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ON A THEOREM BY LIONS AND PEETRE ABOUT INTERPOLATION BETWEEN A BANACH SPACE AND ITS DUAL

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Abstract. We show that if the duality between a Banach space A and its anti-dual A^* is given by the inner product of a Hilbert space H, then $(A, A^*)_{1/2,2} = H = (A, A^*)_{[1/2]}$, provided A satisfies certain mild conditions. We do not assume A is reflexive. Applications are given to normed ideals of operators.

1. Introduction

Let A be a Banach space continuously and densely embedded in a Hilbert space H. We can then consider H as continuously embedded in the anti-dual A^* of A so that, in particular, the couple (A, A^*) is compatible. A classical result of Interpolation Theory states, in case A is reflexive, that

(1) $\qquad (A, A^*)_{1/2,2} = H \qquad$ with equivalence of norms,

(2) $\qquad (A, A^*)_{[1/2]} = H \qquad$ with equality of norms.

Here $(\cdot,\cdot)_{\theta,q}$ denotes the real interpolation functor of order $\theta \in (0,1)$, $q \in [1,\infty]$, and $(\cdot,\cdot)_{[\theta]}$ denotes the complex interpolation functor of order $\theta \in (0,1)$. Formula (1) was established by J.L. Lions and J. Peetre in their famous article on the real interpolation method [13], where it appears as Theorem 3.4.1. For (2), see the paper by Peetre [17], page 175. Reflexivity is essential for the arguments in both [13] and [17]. Without this assumption their methods only give

$$(A, A^*)_{1/2,2} \hookrightarrow H \qquad \text{and} \qquad (A, A^*)_{[1/2]} \hookrightarrow H.$$

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However, in many situations of interest, the space A fails to be reflexive (see, for example, [17], p. 175 or [6], p. 134), or there are no inclusion relationships between the spaces A, A^* and H. It is then important to investigate the validity of the formulae under weaker assumptions. This problem was first considered by Miyazaki [15] in 1968, who gave a somewhat lengthy proof of (1) without assuming reflexivity. No other progress seems to have been made on this matter for several decades. However, during the last few years several papers have appeared dealing with it.

For the complex case, Haagerup and Pisier [10], §3, proved that (2) holds if we are in the situation $A^* \hookrightarrow H \hookrightarrow A$ with dense injections and A^{**} has the analytic Radon-Nikodym property. Later on, Pisier in [18], dealing with operator spaces, gives an argument which shows that (2) holds if $A^* \hookrightarrow H \hookrightarrow A$ with dense injections without need of any further assumptions on A. More recently Watbled [20] showed that (2) is valid assuming only $A \hookrightarrow H$ or $H \hookrightarrow A$ with dense inclusion. She also established some results for the case in which A, A^* and H are not related by inclusions (see [21]). In particular, (2) holds in this more general setting if A is reflexive. Concerning the real case, two short proofs of (1) appeared recently: The techniques used by Xu in [22], working again in the context of operator spaces, yield that (1) remains valid if $A \hookrightarrow H$ with dense injection, while Amrein, Boutet de Monvel and Georgescu use Gagliardo completions to prove this result in [1].

In this paper we address the case in which there is no embedding relationship between the spaces A, A^* and H. We consider the formulae in the more general context of dual pairs, and give mild conditions under which they are valid. We show that these conditions always hold in the cases considered by all the authors mentioned above.

The organization of the paper is as follows. In the next section (Section 1) we develop the notation, define what we understand by a dual pair and verify some generalities. Section 2 deals with the real interpolation case, formula (1). The complex case is studied in Section 3, where we also describe relationships between the real and complex interpolation spaces generated by the couple (A, A^*). Finally, in Section 4 we give some applications.

2. Dual Pairs

All Banach spaces appearing in this paper are over the field of complex numbers. If X is a Banach space, X^* denotes its *anti*-dual, that is, X^* is the space of all bounded conjugate linear functionals of X. By a *dual pair* we shall understand a pair $D = (D_0, D_1)$ of complex vector spaces such that D_0 is a linear subspace of D_1 and there exists a sesquilinear map

$$(a, b) \mapsto (a, b)_D : D_0 \times D_1 \to \mathbb{C}$$

such that

(i): The restriction of $(\cdot, \cdot)_D$ to $D_0 \times D_0$ is an inner product for D_0; i.e., $(a, b)_D = \overline{(b, a)_D}$ for all $a, b \in D_0$ and $(a, a)_D > 0$ if $a \in D_0$, $a \neq 0$.

(ii): If $b \in D_1$ and $(a, b)_D = 0$ for all $a \in D_0$, then $b = 0$.

It follows that $(D_0, (\cdot, \cdot)_D)$ is a pre-Hilbert space with the norm $\|a\|_D = (a, a)_D^{1/2}$ if $a \in D_0$. We shall always assume that our dual pairs $D = (D_0, D_1)$ are *complete*, meaning that whenever $\{x_n\}$ is a Cauchy sequence in the pre-Hilbert space D_0, there exists a (necessarily unique) element $b \in D_1$ such that $\lim_{n\to\infty}(a, x_n)_D = (a, b)_D$ for all $a \in D_0$. The set H of all such b's, normed by

$$\|b\|_H = \sup\{|(a, b)_D| \, : \, a \in D_0, \|a\|_D \leq 1\},$$

can be identified with the Hilbert space completion of D_0. We call H the Hilbert space *associated* with the pair D. In the sequel, given a complete dual pair $D = (D_0, D_1)$, we shall drop the subscript D from the notation for the sesquilinear form, so that (a, b) denotes $(a, b)_D$ if $a \in D_0$, $b \in D_1$ or the inner product of a, b if $a, b \in H$. Note that our concept of a dual pair may be considered as a special case of a more general notion extensively studied in Functional Analysis.

A Banach space A is said to *belong* to the (complete) dual pair $D = (D_0, D_1)$ if

(iii): A is a linear subspace of D_1, and D_0 is a dense subspace of A.

(iv): The anti-dual A^* of A is included in D_1 in the sense that $\psi \in A^*$ if and only if there exists $b_\psi \in D_1$ such that

$$\psi(a) = \overline{(a, b_\psi)}$$

for all $a \in D_0$.

(v): $D_0 \subset A \cap A^* \subset H \subset A + A^*$ and D_0 is dense in $A \cap A^*$ in the topology of A^* (as well as that of A).

Because D_0 is dense in A, the restriction of an element $\psi \in A^*$ to D_0 determines ψ uniquely; the corresponding element b_ψ is necessarily unique. We thus have

$$A^* = \{b \in D_1 \, : \, \exists C_b \geq 0, |(a, b)| \leq C_b \|a\|_A \, \forall a \in D_0\}.$$

Example 2.1. Let A be a Banach space densely and continuously embedded in a Hilbert space H. We can then identify each element $h \in H$ with a unique element of A^* defining $h(a) = (h, a) = \overline{(a, h)}$ for $a \in A$. In this way H becomes continuously included in A^* and we shall set $(a,b) = \overline{b(a)}$ if $a \in A$, $b \in A^*$, noticing that this definition is consistent with the original meaning of (a,b) as the inner product of a,b if $a,b \in H$. It is then clear that (A,A^*) is a dual pair, with associated Hilbert space H, to which A belongs.

Note that if, as usual, we consider the inclusion $A \hookrightarrow A^{**}$ given by the injection $a \mapsto J_a$ where $J_a(b) = \overline{b(a)}$ for $a \in A$ and $b \in A^*$, we see that the action of $a \in A$ on A^* is the conjugate linear map $b \mapsto (a,b)$.

Example 2.2. Assume similarly that H is a Hilbert space which is continuously and densely embedded in a Banach space A. We then consider $A^* \hookrightarrow H$ by identifying $\psi \in A^*$ with the unique $b_\psi \in H$ such that $\psi(a) = \overline{(a, b_\psi)}$ for all $a \in H$. It turns out that A^* is dense in H so that (A^*, A) is a dual pair, with associated Hilbert space H, to which A belongs.

Example 2.3. In the previous examples A, A^* and H are comparable by inclusions. A typical example of a dual pair in which such comparisons do not necessarily hold is given by $D_0 = L^1(\mu) \cap L^\infty(\mu)$, $D_1 = L^1(\mu) + L^\infty(\mu)$, where $(\Omega, \mathcal{M}, \mu)$ is a measure space. Defining

$$(f,g) = \int_\Omega f\bar{g}\,d\mu$$

if $f \in D_0$, $g \in D_1$, (D_0, D_1) becomes a dual pair with associated Hilbert space $L^2(\mu)$; $L^p(\mu)$ belongs to this pair for $1 \le p < \infty$.

Other examples of dual pairs where A, A^* are not comparable by inclusion will be given in Section 4.

In the sequel, we shall always assume that we are given a dual pair with associated Hilbert space H and A shall always denote a Banach space belonging to this pair. By A_1 we shall always understand the closure of $A \cap A^*$ in the topology of A^* so that A_1 is a Banach space with the norm of A^*. Let $\mathcal{D}$ denote the algebraic anti-dual of D_0, where (D_0, D_1) is the basic dual pair. Then D_1 is a subspace of $\mathcal{D}$ (identifying $b \in D_1$ with the functional $a \mapsto \overline{(a,b)}$ of D_0), and so are A, A^*, A_1 and H. Because D_0 is dense in A_1 (by condition (v) of the definition of dual pair), we can identify the anti-dual A_1^* of A_1 with a subspace of $\mathcal{D}$ in a natural way, namely by

$$A_1^* = \{\psi \in \mathcal{D} : \exists C \ge 0 \,\text{such that}\, |\psi(a)| \le C\|a\|_{A^*} \,\text{for all}\, a \in D_0\}.$$

It now makes sense to consider the algebraic sum $A_1^* + A^*$. This space becomes a Banach space when normed in the usual way; that is, by

$$\|a\|_{A_1^*+A^*} = \inf\{\|a_0\|_{A_1^*} + \|a_1\|_{A^*} : a_0 \in A_1^*, a_1 \in A^*, a = a_0 + a_1\}.$$

Indeed, we need to verify that $\|a\|_{A_1^*+A^*} = 0$ implies $a = 0$, since we do not know yet that there is a Hausdorff topological vector space continuously containing both A_1^* and A^*. If $\|a\|_{A_1^*+A^*} = 0$, then there exist sequences $\{a_{0n}\}$, $\{a_{1n}\}$ converging to zero in A_1^*, A^*, respectively, such that $a = a_{0n} + a_{1n}$ for all n. Let $c \in D_0$. Then $c \in A \cap A^* \subset A_1$ and

$$|(c,a)| = |(a_{0n}, c) + \overline{(c, a_{1n})}| \leq \|a_{0n}\|_{A_1^*}\|c\|_{A^*} + \|c\|_A\|a_{1n}\|_{A^*} \to 0$$

as $n \to \infty$. Thus $(c, a) = 0$ for all $c \in D_0$, hence $a = 0$. The remaining properties of a norm and completeness of $A_1^* + A^*$ under this norm are immediate. The Banach space $A_1^* + A^*$ contains the spaces A, A_1, A^*, H and A_1^* as linear subspaces; let us see that each one of them is also continuously embedded in $A_1^* + A^*$. To begin with, it is clear that A, A_1^*, A and A_1^* are continuously included in $A_1^* + A^*$, all with norm ≤ 1. To see that H is also included with inclusion of norm ≤ 1, let $a \in A \cap A^*$. Then

$$\|a\|_H^2 = (a, a) \leq \|a\|_A\|a\|_{A^*} \tag{3}$$

so that $A \cap A^* \hookrightarrow H$ with norm ≤ 1 (we endow $A \cap A^*$ with its natural norm given by $\|a\|_{A\cap A^*} = \max\{\|a\|_A, \|a\|_{A^*}\}$). If $c \in D_0$, $h \in H$, using (3),

$$|(c, h)| \leq \|c\|_H\|h\|_H \leq \|c\|_{A\cap A^*}\|h\|_H$$

hence the norm of h as an element of the dual of $A \cap A^*$ is bounded by $\|h\|_H$. However, $A \cap A^* = A \cap A_1$ and $A \cap A_1$ is dense in both A and A_1 so that we can use the classical duality relationships between sums and intersections (see [4]) and get

$$(A \cap A^*)^* = (A \cap A_1)^* = A_1^* + A^*.$$

In other words, $\|h\|_{A_1^*+A^*} \leq \|h\|_H$, as desired. A consequence of these continuous embeddings is that any pair formed by taking any two of A, A_1, A^*, H or A_1^* is a compatible pair.

All this can be considerably simplified in the case $A \hookrightarrow H$ densely; i.e., Example 2.1. In this situation it is clear that H is densely and continuously included in A_1, hence every element of A_1^* is uniquely determined by its restriction to H and this restriction is a bounded anti-linear functional on H. Since $H = H^*$, we see that A_1^* is continuously embedded in H. Moreover, since A is dense in A_1, an element $h \in H$ is in A_1^* if and only if

$$\|h\|_{A_1^*} = \sup\{|(h, a)| : a \in A, \|a\|_{A^*} \leq 1\} < \infty;$$

the action of h on A_1 is the unique continuous extension to A_1 of the map $a \mapsto (h, a)$ from A to $\mathbb{C}$.

3. The real case

We begin recalling that given a compatible couple $\bar{B} = (B_0, B_1)$ of Banach spaces, Peetre's K-functional is defined by

$$K(t,a) = K(t,a;B_0,B_1) = \inf\{\|a_0\|_{B_0} + t\|a_1\|_{B_1} \,:\, a = a_0 + a_1, a_0 \in B_0, a_1 \in B_1\}$$

if $a \in B_0 + B_1$ and $t > 0$. If $0 < \theta < 1$ and $1 \leq q \leq \infty$, the real interpolation space $\bar{B}_{\theta,q} = (B_0, B_1)_{\theta,q}$ consists of all $a \in B_0 + B_1$ for which the norm

$$\|a\|_{\theta,q} = \begin{cases} \left(\sum_{m=-\infty}^{\infty} \left(2^{-\theta m} K(2^m, a)\right)^q\right)^{1/q} & \text{if } \ 1 \leq q < \infty, \\ \sup_{m\in\mathbb{Z}} \left\{2^{-\theta m} K(2^m, a)\right\} & \text{if } \quad q = \infty, \end{cases}$$

is finite.

As is known, if $1 \leq q < \infty$, $0 < \theta < 1$, then $B_0 \cap B_1$ is dense in $(B_0, B_1)_{\theta,q}$ and if B_i° denotes the closure in B_i of $B_0 \cap B_1$, then

$$(B_0, B_1)_{\theta,q} = (B_0^\circ, B_1)_{\theta,q} = (B_0, B_1^\circ)_{\theta,q} = (B_0^\circ, B_1^\circ)_{\theta,q}. \tag{4}$$

In the case of a space A belonging to a complete dual pair, property (4) implies $(A, A^*)_{\theta,q} = (A, A_1)_{\theta,q}$. We refer to [2], [4], [13] and [19] for further details on this construction. Because $A \cap A_1 = A \cap A^*$ is dense in A and in A_1, and because the arguments used in [4, Theorem 3.7.1] to describe the duals of the spaces obtained by the real interpolation method work as well for the anti-duals, we get that

$$(A, A^*)_{\theta,q}^* = (A^*, A_1^*)_{\theta,q'}, \text{ with equivalence of norms} \tag{5}$$

for $0 < \theta < 1$, $1 \leq q < \infty$ and $\dfrac{1}{q} + \dfrac{1}{q'} = 1$.

Let E, F be Banach spaces with $E \hookrightarrow F$. The *Gagliardo completion* of E with respect to F is defined as the collection $E_F^\sim$ of all those $a \in F$ for which there is a sequence $\{a_n\}$ bounded in E and converging to a in F. The norm of $E_F^\sim$ is given by

$$\|a\|_{E_F^\sim} = \inf\{\sup\{\|a_n\|_E\}\}$$

If (B_0, B_1) is a compatible couple of Banach spaces, we write $B_0^\sim$, $B_1^\sim$ for the Gagliardo completions of B_0, B_1, respectively, with respect to $B_0 + B_1$. It is then easy to show that for every $a \in B_0 + B_1 = B_0^\sim + B_1^\sim$, $t > 0$, we have

$$K(t,a;B_0,B_1) = K(t,a;B_0^\sim,B_1) = K(t,a;B_0,B_1^\sim) = K(t,a;B_0^\sim,B_1^\sim),$$

hence also

$$(B_0, B_1)_{\theta,q} = (B_0^{\sim}, B_1)_{\theta,q} = (B_0, B_1^{\sim})_{\theta,q} = (B_0^{\sim}, B_1^{\sim})_{\theta,q} \tag{6}$$

for all $\theta \in (0,1)$, $q \in [1,\infty]$ (see, for example [7]).

We can now state and prove our theorem for the real case. The criterion may seem a bit involved but, as we shall see, it is quite easy to verify in concrete cases.

Theorem 3.1. *Assume A belongs to a dual pair, let H be the associated Hilbert space and assume the following condition holds: For every $a \in A_1^* \cap A^*$ and for every $\epsilon > 0$ there exist $x \in A$, $y \in A^*$ such that $\|x\|_A \le \|a\|_{A_1^*}$, $\|y\|_{A^*} < \epsilon$ and $a = x + y$. Then*

$$(A, A^*)_{1/2,2} = H \quad \textit{(equivalence of norms).}$$

PROOF. Let E denote the closure of $A_1^* \cap A^*$ in A_1^* (so E is a closed subspace of A_1^*) and let $A^{\sim}$ be the Gagliardo completion of A (with respect to $A + A^*$). Then

$$E \hookrightarrow A^{\sim} \hookrightarrow A_1^* \tag{7}$$

each inclusion being of norm ≤ 1. Indeed, if $a \in A_1^* \cap A^*$, then our assumption provides us with sequences $\{x_n\}$ in A, $\{y_n\}$ in A^* such that $a = x_n + y_n$, $\sup_n \|x_n\|_A \le \|a\|_{A_1^*}$ and $\lim_{n\to\infty} \|y_n\|_{A^*} = 0$. It follows that $a \in A^{\sim}$, $\|a\|_{A^{\sim}} \le \|a\|_{A_1^*}$. By the definition of E, we proved $E \hookrightarrow A^{\sim}$ with norm ≤ 1. Now let $a \in A^{\sim}$ and let $\epsilon > 0$. There is a sequence $\{a_n\}$ in A converging to a in $A + A^*$ and such that

$$\sup_n \|a_n\|_A \le \|a\|_{A^{\sim}} + \epsilon.$$

Passing to a subsequence, we may assume $\{a_n\}$ converges to an element $\alpha \in A^{**}$ in the w^*-topology of A^{**}, where

$$\|\alpha\|_{A^{**}} \le \sup_n \|a_n\|_A \le \|a\|_{A^{\sim}} + \epsilon.$$

It follows that $a \in A_1^*$,

$$\|a\|_{A_1^*} \le \|\alpha\|_{A^{**}} \le \|a\|_{A^{\sim}} + \epsilon,$$

proving the second inclusion in (7) since $\epsilon > 0$ is arbitrary.

Set $B = (A, A^*)_{1/2,2} = (A, A_1)_{1/2,2}$. According to (5) the anti-dual space of B is given by

$$B^* = (A^*, A_1^*)_{1/2,2} = (A_1^*, A^*)_{1/2,2}.$$

Notice that $A \hookrightarrow A_1^*$, so

$$B = (A, A^*)_{1/2,2} \hookrightarrow (A_1^*, A^*)_{1/2,2} = B^*.$$

Conversely, by (4) (since E is the closure of $A_1^* \cap A^*$ in A_1^*) and by (7),

$$B^* = (A_1^*, A^*)_{1/2,2} = (E, A^*)_{1/2,2} \hookrightarrow (A^\sim, A^*)_{1/2,2} = (A, A^*)_{1/2,2}$$

the last equality being due to (6). We proved $B = B^*$, with equivalence of norms. The duality between B and B^* is given by the inner product of H, so that

$$\|c\|_H^2 = (c, c) \leq \|c\|_B \|c\|_{B^*} \leq M\|c\|_B^2;$$

holds for all $c \in A \cap A^* \hookrightarrow H \cap B$ and for some constant $M > 0$. Since $A \cap A^*$ is dense in B this proves $B \hookrightarrow H$. Thus $H \hookrightarrow B^*$, hence $B = H = B^*$. □

An important case in which the criterion of the theorem is satisfied is the case in which A is densely and continuously embedded in the Hilbert space H, so that we have the scheme $A \hookrightarrow H \hookrightarrow A^*$ and A_1 is the closure of A in A^*. As remarked before, A_1^* is then continuously embedded in H. We have

Lemma 3.2. *Assume A is densely and continuously embedded in H. Then the criterion of Theorem 3.1 holds; that is, if $h \in A_1^*$, then for every $\epsilon > 0$, there is $y \in A$, $z \in A^*$ such that $\|y\|_A \leq \|h\|_{A_1^*}$, $\|z\|_{A^*} < \epsilon$ and $h = y + z$.*

PROOF. Since h, as an element of A_1^*, is an A^*-continuous map on the subspace A of A^*, by the Hahn-Banach theorem we can extend it to an element $\psi \in A^{**}$ of the same norm; that is,

$$\psi(a) = (h, a) \qquad \text{for all } a \in A$$

and $\|\psi\|_{A^{**}} = \|h\|_{A_1^*}$. Recalling that the unit ball of A is weak* dense in the unit ball of A^{**}, there exists a net $\{a_\lambda\}_{\lambda \in \Lambda}$ of elements of A such that

$$\|a_\lambda\|_A \leq \|\psi\|_{A^{**}} = \|h\|_{A_1^*} \tag{8}$$

for all $\lambda \in \Lambda$, converging to ψ in the weak* topology of A^{**}. But the inclusion of A in H is continuous, so (8) implies the net $\{a_\lambda\}_{\lambda \in \Lambda}$ is bounded in H. It follows that $\{a_\lambda\}_{\lambda \in \Lambda}$ has a subnet $\{a_{\tau(\mu)}\}_{\mu \in M}$ which converges weakly in H. Here M is a directed set and $\tau : M \to \Lambda$ is a mapping such that for every $\lambda \in \Lambda$ there exists $\mu \in M$ with the property that $\tau(\nu) \geq \lambda$ for all $\nu \geq \mu$. The weak limit of this subnet must be h. In fact, since (a_λ, a) is the effect of the element $a \in A \subset A^*$ on the element $a_\lambda \in A$ and ψ coincides with $(h, \cdot)$ on A, we have for $a \in A$,

$$\psi(a) = \lim_{\lambda \in \Lambda} (a_\lambda, a) = \lim_{\mu \in M} (a_{\tau(\mu)}, a) = (h, a).$$

Since A is dense in H, we established that h is the weak limit of the subnet in H. Now let S be the set of all finite convex combinations of the elements $a_{\tau(\mu)}$ with $\mu \in M$; i.e., $a \in S$ if and only if there exists a finite subset $\{\mu_1, \dots, \mu_k\}$ of M, real numbers $c_1, \dots, c_k \in [0, \infty)$ such that $\sum_{j=1}^k c_j = 1$ such that

$$a = \sum_{j=1}^{k} c_j a_{\tau(\mu_j)}.$$

Then $S \subset A$ and, in fact, $\|a\|_A \le \|h\|_{A_1^*}$ for all $a \in S$. Let S_H be the closure of S in the strong topology of H. Then S_H is a strongly closed convex subset of H, as such, it is also weakly closed, in particular h, which is in the weak closure of S, is in S_H. Since H is continuously embedded in A^*, h is also in the strong closure of S in A^*. It follows that for every $\epsilon > 0$ we can find $y \in S$ such that $\|h - y\|_{A^*} < \epsilon$. Then $\|y\|_A \le \|h\|_{A_1^*}$ because $y \in S$ and therefore the elements y and $z = h - y$ have all the desired properties. □

As an immediate corollary we get (1) without the reflexivity assumption (see also [15, Theorem 11], [22, Theorem 4.2 and Remark], [1, Theorem 2.8.5]).

Corollary 3.3. *Let A be a Banach space, H a Hilbert space, such that A is continuously and densely embedded in H. We have, with equivalence of norms,*

$$(A, A^*)_{1/2,2} = H.$$

Remark. Assume A is densely and continuously embedded in H. Then $A_1^* \hookrightarrow H \hookrightarrow A^*$ and A_1^* coincides with the Gagliardo completion of A with respect to A^*, in view of (7). Another proof of the equality $A_1^* = A^\sim$ can be found in [1].

As a corollary of the last result, we get a somewhat different criterion for the validity of (1) in the general case.

Corollary 3.4. *Let A be a Banach space belonging to some complete dual pair and let H be the associated Hilbert space. Assume the space $A + A^*$ is a closed subspace of $A_1^* + A^*$. Then*

$$(A, A^*)_{1/2,2} = H \quad \textit{(equivalence of norms).}$$

PROOF. We have the scheme

$$A \cap A^* = A \cap A_1 \hookrightarrow H \hookrightarrow A + A^* \hookrightarrow A_1^* + A^*.$$

We get

$$H = (A \cap A^*, A^* + A_1^*)_{1/2,2} = (A \cap A^*, A + A^*)_{1/2,2},$$

the first equality being due to Corollary 3.3, because $A^* + A_1^*$ is the anti-dual of $A \cap A^* = A \cap A_1$ and $A \cap A^*$ is a dense subspace of H; the second equality follows from the assumption that $A + A^*$ is a closed subspace of $A_1^* + A^*$. Next, using Corollary 1 of [14], with $\theta = 1/2$, we conclude that

$$(A \cap A^*, A + A^*)_{1/2,2} = (A, A^*)_{1/2,2}$$

and the result follows. □

Observe that if $A \hookrightarrow H$ with dense range, then the assumption of the last corollary is satisfied, namely: $A + A^* = A^* = A_1^* + A^*$. But verifying that $A + A^*$ is closed in $A_1^* + A^*$ can be quite difficult in more complex situations. The criterion of Theorem 3.1, while less elegant, is usually easier to verify.

4. The complex case

We begin recalling the definition of the complex interpolation spaces $(B_0, B_1)_{[\theta]}$ and $(B_0, B_1)^{[\theta]}$ where $\theta \in (0,1)$ and $\bar{B} = (B_0, B_1)$ is once again a compatible couple of Banach spaces. Let S be the strip in the complex plane consisting of all complex numbers z such that $0 \leq \Re z \leq 1$ and denote by S^0 its interior $0 < \Re z < 1$. We say that a function $f : S \to B_0 + B_1$ is in the Calderón space $\mathcal{F}(\bar{B})$ if $f : S \to B_0 + B_1$ is continuous and bounded, the restriction of f to S^0 is a $B_0 + B_1$-valued analytic function, $f(it) \in B_0$, $f(1 + it) \in B_1$ for all $t \in \mathbb{R}$ and the maps

$$t \mapsto f(it) : \mathbb{R} \to B_0,\ t \mapsto f(1 + it) : \mathbb{R} \to B_1,$$

are continuous and bounded in the topologies of B_0 and of B_1, respectively. A norm is defined on $\mathcal{F}(\bar{B})$ by

$$\|f\|_{\mathcal{F}(\bar{B})} = \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{B_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{B_1}\right\}$$

with respect to which $\mathcal{F}(\bar{B})$ becomes a Banach space. The interpolation space $\bar{B}_{[\theta]} = (B_0, B_1)_{[\theta]}$ is defined for $0 \leq \theta \leq 1$ by

$$\bar{B}_{[\theta]} = \{f(\theta)\ :\ f \in \mathcal{F}(\bar{B})\}$$

and made into a Banach space defining the norm of $a \in \bar{B}_{[\theta]}$ by

$$\|a\|_{[\theta]} = \|a\|_{\bar{B}_{[\theta]}} = \inf\{\|f\|_{\mathcal{F}(\bar{B})}\ :\ f \in \mathcal{F}(\bar{B}), f(\theta) = a\}.$$

We shall also need the space $\mathcal{F}_0(\bar{B}) = \mathcal{F}_0(B_0, B_1)$. We shall say that a function $f : S \to B_0 + B_1$ is in $\mathcal{F}_0(\bar{B})$ if it can be expressed as a finite sum

$$f(z) = e^{\delta z^2} \sum_{k=1}^{n} e^{\lambda_k z} a_k$$

where $\delta > 0$, $\lambda_k \in \mathbb{R}$, $a_k \in B_0 \cap B_1$ for $k = 1, \dots, n$. It is an important result of the theory that $\mathcal{F}_0(\bar{B})$ is a dense subspace of the Banach space $\mathcal{F}(\bar{B})$. One consequence of this density is that $B_0 \cap B_1$ is dense in $\bar{B}_{[\theta]}$ for all $\theta \in (0,1)$ and that we have the following analogue of (4)

$$(9) \qquad (B_0, B_1)_{[\theta]} = (B_0^\circ, B_1)_{[\theta]} = (B_0, B_1^\circ)_{[\theta]} = (B_0^\circ, B_1^\circ)_{[\theta]};$$

in particular $(A, A^*)_{[\theta]} = (A, A_1)_{[\theta]}$ in case A belongs to a complete dual pair.

The space $\mathcal{G}(\bar{B})$ consists of all continuous functions $f : S \to B_0 + B_1$ whose restriction to the interior S^0 of the strip is a B_0+B_1-valued analytic function, such that $|f(z)| \le c|z|+d$ for all $z \in S$, some $c, d \ge 0$ and such that $f(it_1)-f(it_2) \in B_0$, $f(1+it_1) - f(1+it_2) \in B_1$ for all $t_1, t_2 \in \mathbb{R}$ and

$$\|f\|_{\mathcal{G}(\bar{B})} = \max_{j=0,1} \left\{ \sup_{t_1, t_2 \in \mathbb{R}, t_1 \ne t_2} \frac{1}{|t_1 - t_2|} \|f(j+it_1) - f(j+it_2)\|_{B_j} \right\} < \infty.$$

With $\|f\|_{\mathcal{G}(\bar{B})}$ as defined, $\mathcal{G}(\bar{B})$, reduced modulo constant functions, becomes a Banach space. For $0 \le \theta \le 1$, the interpolation space $\bar{B}^{[\theta]}$ is given by

$$\bar{B}^{[\theta]} = \{f'(\theta) \, : \, f \in \mathcal{G}(\bar{B})\}$$

and made into a Banach space defining the norm of $a \in \bar{B}^{[\theta]}$ by

$$\|a\|^{[\theta]} = \|a\|_{\bar{B}^{[\theta]}} = \inf\{\|f\|_{\mathcal{G}} \, : \, f \in \mathcal{G}(\bar{B}), f'(\theta) = a\}.$$

We refer to [4], [5], and [19] for further details on the complex interpolation method. We shall need the following duality result, which is a direct consequence of the results proved in [5].

Lemma 4.1. *Let $0 < \theta < 1$. Let A belong to a complete dual pair (D_0, D_1) with associated Hilbert space H. The anti-dual of $(A, A^*)_{[\theta]}$ is given by $(A^*, A_1^*)^{[\theta]}$, the duality being defined as follows: Let $a \in (A, A^*)_{[\theta]}$, $b \in (A^*, A_1^*)^{[\theta]}$. Let $f \in \mathcal{F}(A, A^*)$, $g \in \mathcal{G}(A^*, A_1^*)$ be such that $f(\theta) = a$, $g'(\theta) = b$. Then $b(a) = \overline{(a, b)}$ where*

$$(a, b) = \sum_{j=0}^{1} \int_{-\infty}^{\infty} \mu_j(\theta, t)(f(j+it), dg(j-it)),$$

where μ_0, μ_1 denote the Poisson kernels for the strip S. This "new" value of (a,b) coincides with the "old" value wherever the old value is defined; i.e., it coincides with the inner product of H if $a, b \in H$ and with $\overline{b(a)}$ if $a \in A$, $b \in A^*$

PROOF. For the purpose of this proof, we write X' to denote the regular (linear) dual of a Banach space X. The map $J : A^* \to A'$, defined by $(Jb)(a) = \overline{b(a)} = (a,b)$ is an anti-linear isometric isomorphism of A^* onto A' and allows us to identify A^* and A'. If $g \in \mathcal{G}(A^*, A_1^*)$, we define $\tilde{J}g : S \to A'$ by $\tilde{J}g(z) = J(g(\bar{z}))$. We see that $\tilde{J}$ is an antilinear, isometric map of $\mathcal{G}(A^*, A_1^*)$ onto $\mathcal{G}(A', A_1')$ such that $(\tilde{J}g)'(z) = J(g'(\bar{z}))$, in particular $(\tilde{J}g)'(\theta) = J(g'(\theta))$ and we see that the map J is also an isometric isomorphism from $(A^*, A_1^*)^{[\theta]}$ onto $(A', A_1')^{[\theta]}$. The lemma is now an immediate consequence of the duality results stated and proved in [5, Sections 12.1, 32.1]. □

We are ready for the complex interpolation result.

Theorem 4.2. *Let A be a Banach space belonging to some complete dual pair and let H be the associated Hilbert space. Assume either that $A_1 = A^*$ or that*

$$A_1^* \cap A^* = A \cap A^*$$

and

$$\|a\|_A = \sup\{|b(a)| \,:\, b \in A_1, \|b\|_{A^*} \leq 1\} \tag{10}$$

holds for all $a \in A$. Then

$$(A, A^*)_{[1/2]} = H \quad \textit{with equality of norms}$$

Remark. Condition (10) is equivalent to saying that the norms of A and of A_1^* coincide on $A \cap A^*$. If we have $A_1 = A^*$ so that $A_1^* = A^{**}$, then (10) holds automatically for all $a \in A$.

PROOF. Set $B = (A, A^*)_{[1/2]}$. First of all we observe that the bilinear interpolation theorem yields that $B \hookrightarrow H$ with

$$(a,a)^{1/2} = \|a\|_H \leq \|a\|_B$$

for all $a \in B$. Since $A \cap A^*$ is dense in B and in H, in order to establish the result it suffices to show that

$$\|a\|_B \leq \|a\|_H \tag{11}$$

for every $a \in A \cap A^*$. Assume first $A_1^* \cap A^* = A \cap A^*$ and (10) hold. Then $\mathcal{F}_0(A, A^*) = \mathcal{F}_0(A_1^*, A^*)$. Due to (10) (see also the remark preceding this proof), the restriction of the norm of $\mathcal{F}(A, A^*)$ to its dense subspace $\mathcal{F}_0(A, A^*)$ coincides with the restriction of the norm of $\mathcal{F}(A_1^*, A^*)$ to its dense subspace $\mathcal{F}_0(A_1^*, A^*)$; it follows that $\mathcal{F}(A, A^*) = \mathcal{F}(A_1^*, A^*)$, with equal norms, implying that every complex interpolate of the pair (A, A^*) coincides (with equal norms) with the corresponding interpolate of (A_1^*, A^*). Thus $B = (A_1^*, A^*)_{[1/2]}$ and since, by Lemma 4.1, $B^* = (A_1^*, A^*)^{[1/2]}$, we get that B is a closed subspace of B^*, with the same norm as a consequence of Bergh's theorem (see [3]). Given $a \in A \cap A^*$ we then have

$$\begin{aligned}\|a\|_B &= \|a\|_{B^*} = \sup\{|(c,a)| : c \in B, \|c\|_B \le 1\} \\ &\le \sup\{(c,c)^{1/2}(a,a)^{1/2} : c \in B, \|c\|_B \le 1\} \le \|a\|_H\end{aligned}$$

(since $\|c\|_H \le \|c\|_B$). This establishes (11).

Assume now $A_1 = A^*$, so $A_1^* = A^{**}$. Then $A \cap A^*$ is not only dense in A but also in A^*; (A^*, A^{**}) is a compatible couple and $B^* = (A^*, A^{**})^{[1/2]} = (A^{**}, A^*)^{[1/2]}$. Moreover $A^* \cap A^{**} \hookrightarrow H$. In fact, by the definition of belonging to a dual pair, $A \cap A^* \subset H \subset A + A^*$, since $A \cap A^*$ is dense in A, A^*, hence also in $A + A^*$, we see that H is dense in $A + A^*$ so that taking duals in $H \subset A + A^*$ we get $H = H^* \supset (A + A^*)^* = A^* \cap A^{**}$. Because of this, the mapping

$$(a, b_0 + b_1) \mapsto a(b_0) + \overline{b_1(a)} : (A^{**} \cap A^*) \times (A^* + A^{**}) \to \mathbb{C}$$

is a well defined mapping of norm ≤ 1. By the bilinear interpolation theorem of [5, Sections 11.2] or [4, Theorem 4.4.2] it has a unique extension to a sesquilinear mapping

$$L : (A^{**}, A^*)_{[1/2]} \times (A^*, A^{**})^{[1/2]} = (A^{**}, A^*)_{[1/2]} \times B^* \to \mathbb{C}$$

of norm ≤ 1. Because $A \hookrightarrow A^{**}$ is of norm 1, we see that $B = (A, A^*)_{[1/2]} \hookrightarrow (A^{**}, A^*)_{[1/2]}$ is of norm ≤ 1 and L extends the duality $B \times B^* \to \mathbb{C}$; that is, $\overline{b(a)} = L(a, b)$ for $a \in B$, $b \in B^*$. Now let $a \in B$. There is $f \in B^*$ such that $\|f\|_{B^*} = 1$ and $f(a) = \|a\|_B$. Thus,

$$\|a\|_B \le |f(a)| = |L(a,f)| \le \|a\|_{(A^{**},A^*)_{[1/2]}} \|f\|_{(A^*,A^{**})^{[1/2]}} = \|a\|_{(A^{**},A^*)_{[1/2]}}.$$

Applying again Bergh's theorem, we proved $\|a\|_B \le \|a\|_{B^*}$ and B is once more a closed subspace of B^*, with the same norm. We now proceed as in the previous case to obtain (11) and complete the proof of the theorem. □

Other sufficient conditions for the validity of (2) can be be found in [21].

The proof of the preceding theorem is, in essence, not too different from the proof given by Pisier of the same result assuming that the Hilbert space H is densely and continuously embedded in A (cf. [18] and also [20]). We can recover it as a direct consequence of Theorem 4.2.

Corollary 4.3. *Assume the Hilbert space H is densely and continuously embedded in the Banach space A. Then $(A, A^*)_{[1/2]} = H$.*

As Watbled [20] has shown, one can use the result of the corollary to settle the case in which $A \hookrightarrow H$. Her proof is based on the equality of the upper and lower complex interpolation methods when applied to a dual couple (B_0^*, B_1^*) such that $(B_0^*, B_1^*)_{[\theta]}$ is reflexive (see [20, Lemme 1]). It is also possible to base the proof on the following result (of independent interest).

Lemma 4.4. *Let $\bar{B} = (B_0, B_1)$ be a compatible pair of Banach spaces such that $B_0 \hookrightarrow B_1$ (continuous inclusion). Assume that for some $\theta \in (0,1)$ the space $\bar{B}_{[\theta]}$ is reflexive. Then $\bar{B}^{[\eta]} = \bar{B}_{[\eta]}$ for all $\eta \in (0,1)$.*

PROOF. It suffices to prove $\bar{B}^{[\theta]} = \bar{B}_{[\theta]}$ since reflexivity of $\bar{B}_{[\theta]}$ and reiteration imply (via Calderón's theorem on reflexivity of the spaces obtained by the complex method, cf. [5, 12.2]) the reflexivity of $\bar{B}_{[\eta]}$ for all $\eta \in (0,1)$. Let $a \in \bar{B}^{[\theta]}$; there exists $g \in \mathcal{G}(\bar{B})$ such that $g'(\theta) = a$. By substracting a constant element in B_1, we may assume $g(it) \in B_0$ for all $t \in \mathbb{R}$, then $t \mapsto g(j+it)$ are continuous (Lipschitz) maps from $\mathbb{R}$ to B_j, $j = 0,1$. For $\epsilon > 0$ set

$$g_\epsilon(z) = \int_{-\infty}^{\infty} \psi_\epsilon(\tau) g(z - i\tau)\, d\tau = \int_{-\infty}^{\infty} \psi_\epsilon(t-\tau) g(s + i\tau)\, d\tau$$

for $z = s + it \in S$, where $\psi_\epsilon(\tau) = \epsilon^{-1}\psi(\frac{\tau}{\epsilon})$ and ψ is a C^∞, non-negative function on $\mathbb{R}$, supported by the interval $[-1,1]$ and such that

$$\int_{-\infty}^{\infty} \psi(\tau)\, d\tau = 1.$$

Then $g_\epsilon : S \to B_1$ is continuous, with analytic restriction to S^0. Moreover, the restriction of g_ϵ to $i\mathbb{R}$ (respectively, $1 + i\mathbb{R}$) is an infinitely differentiable map with values in B_0 (respectively, B_1). Moreover, we have for $z = s + it \in S^0$,

$$g_\epsilon'(z) = \int_{-\infty}^{\infty} \frac{d\psi_\epsilon}{d\tau}(\tau) g(z - i\tau)\, d\tau = \int_{-\infty}^{\infty} \psi_\epsilon(\tau) g'(z - i\tau)\, d\tau.$$

The first equality above can be used to extend g_ϵ' to the boundary of the strip S. It is then easy to see that $g_\epsilon' \in \mathcal{F}(\bar{B})$ and

$$\|g_\epsilon'\|_{\mathcal{F}(\bar{B})} \leq \|\psi_\epsilon\|_{L^1} \|g\|_{\mathcal{G}(\bar{B})} = \|g\|_{\mathcal{G}(\bar{B})}$$

for all $\epsilon > 0$. It follows that

$$a_\epsilon = g'_\epsilon(\theta) \in \bar{B}_{[\theta]}$$

and

$$\|a_\epsilon\|_{\bar{B}_{[\theta]}} \leq \|g\|_{\mathcal{G}(\bar{B})}$$

for all $\epsilon > 0$ and the family $\{a_\epsilon\}_{\epsilon>0}$ is bounded in $\bar{B}_{[\theta]}$. Since $\bar{B}_{[\theta]}$ is reflexive there is a sequence $\{\epsilon_n\}$ converging to 0 such that $\{a_{\epsilon_n}\}$ converges weakly to some $\tilde{a} \in \bar{B}_{[\theta]}$ as $n \to \infty$. On the other hand, because $t \mapsto g'_\epsilon(\theta + it) : \mathbb{R} \to B_1$ is continuous, we see that

$$\lim_{\epsilon\to 0} a_\epsilon = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \psi_\epsilon(\tau) g'(\theta - i\tau)\, d\tau = g'(\theta) = a$$

in B_1. By compatibility we must have $a = \tilde{a}$, proving that $a \in \bar{B}_{[\theta]}$. □

Remark. Other sufficient conditions for the equality $\bar{B}_{[\theta]} = \bar{B}^{[\theta]}$ to hold can be found in the paper by Haagerup and Pisier [10]. We remark, however, that the spirit of these results is quite different.

For the sake of completeness, we now state Watbled's result.

Corollary 4.5. *Let A be continuously and densely embedded in the Hilbert space H. Then*

$$(A, A^*)_{[1/2]} = H$$

with equal norms.

Concerning this last result, let us point out that it can also be proved following the lines of Corollary 3.3 (the corresponding real case), using Lemma 3.2 which (as mentioned above) has as a consequence that A_1^* is the Gagliardo completion $A^\sim$ of A when A is densely and continuously embedded in H. In fact, by the result of Cwikel and Sharif [8] on complex interpolation between Gagliardo completions, we obtain

$$B^* = (A_1^*, A^*)_{[1/2]} = (A^\sim, A^*)_{[1/2]} = (A, A^*)_{[1/2]} = B,$$

which implies $B = H$ with equality of norms.

Example 4.6. The simplest illustration of these theorems is provided by the scheme

$$l^1 \hookrightarrow l^2 \hookrightarrow l^\infty.$$

Writing down Corollary 3.3 and Corollary 4.5 for this example, we recover the well known formulae

$$(l^1, l^\infty)_{1/2,2} = l^2 \qquad \text{(equivalence of norms)}$$
$$(l^1, l^\infty)_{[1/2]} = l^2 \qquad \text{(equality of norms).}$$

In this example we have for $\theta \in (0,1)$ and $\frac{1}{p} = 1 - \theta$

$$(l^1, l^\infty)_{\theta,p} = l^p = (l^1, l^\infty)_{[\theta]}.$$

It is natural to wonder whether in our abstract setting there is still any relationship between $(A, A^*)_{\theta,p}$ and $(A, A^*)_{[\theta]}$ besides the well known one

$$(\dagger\dagger) \qquad (A, A^*)_{\theta,1} \hookrightarrow (A, A^*)_{[\theta]} \hookrightarrow (A, A^*)_{\theta,\infty}.$$

In fact, the theorems of Section 2 and those of this section allow us to improve the relationship $(\dagger\dagger)$. Assume $A \hookrightarrow H$ densely. According to the reiteration theorem and Corollary 3.3, we get for $1 < p < 2$ and $\frac{1}{p} = 1 - \theta$

$$(A, A^*)_{\theta,p} = \big(A, (A, A^*)_{1/2,2}\big)_{2\theta,p} = (A, H)_{2\theta,p}$$

and similar formulae hold for the complex method. Since the Hilbert space H has Fourier type 2 (see [16] for details on this notion), we can use the argument given in [6, Theorem 4.1], where it is applied to spaces of multilinear forms, and derive

Corollary 4.7. *Let A be continuously and densely embedded in the Hilbert space H, let $0 < \theta < 1$, $\frac{1}{p} = 1 - \theta$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then*

$$(A, A^*)_{\theta,p} \hookrightarrow (A, A^*)_{[\theta]} \hookrightarrow (A, A^*)_{\theta,p'} \qquad \text{if } 1 < p < 2,$$
$$(A, A^*)_{\theta,p'} \hookrightarrow (A, A^*)_{[\theta]} \hookrightarrow (A, A^*)_{\theta,p} \qquad \text{if } 2 < p < \infty.$$

Remark. In the proof indicated above no information is required about A except that it has the trivial Fourier type 1. This is just the case of the space A in Example 4.6: l^1 has Fourier type 1 and it is not of Fourier type q for any $q > 1$. If the space A has a non-trivial Fourier type, say q where $1 < q \leq 2$ and if A is not a closed subspace of A^* (otherwise all interpolation spaces coincide with A), then all inclusions in Corollary 4.7 are strict.

Indeed, let $1 < p < 2$, $\frac{1}{p} = 1 - \theta$; put $\eta = 2\theta$ and define r by $\frac{1}{r} = \frac{1-\eta}{q} + \frac{\eta}{2}$. Then $1 < p < r \leq 2 \leq r' < p' < \infty$, so that $(A, A^*)_{\theta,p} \hookrightarrow (A, A^*)_{\theta,r}$ while $(A, A^*)_{\theta,r'} \hookrightarrow (A, A^*)_{\theta,p'}$, and these inclusions are strict by [11, Theorem 3.1]. Using now the reiteration theorem and Peetre's theorem [16, Theorem 3.1] on the relationship between real and complex interpolation spaces generated by a couple of spaces with a certain Fourier type, we conclude

$$(A, A^*)_{\theta,p} \underset{\neq}{\hookrightarrow} (A, A^*)_{\theta,r} \hookrightarrow (A, A^*)_{[\theta]} \hookrightarrow (A, A^*)_{\theta,r'} \underset{\neq}{\hookrightarrow} (A, A^*)_{\theta,p'}.$$

In the argument above, the case $2 < p < \infty$ can be treated similarly recalling that if A has Fourier type q then so does A^* (see [16, Theorem 2.2] for a proof in the case when A is reflexive, [12, p.222] for the general case).

We close this section with another property of the interpolates $(A, A^*)_{\theta,p}$, $(A, A^*)_{[\theta]}$ which is generally valid, not only when $\theta = \frac{1}{2} = \frac{1}{p}$.

Corollary 4.8. *Let A be continuously and densely included in the Hilbert space H. The spaces $(A, A^*)_{\theta,p}$ and $(A, A^*)_{[\theta]}$ are reflexive for all values of $\theta \in (0,1)$, $p \in (1, \infty)$.*

PROOF. The inclusion $A = A \cap A^* \hookrightarrow A + A^* = A^*$ factors through the Hilbert space H, thus is weakly compact. Reflexivity of $(A, A^*)_{\theta,p}$ follows now from [2, Proposition 2.2.3].

In the complex case, if $0 < \theta < 1/2$, combining the reiteration theorem and Corollary 4.5, we obtain

$$(A, A^*)_{[\theta]} = \big(A, (A, A^*)_{[1/2]}\big)_{[2\theta]} = (A, H)_{[2\theta]}$$

therefore the reflexivity of H implies, via [5, 12.2], the reflexivity of $(A, A^*)_{[\theta]}$. A similar argument works for $1/2 < \theta < 1$. □

5. APPLICATIONS

Let K be a Hilbert space and let $\mathcal{L}$ be the collection of all bounded linear operators of K. For $1 \leq p \leq \infty$ let S_p be the family of all compact operators T of K such that

$$\|T\|_p = \left(\sum_{n=1}^{\infty} s_n(T)^p\right)^{1/p} < \infty,$$

where

$$s_n(T) = \inf\{\|T - R_n\| \ : \ R_n \in \mathcal{L} \text{ with rank } R_n < n\}.$$

The space S_p is called the Schatten-von Neumann p-class and is an important example of a symmetrically normed ideal (see [9]). As is well known, the spaces S_1, S_2 and S_∞ coincide with the spaces of nuclear, Hilbert-Schmidt and compact operators, respectively. In particular, S_2 is a Hilbert space and S_1 is densely embedded in S_2. For an operator $T \in S_1$, one defines the *spectral trace* of T by $\mathrm{sp}(T) = \sum_{n=1}^{\infty}(T\phi_n, \phi_n)_K$, where $(\phi_n)_{n=1}^{\infty}$ is a complete orthonormal system in K. The inner product of S_2 is then given by $(T_1, T_2) = \mathrm{sp}(T_2^* T_1)$ and, with respect to this inner product, $S_1^* = \mathcal{L}$. Writing down Corollary 3.3 and Corollary 4.5 for the case $A = S_1$ and $H = S_2$, we get the classical formulae

$$(S_1, \mathcal{L})_{1/2,2} = S_2 = (S_1, \mathcal{L})_{[1/2]}.$$

The results of Sections 2 and 3 can also be used to analyze other families of symmetrically-normed ideals and to derive new information on them. Let $\pi = \{\pi_n\}_{n=1}^{\infty}$ be an arbitrary non-increasing sequence of positive numbers such that $\pi_1 = 1$,

$$\sum_{n=1}^{\infty} \pi_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \pi_n = 0.$$

Consider the operator spaces defined by

$$\begin{aligned} S_\Pi &= \left\{ T \in S_\infty : \|T\|_\Pi = \sup_n \frac{\sum_{j=1}^n s_j(T)}{\sum_{j=1}^n \pi_j} < \infty \right\}, \\ S_\Pi^0 &= \left\{ T \in S_\Pi : \lim_{n\to\infty} \frac{\sum_{j=1}^n s_j(T)}{\sum_{j=1}^n \pi_j} = 0 \right\}, \\ S_\pi &= \left\{ T \in S_\infty : \|T\|_\pi = \sum_{n=1}^{\infty} \pi_n s_n(T) < \infty \right\}. \end{aligned}$$

It can be checked that S_Π^0 is the closure in S_Π of the space of finite rank operators. Moreover, if $T \in S_\Pi$, $R \in S_\pi$, then $RT, TR \in S_1$ and it makes sense to define

$$(R, T) = \overline{(T, R)} = \mathrm{sp}(T^* R),$$

putting S_Π and S_π into duality with respect to which $\left(S_\Pi^0\right)^* = S_\pi$, $S_\pi^* = S_\Pi$ (cf. [9, Theorem 3.15.2]). Note that S_Π^0, S_2 and S_π *are not necessarily related by inclusions.* However, our previous results apply to the effect that

$$(S_\Pi^0, S_\pi)_{1/2,2} = S_2 = (S_\Pi^0, S_\pi)_{[1/2]}. \tag{12}$$

In fact, we are taking here $A = S_\Pi^0$, hence $A^* = S_\pi$ and (because finite rank operators are dense in S_π and contained in S_Π^0) $A_1 = S_\pi = A^*$. The complex interpolation follows from Theorem 4.2. For the real interpolation formula we verify that the conditions of Theorem 3.1 hold. Let $T \in A_1^* \cap A^* = S_\Pi \cap S\pi$ and let $\epsilon > 0$. From $\|T\|_\pi = \sum_{n=1}^\infty \pi_n s_n(T) < \infty$, there exists N such that

$$\sum_{n=1}^{\infty} \pi_n s_{N+n}(T) < \epsilon.$$

Let $T = \sum_{n=1}^\infty s_n(T)(\cdot, \phi_n)_K \psi_n$ be the Schmidt series of T, then setting $T_1 = \sum_{n=1}^N s_n(T)(\cdot, \phi_n)_K \psi_n$, $T_2 = \sum_{n=N+1}^\infty s_n(T)(\cdot, \phi_n)_K \psi_n$, we see that $T = T_1 + T_2$, $T_1 \in S_\Pi^0$, $\|T_1\|_\Pi \leq \|T\|_\Pi$ and $\|T_2\|_\pi = \sum_{n=1}^\infty \pi_n s_{N+n}(T) < \epsilon$.

Combining (7) and (12) it is not hard to derive now that we also have

$$(S_\Pi, S_\pi)_{1/2,2} = S_2 = (S_\Pi, S_\pi)_{[1/2]}.$$

In the special case $\pi_n = \dfrac{1}{2n-1}$ the resulting spaces play an important role for treating certain problems of perturbation theory and of invariant subspaces (see [9] for precise references).

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