0$ for all vectors X on M.*

PROOF. Let $U \in T_\xi(TM)$ be a null vector, and take X, $Y \in T_{\pi(\xi)}(M)$ orthogonal vectors such that $U = X^H + Y^V$. Then

$$\tilde{R}(U, J^H U, U, J^H U) = g((\nabla_\xi R)(X, JX)X, JX)^V + 2g(R(X, JX)X + JR(X, JX)JX, JY)^V. \tag{6.1}$$

If (TM, g^H, J^H) is isotropic, since X^H is a null vector for each vector field X on M, we have $\tilde{R}(X^H, J^H X^H, X^H, J^H X^H) = 0$, and hence, from (6.1)

$$g((\nabla_\xi R)(X, JX)X, JX)^V = 0. \tag{6.2}$$

Next, take $\{X, Y\}$ orthogonal vectors on M. Then $U = X^H + Y^V$ is a null vector on TM and thus $\tilde{R}(U, J^H U, U, J^H U) = 0$. Hence, from (6.1) and (6.2) it follows that $g(R(X, JX)X + JR(X, JX)JX, JY)^V = 0$, and thus $R(X, JX)X + JR(X, JX)JX \sim JX$. This shows that the paraholomorphic sectional curvature is pointwise constant on M as an application of Theorem 3.1.

Conversely, let U be a null vector, $U \in T_\xi(TM)$ and decompose it as $U = X^H + Y^V$, with $X, Y \in T_{\pi(\xi)}(M)$. Since the paraholomorphic sectional curvature is pointwise constant, it must be $g(R(X, JX)X + JR(X, JX)JX, JY)^V = 0$. Hence, $\tilde{R}(U, J^H U, U, J^H U) = 0$, since we assume that $g((\nabla_\xi R)(X, JX)X, JX)^V = 0$. This shows that (TM, g^H, J^H) is isotropic. □

Further, note that the paraholomorphic sectional curvature of (TM, g^H, J^H) is not constant unless be flat.

Remark. In the special case of being M a para-Kähler manifold, it follows that *(M, g, J) is a para-Kähler manifold of constant paraholomorphic sectional curvature c if and only if (TM, g^H, J^H) is an isotropic para-Kähler manifold.* Note also that (M, g) is locally symmetric if and only if (TM, g^H) is so, which shows that the tangent bundle of any paracomplex space form is an isotropic locally symmetric para-Kähler manifold. However, (TM, g^H, J^H) does not correspond to any of the cases listed in Theorem 4.6. This is due to the nondiagonalizability of the Ricci operator corresponding to (TM, g^H) unless be Ricci flat [CV-GR-VA, Thm.2.2], which shows the impossibility of removing that condition in Theorem 4.6.

Example 6.8. Locally symmetric isotropic almost para-Hermitian manifolds with nonparallel Q_φ–operator.

Let $(\mathbb{R}^{2n+2}, \tilde{g}, \tilde{J})$ be the $(2n+2)$–real space with the standard para-Kähler structure. Let $H_n^{2n+1}(c)$ be the pseudohyperbolic space of constant sectional curvature $c < 0$, [Ne]. Let N denote the timelike unit normal vector field and $\xi = \tilde{J}(N)$.

For each vector X on $H_n^{2n+1}(c)$ decompose $\tilde{J}X = \phi X + g(X, \xi)N$ in tangential and normal components, where ϕX denotes the tangential component of $\tilde{J}(X)$. Now, if $\eta(X) = g(X, \xi)$, it follows that $\phi^2 = Id + \eta \otimes \xi$ and $\eta(\xi) = 1$. Moreover, if g denotes the induced metric on $H_n^{2n+1}(c)$ it immediately follows that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

which shows that $H_n^{2n+1}(c)$ has an induced almost paracontact structure [K].

Next, let $(M_i, \phi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$ denote the pseudohyperbolic spaces of dimension $2p + 1$ and $2q + 1$ with the structures $(\phi_i, \xi_i, \eta_i, g_i)$ constructed as above. Let M be the product manifold, $M = H_p^{2p+1}(c) \times H_q^{2q+1}(c)$ and consider the para-Hermitian structure

$$J(X_1, X_2) = (\phi_1 X_1 + \eta_2(X_2)\xi_1, \phi_2 X_2 + \eta_1(X_1)\xi_2),$$

$$g[(X_1, X_2), (Y_1, Y_2)] = g_1(X_1, Y_1) - g_2(X_2, Y_2).$$

Now, it is a straightforward calculation to show that the product manifold (M, g, J) is an isotropic para-Hermitian manifold. Further note that M is a locally symmetric space, and hence, that the Ricci operator is parallel. However, the Q_φ–operator $(g(Q_\varphi X, Y) = \varphi(X, Y)$, φ defined by (4.7)) is not parallel although it has constant eigenvalues, $4(p+3)c$, $2(p-q)c$ and $4(q+3)c$ with multiplicities $2p$, 2 and $2q$ respectively.

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