HOUSTON JOURNAL OF MATHEMATICS

Volume 24, No. 2, 1998

SUPPORTS OF QUASI-MEASURES

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ABSTRACT. We define and investigate a notion of support for quasi-measures that generalizes the usual notion of support for Borel measures. Also, for certain spaces we prove a decomposition theorem that implies, for example, that a space that carries a simple quasi-measure with full support is connected and has no cut points.

1. INTRODUCTION

All spaces under consideration are compact Hausdorff. By $\mathcal{F} = \mathcal{F}(X)$, we will denote the collection of closed subsets of a space X. We use $\mathcal{A} = \mathcal{A}(X)$ to denote those subsets of X that are either closed or open. A *quasi-measure* on X is a real-valued, finite, non-negative set-function μ defined on $\mathcal{F}$ that satisfies the following three axioms:

1. Whenever $F, F' \in \mathcal{F}$ and $F \subseteq F'$, then $\mu(F) \leq \mu(F')$.
2. If $F, F' \in \mathcal{F}$ and $F \cap F' = \emptyset$, then $\mu(F \cup F') = \mu(F) + \mu(F')$.
3. If $F \in \mathcal{F}$ and $\epsilon > 0$, then there is an $F' \in \mathcal{F}$ such that $F' \cap F = \emptyset$ and $\mu(F) + \mu(F') > \mu(X) - \epsilon$.

Given a μ satisfying these axioms, one can extend μ to a set-function on $\mathcal{A}$ by setting $\mu(U) = \mu(X) - \mu(X \setminus U)$ for every open $U \subseteq X$. The μ so obtained will satisfy the quasi-measure axioms given by Aarnes in [A1], where quasi-measures were first introduced:

1. Whenever $A \in \mathcal{A}$, then $\mu(A) + \mu(X \setminus A) = \mu(X)$.
2. If A_1, $A_2 \in \mathcal{A}$ and $A_1 \subseteq A_2$, then $\mu(A_1) \leq \mu(A_2)$.
3. If A_1, $A_2 \in \mathcal{A}$, $A_1 \cup A_2 \in \mathcal{A}$, and $A_1 \cap A_2 = \emptyset$, then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.
4. If $U \in \mathcal{A}$ is open, then $\mu(U) = \sup\{\mu(F) : F \text{ is closed and } F \subseteq U\}$.

1991 *Mathematics Subject Classification*. 28C15.

We will use the closed set version (which is due to Wheeler [W]) because it is more convenient for our purposes, although we will always consider quasi-measures to be defined on both open and closed sets. For ease of notation and to avoid trivialities, we will also assume that a quasi-measure satisfies $\mu(X) > 0$. Thus, for the purposes of this paper, the function that is identically equal to zero on $\mathcal{A}(X)$ is not a quasi-measure.

It is important to emphasize that a quasi-measure does not necessarily extend to a Borel measure. Indeed, as Wheeler shows in [W], a quasi-measure extends to a Borel measure if and only if it is subadditive. The first examples of quasi-measures that are not subadditive were given by Aarnes in [A1].

In this paper, we will define a notion of support for quasi-measures. Recall that for a Borel measure λ, the support of λ is the closed set $\bigcap\{K \subseteq X : K$ is closed and $\lambda(K) = 1\} = X \setminus \bigcup\{U \subseteq X : U$ is open and $\lambda(U) = \emptyset\}$. This definition of support is unsatisfactory when applied to quasi-measures. For example, let X be the unit square and μ any proper simple quasi-measure on X. Then for every $x \in X$, there is an open $U \ni x$ such that $\mu(U) = 0$. So in the usual sense, the support of μ is empty, even though X is compact and μ is two-valued.

After we give a more useful notion of support for quasi-measures, we will prove some basic facts about supports, investigate the relationship between supports and connectivity, and prove a theorem on decompositions of quasi-measures. The decomposition theorem is a measure-theoretic result that also allows one to deduce topological information about the underlying space. For example, if there is a fully supported simple quasi-measure on X, then X must be connected and (if also locally connected) no closed 0-dimensional set disconnects X. Together with Example 3, this indicates that, unlike Borel measures, proper quasi-measures are a global phenomena that interact with the connectivity of X in subtle ways.

Before continuing, we pause to collect some of the terminology we will use.

A *measure* is a subadditive quasi-measure. By results from [W], such a quasi-measure extends uniquely to a Borel measure on X, so the abuse of notation is minor.

A *proper* quasi-measure is a quasi-measure with no measure beneath it. I.e., if ν is proper, λ is a measure, and $0 \leq \lambda \leq \nu$, then $\lambda = 0$. By Grubb's decomposition theorem (see [GL]), every quasi-measure decomposes uniquely into the sum of a measure and a proper quasi-measure.

An *extremal* quasi-measure is a normalized quasi-measure μ such that whenever μ is a convex sum of normalized quasi-measures μ_1 and μ_2, then $\mu = \mu_1 = \mu_2$.

A quasi-measure is *simple* if it takes only the values 0 and 1. Clearly, a simple quasi-measure is extremal, but the converse fails in general (see [A2]).

Let $f : X \to Y$ be continuous and let μ be a quasi-measure on X. Then the *image of* μ *under* f is the quasi-measure μ^* on Y defined by $\mu^*(F) = \mu(f^{-1}(F))$ for each $F \in \mathcal{F}(Y)$.

Given a space X, let X^* denote the collection of all simple quasi-measures on X. For $U \subseteq X$ open, let $U^* = \{\mu \in X^* : \mu(U) = 1\}$. Give X^* the topology generated by using the collection of all U^* for U open in X as a subbasis. With this topology, X^* is compact Hausdorff and (after identifying points of X with point-masses) X is a closed subspace of X^*. See [A2] for more details.

We will also use Aarnes' notion of solid set function (see [A3]) to define quasi-measures on spaces that are connected and locally connected.

2. The definition of support

Definition 1. Let μ be a quasi-measure on a compact Hausdorff X.

1. A closed set $K \subseteq X$ *supports* μ if whenever $F \in \mathcal{F}$, then $\mu(F \cap K) = \mu(F)$.
2. $\mathcal{S}(\mu) = \{K : K \text{ supports } \mu\}$.

Clearly X supports μ, so that $\mathcal{S}(\mu)$ is never empty. Also, if K supports μ, then $\mu(K) = \mu(X)$. Given any $K \in \mathcal{F}$, define $\mu \restriction K = \mu \restriction \mathcal{F}(K)$; i.e., $\mu \restriction K$ is μ restricted to the closed subsets of K. Notice that because quasi-measures are not generally subadditive, $\mu \restriction K$ is not always a quasi-measure—set μ equal to Aarnes measure (see Example 1) and K equal to the boundary of the unit square for a counter-example.

Lemma 2.1. *Let μ be a quasi-measure on X and $K \in \mathcal{F}$. Then $K \in \mathcal{S}(\mu)$ if and only if $\mu \restriction K$ is a quasi-measure and $\mu(K) = \mu(X)$.*

Proof. [$\Rightarrow$] Suppose $K \in S(\mu)$. We already have $\mu(K) = \mu(X)$. Clearly, $\mu \restriction K$ satisfies axioms 1 and 2, so we need only check 3. Fix a closed $F \subseteq K$ and an $\epsilon > 0$. Because μ is a quasi-measure, there is a closed $C \subseteq X$ such that $\mu(F) + \mu(C) > \mu(X) - \epsilon$. Set $F' = C \cap K$. Because K supports μ, we have $\mu(F') = \mu(C)$, so $\mu(F) + \mu(F') > \mu(X) - \epsilon = \mu(K) - \epsilon$.

[$\Leftarrow$] Suppose $\mu(K) = \mu(X)$ and $\mu \restriction K$ is a quasi-measure. Fix a closed $F \subseteq X$ and an $\epsilon > 0$ arbitrary. There is a closed $F' \subseteq K$ disjoint from $F \cap K$ such that

$\mu(F \cap K) > \mu(K) - \mu(F') - \epsilon$. Thus,

$$\begin{aligned}\mu(F) \geq \mu(F \cap K) &> \mu(K) - \mu(F') - \epsilon \\ &= \mu(X) - \mu(F') - \epsilon \\ &= \mu(X \setminus F') - \epsilon \\ &\geq \mu(F) - \epsilon.\end{aligned}$$

Because $\epsilon > 0$ was arbitrary, we have $\mu(F) = \mu(F \cap K)$, so K supports μ. ☐

Lemma 2.2. *If μ is a quasi-measure on X, then $\mathcal{S}(\mu)$ is closed under finite intersections.*

PROOF. Clearly, it suffices to show that if K and K' are in $\mathcal{S}(\mu)$, then so is $K \cap K'$. So fix $K, K' \in \mathcal{S}(\mu)$ and a closed F. Then $\mu(F \cap (K \cap K')) = \mu((F \cap K) \cap K') = \mu(F \cap K') = \mu(F)$, where the second equality follows from $K \in \mathcal{S}(\mu)$ and the third from $K' \in \mathcal{S}(\mu)$. ☐

Definition 2. Let μ be a quasi-measure on X. The *support of μ* is the closed set $\text{suppt}(\mu) = \bigcap \mathcal{S}(\mu)$. If $\text{suppt}(\mu) = X$, we say that μ is *fully supported*.

Notice that because X is compact and $\mathcal{S}(\mu)$ is non-empty and closed under finite intersections, $\text{suppt}(\mu)$ is always non-empty.

Theorem 2.3. *Suppt(μ) supports μ.*

PROOF. Set $S = \text{suppt}(\mu)$. Suppose $F \subseteq X$ is closed and, by way of contradiction, $\mu(F \cap S) < \mu(F)$. Let $U \subseteq X$ be open with $F \cap S \subseteq U$ and $\mu(U) < \mu(F)$. Notice $(F \setminus U) \cap S = \emptyset$; because $\mathcal{S}(\mu)$ is closed under finite intersections and $F \setminus U$ is compact, there is a $K \in \mathcal{S}(\mu)$ such that $(F \setminus U) \cap K = \emptyset$. But then $\mu(F) = \mu(F \cap K) \leq \mu(U) < \mu(F)$, a contradiction. ☐

Thus, the support of a quasi-measure μ is the smallest closed set S such that $\mu(S) = \mu(X)$ and $\mu \restriction S$ is still a quasi-measure. This is clearly a generalization of the usual notion of support for Borel measures.

3. Basic results

We begin this section with an example.

Example 1. Let X be the unit square, ∂X its boundary, and p the point $(\frac{1}{2}, \frac{1}{2})$. Aarnes measure on X is the quasi-measure determined by the solid set-function

$$\mu(A) = \begin{cases} 0 & \text{if } A \cap \partial X = \emptyset, \\ 1 & \text{if either } \partial X \subseteq A \text{ or } p \in A \text{ and } A \cap \partial X \neq \emptyset. \end{cases}$$

We claim that μ is fully supported. Let U be any non-empty open subset of X; we will show that $X \setminus U$ does not support μ. If necessary, we can shrink U to an open disk with $p \notin U$ and $U \cap \partial X = \emptyset$. Let C be a line segment with one endpoint at p and the other on ∂X that intersects U. Then $\mu(C) = 1$, but $\mu(C \cap (X \setminus U)) = 0$ (both components of $C \cap (X \setminus U)$ have measure 0), so $X \setminus U$ does not support μ.

Similarly, one can show that on the unit square, any of Knudsen's generalized point-masses (see [K]) are fully supported. Thus, the collection of simple quasi-measures with full support form a dense and connected subset of the collection of all simple quasi-measures on the square.

Problem 1. In general, when is the collection of fully supported simple quasi-measures on X non-empty, dense, or connected as a subset of X^*?

For locally connected spaces, Theorem 5.3 provides a necessary condition for the existence of a simple quasi-measure that is fully supported: if there is a simple quasi-measure on X that has full support, then X is connected and no 0-dimensional closed subset of X disconnects X.

We now prove some basic results about supports of quasi-measures and how they behave under standard operations. We consider subspaces first.

Lemma 3.1. *If μ and ν are quasi-measures on X with μ extremal and $\nu \leq \mu$, then $\nu = \alpha\mu$, with $0 \leq \alpha \leq 1$.*

PROOF. By way of contraposition, suppose that μ and ν are as above with $\nu < \mu$. Set $\alpha = \nu(X)$, then $\frac{1}{\alpha}\nu$ and $\frac{1}{1-\alpha}(\mu - \nu)$ are normalized quasi-measures and

$$\mu = \alpha\left(\frac{1}{\alpha}\nu\right) + (1-\alpha)\left[\frac{1}{1-\alpha}(\mu - \nu)\right],$$

so that μ is not extremal. □

Theorem 3.2. *Let μ be a quasi-measure on X. Suppose $C \subseteq X$ is closed and that $\mu \restriction C$ is a quasi-measure. Then suppt$(\mu \restriction C) \subseteq$ suppt$(\mu) \cap C$. Equality holds if μ is extremal.*

PROOF. We show $\text{suppt}(\mu) \cap C$ supports $\mu \restriction C$. Let $F \subseteq C$ be a closed subset of C. Then $\mu \restriction C(F \cap (\text{suppt}(\mu) \cap C)) = \mu(F \cap \text{suppt}(\mu)) = \mu(F) = \mu \restriction C(F)$.

If μ is extremal, then Lemma 3.1 implies $\mu \restriction C = \mu$, so that $\text{suppt}(\mu) \cap C = \text{suppt}(\mu) = \text{suppt}(\mu \restriction C)$. □

If μ is not extremal, then the containment in the previous result may be strict, as the following example shows.

Example 2. Let X be the unit square $[0,1] \times [0,1]$. Set $Y = [0, \frac{1}{2}] \times [0,1]$, and $Z = [\frac{1}{2}, 1] \times [0,1]$. Let ∂Y and ∂Z denote the boundaries of Y and Z, respectively. Set $p_Y = (\frac{1}{4}, \frac{1}{2})$ and $p_Z = (\frac{3}{4}, \frac{1}{2})$. Let μ_Y be Aarnes measure on Y with p_Y and ∂Y (so that a closed solid C gets measure 1 if and only if $\partial Y \subseteq C$ or $p \in C$ and $C \cap \partial Y \neq \emptyset$). Similarly, let μ_Z be Aarnes measure on Z with p_Z and ∂Z. Let $f_Y : Y \to X$ and $f_Z : Z \to X$ be the natural embeddings, and set $\mu = \mu_Y^* + \mu_Z^*$. Because $\text{suppt}(\mu_Y^*) = Y$ and $\text{suppt}(\mu_Z^*) = Z$, Theorem 3.3 below implies that $\text{suppt}(\mu) = Y \cup Z = X$. Let $W = [0, \frac{5}{8}] \times [0,1]$. Then $\mu \restriction W$ is a quasi-measure and $\mu \restriction W = \mu_Y^* \restriction W$, but $\text{suppt}(\mu \restriction W) = Y \neq \text{suppt}(\mu) \cap W$.

Theorem 3.3. *Let μ_1 and μ_2 be quasi-measures on X. Then suppt$(\mu_1 + \mu_2) =$ suppt$(\mu_1) \cup$ suppt(μ_2).*

Proof. Suppose K supports $\mu_1 + \mu_2$. Let $F \subseteq X$ be closed. Then $(\mu_1 + \mu_2)(F \cap K) = (\mu_1 + \mu_2)(F) = \mu_1(F) + \mu_2(F)$. So $\mu_1(F) + \mu_2(F) - \mu_1(F \cap K) - \mu_2(F \cap K) = 0$, so K supports both μ_1 and μ_2.

Conversely, suppose a closed K supports both μ_1 and μ_2. Let $F \subseteq X$ be closed. Then $(\mu_1 + \mu_2)(F \cap K) = \mu_1(F \cap K) + \mu_2(F \cap K) = \mu_1(F) + \mu_2(F) = (\mu_1 + \mu_2)(F)$, so K supports $\mu_1 + \mu_2$. □

D.J. Grubb has informed us of the next two results, and has graciously allowed us to include them here.

Theorem 3.4. *Let μ be a quasi-measure on X and suppose $f : X \to Y$ is continuous. Then suppt$(\mu^*) \subseteq f($suppt$(\mu))$.*

The proof of this result is straightforward, but again the containment may be strict. For example, let μ be Aarnes measure on the unit square X. Let Y be the quotient space obtained by identifying the closed set $\{\frac{1}{2}\} \times [0, \frac{1}{2}]$ to a point p and let $f : X \to Y$ be the induced quotient map. Then μ^* is a point-mass at p, so $\text{suppt}(\mu^*) \neq f(\text{suppt}(\mu))$.

The next theorem, which we state without proof, uses the notation of [G].

Theorem 3.5. *If $\mu \times_l \nu$ is a product quasi-measure on $X \times Y$, then suppt$(\mu \times_l \nu) =$ suppt$(\mu) \times$ suppt(ν).*

Since the support of a Borel measure is always ccc (i.e., any family of pairwise disjoint non-empty open subsets is at most countable), to study Borel measures on compact spaces, it suffices (roughly speaking) to study compact ccc spaces. The next example shows that there may be an uncountable pairwise disjoint family of open sets in the support of a quasi-measure.

Example 3. Let ω_1 be the first uncountable ordinal. Let Y be the one-point compactification of the long line. I.e., Y is $\omega_1 + 1$ with a copy of the unit interval inserted between each pair of ordinals α and $\alpha + 1$. Set $X = Y \times [0,1]$ and $\partial X = (Y \times \{0,1\}) \cup (\{0, \omega_1\} \times [0,1])$, the boundary of X. Define a quasi-measure on X similar to Aarnes measure via the following solid set function:

$$\mu(A) = \begin{cases} 0 & \text{if } A \cap \partial X = \emptyset, \\ 1 & \text{if either } \partial X \subseteq A \text{ or } (\frac{1}{2}, \frac{1}{2}) \in A \text{ and } A \cap \partial X \neq \emptyset. \end{cases}$$

As before, $\text{suppt}(\mu) = X$, even though X is not ccc.

By replacing ω_1 with any uncountable cardinal in the previous example, one can show that there is no bound on the cardinality of a pairwise disjoint family of open subsets of the support of a quasi-measure.

4. Connectivity and supports

In this section, we explore some of the connections between connectivity and supports of quasi-measures. In the next result, the operator "dim" is Čech-Lebesgue covering dimension.

Theorem 4.1. *Let μ be a proper quasi-measure on X. Then $\dim(\text{suppt}(\mu)) \geq 2$.*

PROOF. If $\dim(\text{suppt}(\mu)) \leq 1$, then by the results in [W], $\mu \restriction \text{suppt}(\mu)$ is a measure, whence μ is not proper. □

Lemma 4.2. *Let μ be a quasi-measure on X. Suppose W is a proper clopen subset of suppt(μ). Then $\mu \restriction W$ is a quasi-measure and $0 < \mu(W) < \mu(X)$.*

PROOF. Let W be a proper subset of $\text{suppt}(\mu)$ (so that $0 \neq W \neq \text{suppt}(\mu)$) that is both open and closed as a subset of $\text{suppt}(\mu)$. Then $W' = \text{suppt}(\mu) \setminus W$ is also a proper clopen subset of $\text{suppt}(\mu)$ and $\mu(X) = \mu(\text{suppt}(\mu)) = \mu(W) + \mu(W')$.

To show that $\mu \restriction W$ is a quasi-measure, we need only check 3 of Wheeler's axioms. So fix $\epsilon > 0$ and a closed $F \subseteq W$. Because F is closed as a subset of X, there is a closed $C \subseteq X$ such that $F \cap C = \emptyset$ and $\mu(F) + \mu(C) > \mu(X) - \epsilon$. Without loss of generality, we may assume that $W' \subseteq C$. Then

$$\begin{aligned} \mu(F) + \mu(C \cap W) &= \mu(F) + \mu(C \cap W) + \mu(C \cap W') - \mu(C \cap W') \\ &\geq \mu(F) + \mu(C \cap (\text{suppt}(\mu))) - \mu(W') \\ &= \mu(F) + \mu(C) - \mu(W') \\ &> \mu(X) - \epsilon - \mu(W') \\ &= \mu(W) - \epsilon. \end{aligned}$$

Because $\mu \restriction W$ is a quasi-measure and W does not support μ (it is a proper subset of suppt(μ)), by Lemma 2.1 we must have $\mu(W) < \mu(X)$. By symmetry, we also have $\mu(W') < \mu(X)$, so in fact $0 < \mu(W) < \mu(X)$. □

Suppose $C \subseteq X$ is closed and $\mu \restriction C$ is a quasi-measure. We can consider $\mu \restriction C$ to be a quasi-measure on X with support contained in C by defining $\mu \restriction C(F) = \mu(F \cap C)$ for all $F \in \mathcal{F}$. This is equivalent to embedding C in X in the natural way and then taking the image of $\mu \restriction C$ under this embedding. We will use this convention freely in the sequel.

Corollary 4.3. *If suppt(μ) $= C_1 \cup C_2$, with C_1 and C_2 disjoint and closed, then $\mu = \mu \restriction C_1 + \mu \restriction C_2$.*

PROOF. Let $F \subseteq X$ be closed. Then $\mu(F) = \mu(F \cap (C_1 \cup C_2)) = \mu(F \cap C_1) + \mu(F \cap C_2) = \mu \restriction C_1(F) + \mu \restriction C_2(F)$. □

Corollary 4.4. *If μ is an extremal quasi-measure, then suppt(μ) is connected.*

PROOF. By way of contradiction, suppose suppt(μ) $= C_1 \cup C_2$, with C_1 and C_2 disjoint non-empty closed subsets of suppt(μ). By Lemma 4.2, both $\mu(C_1)$ and $\mu(C_2)$ are quasi-measures strictly less than μ, which contradicts Lemma 3.1. □

Corollary 4.5. *If μ is a simple quasi-measure, then suppt(μ) is connected.*

Notice that this corollary generalizes a (trivial) property of 0–1 measures. The support of a proper quasi-measure can be very disconnected, as the following example shows.

Example 4. Let X be the unit square and μ Aarnes measure on X. Let Y be the Cantor set and let λ be Haar probability measure on Y. Set $\nu = \mu \times_l \lambda$; then by Theorem 3.5 suppt(ν) $= X \times Y$. Notice that for each $y \in Y$, $X \times \{y\}$ is a connected component of suppt(ν).

5. DECOMPOSITIONS OF QUASI-MEASURES

We will need the following corollary of the following lemma, which is of independent interest and generalizes the result (implicit in [W]) that proper quasi-measures vanish on 0-dimensional sets.

Lemma 5.1. *Suppose μ is a quasi-measure on X, W and F are closed subsets of X, and that W is 0-dimensional. Then $\mu(W \cup F) \leq \mu(W) + \mu(F)$.*

PROOF. By Grubb's decomposition theorem, we can write $\mu = \lambda + \nu$, where λ is a measure and ν is a proper quasi-measure. Suppose $V \subseteq W \setminus F$ is clopen-in-W. Then $W \cup F = V \cup [(W \cup F) \setminus V]$, so because V is 0-dimensional, we have $\nu(W \cup F) = \nu(V) + \nu((W \cup F) \setminus V) = 0 + \nu((W \cup F) \setminus V) = \nu((W \cup F) \setminus V)$.

Let $\mathcal{V} = \{V : V \subseteq W \setminus F \text{ is closed and clopen-in-}W\}$. Then

$$F = \bigcap_{V \in \mathcal{V}} [(W \cup F) \setminus V],$$

and this intersection is directed. Thus, by results in [A2], we have $\nu(F) = \lim_{V \in \mathcal{V}} \nu((W \cup F) \setminus V) = \nu(W \cup F)$. Therefore,

$$\begin{aligned} \mu(W \cup F) &= \lambda(W \cup F) + \nu(W \cup F) \\ &\leq \lambda(W) + \lambda(F) + \nu(F) \\ &= \lambda(W) + \nu(W) + \lambda(F) + \nu(F) \\ &= \mu(W) + \mu(F), \end{aligned}$$

where the second equality follows from the fact that $\nu(W) = 0$. □

Corollary 5.2. *If ν is a proper quasi-measure on X, W and F are closed in X, and W is 0-dimensional, then $\nu(W \cup F) = \nu(F)$*

Suppose now that X is connected and locally connected and that $W \subseteq X$ is a closed 0-dimensional set that disconnects X. Write $X \setminus W = \bigcup_{\alpha \in A} U_\alpha$, where the U_α's are the (open) connected components of $X \setminus W$. For each $\alpha \in A$, set $K_\alpha = U_\alpha \cup W$; notice that each K_α is closed and connected.

Let ν be a proper quasi-measure on X, and for each $\alpha \in A$, let $\nu_\alpha = \nu \restriction K_\alpha$.

Theorem 5.3. *With notation as above, each ν_α is a quasi-measure on K_α and (construing the ν_α's as quasi-measures on X) $\nu = \sum_{\alpha \in A} \nu_\alpha$.*

PROOF. Fix $\alpha \in A$; we show ν_α is a quasi-measure on K_α. Let $F \subseteq K_\alpha$ be closed and fix $\epsilon > 0$. We seek a closed $F' \subseteq K_\alpha$ disjoint from F with $\nu(F) + \nu(F') > \nu(K_\alpha) - \epsilon$.

In X, there is a closed H disjoint from F with $\nu(F) + \nu(H) > \nu(X) - \epsilon/2$. Set $F' = H \cap K_\alpha$ and find an open V with $F \cup F' \cup W \subseteq V$ and $\nu(V) < \nu(F) + \nu(F') + \epsilon/2$. (This uses Corollary 5.2.) Then $X = (K_\alpha \setminus V) \cup V \cup (\bigcup_{\beta \neq \alpha} (U_\beta \setminus V))$, where the first and last sets in this pairwise disjoint union are closed. Also notice that because $(K_\alpha \setminus V) \cap (F \cup H) = \emptyset$, we have $\nu(K_\alpha \setminus V) < \epsilon/2$, and that because $\bigcup_{\beta \neq \alpha} U_\beta \subseteq X \setminus K_\alpha$, we have $\nu(\bigcup_{\beta \neq \alpha} U_\beta \setminus V) \leq \nu(X \setminus K_\alpha)$. Thus, $\nu(X) < \epsilon/2 + \nu(F) + \nu(F') + \epsilon/2 + \nu(X \setminus K_\alpha)$, so that $\nu(F) + \nu(F') > \nu(K_\alpha) - \epsilon$.

To see that $\nu = \sum_{\alpha \in A} \nu_\alpha$, fix $F \subseteq X$ closed. Then for any $\alpha \in A$, $\nu_\alpha(F \cap K_\alpha) = \nu(F \cap K_\alpha) = \nu(W \cup (F \cap K_\alpha)) = \nu(K_\alpha) - \nu(U_\alpha \setminus F)$, because $K_\alpha = (U_\alpha \setminus F) \cup (W \cup (F \cap K_\alpha))$, and this union is disjoint.

Also notice that $X = \bigcup_{\alpha \in A} U_\alpha \cup W$, so $\nu(X) = \sum_{\alpha \in A} \nu(U_\beta) = \sum_{\alpha \in A} \nu(U_\beta) + 0 = \sum_{\alpha \in A} \nu(U_\beta) + \nu(W) = \sum_{\alpha \in A} \nu(K_\alpha)$.

Thus,

$$\begin{aligned}\sum_{\alpha \in A} \nu_\alpha(F) &= \sum_{\alpha \in A} \nu(F \cap K_\alpha) \\ &= \sum_{\alpha \in A} \nu(K_\alpha) - \nu(U_\alpha \setminus F) \\ &= \nu(X) - \sum_{\alpha \in A} \nu(U_\alpha \setminus F) \\ &= \nu(X) - \sum_{\alpha \in A} \nu(U_\alpha \setminus (W \cup F)) \\ &= \nu(X) - \nu(X \setminus (W \cup F)) \\ &= \nu(W \cup F) \\ &= \nu(F),\end{aligned}$$

where the fifth equality follows from [A2] and the fact that the $U_\alpha \setminus (F \cup W)$'s are pairwise disjoint. □

Corollary 5.4. *With notation as above, if μ is any quasi-measure on X that vanishes on W, then $\mu = \sum_{\alpha \in A} \mu \restriction K_\alpha$.*

PROOF. Use Grubb's decomposition theorem to write $\mu = \lambda + \nu$. The preceding result for measures is trivial, so the result follows immediately. □

Corollary 5.5. *With notation as above, if ν is a proper extremal quasi-measure, then there is an $\alpha \in A$ such that suppt(ν) $\subseteq K_\alpha$. Moreover, suppt(ν) cannot be disconnected by the removal of any closed 0-dimensional set.*

Theorem 5.3 and its corollaries are sharp, in the sense that "W is 0-dimensional" cannot be replaced by "W is 1-dimensional". This can be seen by letting μ be Aarnes measure on the square and setting $W = \{\frac{3}{4}\} \times [0, 1]$.

We now illustrate the above results with a concrete example.

Example 5. Let $Y = \{(x, y) : x^2 + y^2 = 1\}$, $Z = \{(x, y) : (x - 2)^2 + y^2 = 1\}$, and $X = Y \cup Z$. I.e., X is the union of two tangent circles in the plane. Because the point $(0, 1)$ disconnects X, any quasi-measure on X is the sum of a quasi-measure with support contained in Y and a quasi-measure with support contained in Z.

We conclude with a construction that suggests the possibility of a representation theory for quasi-measures based on connectivity of supports. Let μ be a quasi-measure on X and $\{K_\alpha : \alpha \in A\}$ the set of connected components of $\text{suppt}(\mu)$. Let Y be the quotient space obtained from X by identifying each K_α to a point. Let $\pi : X \to Y$ be the associated quotient map, and for convenience, identify $\pi(K_\alpha)$ with α, so that $\pi(\text{suppt}(\mu)) = A$. Then as in [GL], Y is compact Hausdorff and A is 0-dimensional. Let μ^* be the image of μ under π and write $\mu^* = \lambda + \nu$, where λ is a measure and ν is proper. Then because $\text{suppt}(\mu^*) \subseteq A$ and A is 0-dimensional, we have $\nu = 0$, so $\mu^* = \lambda$. In view of Lemma 4.2, λ is essentially a measure on the Boolean algebra of clopen subsets of $\text{suppt}(\mu)$.

We illustrate this construction with some examples. Suppose that $\text{suppt}(\mu) = \bigcup_{n\in\mathbb{N}} C_n$, with the C_n's pairwise disjoint, closed, and connected. Then λ is a countable sum of weighted point-masses at the points of A. If μ is the product quasi-measure described in Example 4, then λ is essentially Haar measure on the Cantor set. Notice that in both of these examples, we can represent μ as an integral with respect to λ of an appropriate function defined on A.

Problem 2. With the notation described above, when is it possible to find, for each $\alpha \in A$, quasi-measures μ_α on K_α so that if for each $F \in \mathcal{F}$, g_F is defined on A by $g_F(\alpha) = \mu_\alpha(F \cap K_\alpha)$, then

$$\mu(F) = \int_A g_F(\alpha)\, d\lambda(\alpha)?$$

6. Coda

After this paper had been submitted for publication, Bob Wheeler has informed the author of the following result which follow easily from the results presented in this paper. He has graciously consented to allow us to include it here.

Theorem 6.1. *Suppose $Y \subseteq X$ is closed and the boundary of Y is 0-dimensional. Then whenever μ is a quasi-measure on X, $\mu \restriction Y$ is a quasi-measure on Y.*

Wheeler has also noticed that the fact that a quasi-measure μ on X is a measure iff $\mu \restriction F$ is a quasi-measure for all closed $F \subseteq X$ can be used to provide an alternate proof of the following theorem first proved by Aarnes in [A2].

Theorem 6.2. *If μ is a measure on X and λ is a quasi-measure on X that satisfies $0 \le \lambda \le \mu$, then λ is a measure on X.*

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Received June 16, 1997

Revised version received February 23, 1998

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