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A NEW EXAMPLE OF HIGHER ORDER ALMOST FLAT AFFINE CONNECTIONS ON THE THREE-DIMENSIONAL SPHERE

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ABSTRACT. We give a new example of an almost flat affine connection on the three-dimensional sphere which satisfies not only $R \approx 0$ but also the higher order condition $\nabla^k R \approx 0$ $(k = 1,\ 2)$. This example is obtained by modifying the connection which we constructed previously in [2]. We also show that the three-dimensional Brieskorn manifold $M(p, q, r)$ is almost affinely flat, by giving a new left invariant affine connection on the Lie group $SL(2, \mathbf{R})$.

1. INTRODUCTION

In affine differential geometry, it is one of the fundamental problem whether there exists a torsion free flat affine connection on a given manifold M, and concerning this problem there are several existence or non-existence results such as [3], [4], [5], [10], etc. Related to this problem, in the previous paper [2], we gave an example of "almost flat" affine connection on the three-dimensional sphere S^3, i.e., we showed that for any positive number ε, there exists a torsion free affine connection ∇ on S^3 such that the norm of the curvature satisfies the inequality $\|R\| < \varepsilon$ at every point of S^3. (This property does not depend on the choice of the norm $\|\ \|$. For the precise definition, see section 2.) This example shows that the well-known result of Auslander-Markus [4, p.145] on the non-existence of torsion free flat affine connections on compact connected manifolds with finite fundamental group is a subtle result for the case S^3. In addition, from this example, we know the striking difference between "Riemannian" and "affine" category in considering the concept "almost flatness".

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In fact, in the Riemannian category, the three-dimensional sphere S^3 cannot be almost flat in the sense of Gromov because it is not covered by a nilpotent Lie group (cf. [6], [7], [12]). But in the affine category, S^3 is almost flat as we showed in [2]. In general, a manifold which is almost flat in the sense of Gromov is necessarily almost affinely 0-flat in our sense, but the converse is not true as the example S^3 in [2] shows. (For details, see the explanation at the end of section 2.) And this difference indicates that it is worth studying almost flat manifolds not only in the Riemannian but also in the affine category.

Once we found such an example, it is natural to consider the "higher order almost flatness" as a next problem. In the previous paper [2], we used only pointwise value of the curvature of ∇ in defining the concept "almost affinely flatness". But, if we take covariant derivatives of the curvature into consideration, the example in [2] is "not" almost affinely flat because ∇R is not almost zero. (See section 4 (A). Note that the almost flat condition $\|R\| \approx 0$ does not in general imply $\|\nabla^k R\| \approx 0$ for $k \geq 1$.) Hence, it is natural and interesting to ask whether there exists a torsion free affine connection on S^3 such that the norms $\|\nabla^k R\|$ $(k \geq 0)$ are simultaneously almost zero. In this paper, concerning this question, we give a new example of almost affinely flat connection on S^3 such that $\|\nabla^k R\|$ is almost zero for $k = 0 \sim 2$, by modifying the example in [2]. At present, it is an open question whether there exists an almost affinely flat connection on S^3 such that $\|\nabla^k R\| \approx 0$ for all $k \geq 0$, or conversely, there is an obstruction to the existence of such a connection at some order k. But from the example in this paper, we know that if the latter is the case, the obstruction must be related to the derivative of the curvature tensor of order at least three.

2. Main theorem

We first give the exact definition of almost flatness which we use in this paper, and explain the relation between two types of almost flatness in affine and Riemannian categories.

Definition. A Riemannian manifold (M, g) is called *almost affinely k-flat* if for any real positive number ε, there exists a torsion free affine connection ∇ on M such that

$$\|R\| < \varepsilon, \quad \|\nabla R\| < \varepsilon, \quad \cdots \quad, \quad \|\nabla^k R\| < \varepsilon$$

at every point of M, where R is the curvature of ∇, $\nabla^i R$ is the i-th covariant derivative of R, and $\| \ \|$ is the norm defined by the Riemannian metric g.

It is easy to see that this concept does not depend on the choice of the Riemannian metric on M if M is compact (cf. [2]), and in the following we simply say that M is *almost affinely k-flat* in this situation.

Clearly, the notion "almost affinely flat" introduced in [2] corresponds to the case $k = 0$, and we showed there that S^3 is almost affinely 0-flat. Now, our main result of this paper is stated as follows.

Theorem. *The three-dimensional sphere S^3 is almost affinely 2-flat.*

Before the proof, we briefly explain the difference and also a relation between two types of "almost flatness" (in the sense of Gromov and ours). The definition of Gromov's almost flat manifold belongs essentially to the Riemannian category, and is stated as follows: A Riemannian manifold (M, g) is called *ε-flat* if it satisfies the inequality $d(M)^2 |K| < \varepsilon$, where $d(M)$ is the diameter of M, and K is the sectional curvature. A manifold M is *almost flat* if for any $\varepsilon > 0$, there exists a Riemannian metric g on M such that (M, g) is ε-flat. And the fundamental theorem of Gromov and Ruh says that there exists a positive number ε_n depending only on the dimension n of M such that if (M, g) is ε_n-flat, then M is covered by a nilpotent Lie group (cf. [6], [7], [12]).

On the contrary, in the affine category, the concepts "diameter" nor "sectional curvature" do not exist, and as the above definition shows, the concept "almost flatness" must be different from that of Riemannian case. These two distinct definitions are related in the following way: For a Riemannian manifold (M, g), the Riemannian connection ∇ determined by g is torsion free, and hence it defines the affine structure on M. From this viewpoint, if M is almost flat in the sense of Gromov, then M is almost affinely 0-flat in our sense. In fact, if a Riemannian manifold (M, g) satisfies the condition $d(M)^2 |K| < \varepsilon$ for a given $\varepsilon > 0$, then we can uniquely normalize the metric such that $d(M) = 1$, and hence we have $|K| < \varepsilon$. (Note that the Riemannian connection ∇ and the curvature tensor $R^i{}_{jkl}$ are unchanged by this modification though the tensor R_{ijkl} is multiplied by a constant.) Then, in terms of an orthonormal basis of the new metric, the components of the curvature tensor $R^i{}_{jkl} = R_{ijkl}$ can be expressed as a linear sum of sectional curvatures, and hence we have $\|R\|^2 = 1/2 \cdot \Sigma(R^i{}_{jkl})^2 < c\,|K|^2 < c\varepsilon^2$ for some constant c, which depends only on the dimension of M. Therefore, M is almost affinely 0-flat in our sense, as desired.

But the converse inclusion relation does not hold in general. In fact, in defining the almost affinely 0-flatness, the "affine connection" and the "Riemannian metric" by which we define the norm of tensors have no geometric relation at all in contrast to the Riemannian case. Hence, the tensor $R_{ijij} = \Sigma g_{ik} R^k{}_{jij}$ does not give the sectional curvature, and we can say nothing about K for almost affinely 0-flat manifolds. And our example S^3 in this paper (or in [2]) shows that these two sets of almost flat manifolds are actually different.

Intuitively, this difference may be considered as a consequence of the degree of freedom of each geometric structures: Riemannian connections are determined by $1/2 \cdot n(n+1)$ functions g_{ij}, while general torsion free affine connections depend on $1/2 \cdot n^2(n+1)$ functions Γ^i_{jk}, which is greater than $1/2 \cdot n(n+1)$. And hence, in the affine case, the norm $\|R\|$ may move in a wider range than in the Riemannian case, and it may be happen that there exists an example of almost affinely 0-flat manifold (such as S^3) that cannot be almost flat in the sense of Gromov.

3. Construction of an almost flat affine connection

Now, we prove the Theorem.

PROOF. We consider S^3 as a Lie group as in the paper [2], and construct a desired connection, by using left invariant vector fields on S^3. Let $\{X_1, X_2, X_3\}$ be the orthonormal left invariant vector fields on S^3 such that the bracket operation is given by

$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2.$$

In terms of these vector fields, we define the left invariant torsion free affine connection on S^3 by

(3.1)

$$\begin{array}{lll} \nabla_{X_1} X_1 = 4tX_1, & \nabla_{X_2} X_1 = 4tX_2 - sX_3, & \nabla_{X_3} X_1 = sX_2 + 4tX_3, \\ \nabla_{X_1} X_2 = 4tX_2 - (s-2)X_3, & \nabla_{X_2} X_2 = -1/t \cdot X_1, & \nabla_{X_3} X_2 = -X_1, \\ \nabla_{X_1} X_3 = (s-2)X_2 + 4tX_3, & \nabla_{X_2} X_3 = X_1, & \nabla_{X_3} X_3 = -1/t \cdot X_1, \end{array}$$

where $t \in \mathbf{R} \setminus \{0\}$ and $s = 3t^2$. In the following, we show that this connection has the desired property.

Before the calculations of $\nabla^k R$, we state some properties of ∇. First, from the above definition, each component $\nabla_{X_i} X_j$ is $\langle X_1 \rangle$- or $\langle X_2, X_3 \rangle$-valued, and it is $\langle X_1 \rangle$-valued if and only if the set of indices $\{i, j\}$ contains even number of 1. Next, we define the automorphism I of the Lie algebra by

$$IX_1 = X_1, \quad IX_2 = X_3, \quad IX_3 = -X_2.$$

Then, the above connection (3.1) satisfies the equality

$$I\nabla_{X_i}X_j = \nabla_{IX_i}IX_j \text{ for } i,\ j = 1 \sim 3, \tag{3.2}$$

which enables us to reduce the calculations of $\nabla^k R$ to half. In fact, from this equality, we have

$$R(IX, IY)IZ = IR(X,Y)Z, \tag{3.3}$$

and by using this property combined with the Bianchi identity, we can calculate the curvature of ∇ as follows:

$$\begin{aligned}
R(X_1, X_2)X_1 &= -9t^4X_2 - 12t^3X_3,\\
R(X_1, X_2)X_2 &= 3t^2X_1,\\
R(X_1, X_2)X_3 &= 3tX_1,\\
R(X_2, X_3)X_2 &= -9tX_2 + 9t^2X_3,\\
R(X_1, X_3)X_1 &= IR(X_1, X_2)X_1 = 12t^3X_2 - 9t^4X_3,\\
R(X_1, X_3)X_3 &= IR(X_1, X_2)X_2 = 3t^2X_1,\\
R(X_1, X_3)X_2 &= -IR(X_1, X_2)X_3 = -3tX_1,\\
R(X_2, X_3)X_3 &= IR(X_2, X_3)X_2 = -9t^2X_2 - 9tX_3,\\
R(X_2, X_3)X_1 &= R(X_1, X_3)X_2 - R(X_1, X_2)X_3 = -6tX_1.
\end{aligned}$$

Next, we express the covariant derivative of R as

$$Q(X, Y, Z, W) = (\nabla_X R)(Y, Z)W.$$

Then, from the property of ∇ and $[\ ,\ ]$, it is easy to check that $Q(X_i,\ X_j,\ X_k,\ X_l)$ takes value in $\langle X_1\rangle$ or $\langle X_2, X_3\rangle$ and it is $\langle X_1\rangle$-valued if and only if the set of indices $\{i,\ j,\ k,\ l\}$ contains even number of 1. We use this property later. Now, by using the definition

$$\begin{aligned}
&Q(X, Y, Z, W) =\\
&\quad \nabla_X(R(Y, Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y, Z)\nabla_X W,
\end{aligned}$$

we can calculate the following 10 components of Q directly:

$$\begin{aligned}
Q(X_1,X_1,X_2,X_1) &= 72t^5X_2+96t^4X_3,\\
Q(X_1,X_1,X_2,X_2) &= -24t^3X_1,\\
Q(X_1,X_1,X_2,X_3) &= -24t^2X_1,\\
Q(X_2,X_1,X_2,X_1) &= 12t^3X_1,\\
Q(X_2,X_1,X_2,X_2) &= 30t^3X_2-12t^2(3t^2+1)X_3,\\
Q(X_2,X_1,X_2,X_3) &= 12t^2(3t^2+1)X_2+30t^3X_3,\\
Q(X_2,X_1,X_3,X_2) &= 36t^2X_2-36t^3X_3,\\
Q(X_2,X_1,X_3,X_3) &= 36t^3X_2+36t^2X_3,\\
Q(X_2,X_2,X_3,X_2) &= 12t^2X_1,\\
Q(X_2,X_2,X_3,X_3) &= 12tX_1.
\end{aligned}$$

Then, the remaining 17 components of Q are obtained in the following way without explicit calculations. First, from the property (3.2) and (3.3), we have

$$Q(IX,IY,IZ,IW)=IQ(X,Y,Z,W), \tag{3.4}$$

and from this equality, we know the value of the following components of Q:

$$\begin{aligned}
Q(X_1,X_1,X_3,X_1) &= IQ(X_1,X_1,X_2,X_1),\\
Q(X_1,X_1,X_3,X_2) &= -IQ(X_1,X_1,X_2,X_3),\\
Q(X_1,X_1,X_3,X_3) &= IQ(X_1,X_1,X_2,X_2),\\
Q(X_3,X_1,X_2,X_2) &= IQ(X_2,X_1,X_3,X_3),\\
Q(X_3,X_1,X_2,X_3) &= -IQ(X_2,X_1,X_3,X_2),\\
Q(X_3,X_1,X_3,X_1) &= IQ(X_2,X_1,X_2,X_1),\\
Q(X_3,X_1,X_3,X_2) &= -IQ(X_2,X_1,X_2,X_3),\\
Q(X_3,X_1,X_3,X_3) &= IQ(X_2,X_1,X_2,X_2),\\
Q(X_3,X_2,X_3,X_2) &= -IQ(X_2,X_2,X_3,X_3),\\
Q(X_3,X_2,X_3,X_3) &= IQ(X_2,X_2,X_3,X_2).
\end{aligned}$$

Next, by using the equalities

$$\mathfrak{S}_{X,Y,Z}Q(X,Y,Z,W)=\mathfrak{S}_{Y,Z,W}Q(X,Y,Z,W)=0,$$

we have

$$\begin{aligned} Q(X_1, X_2, X_3, X_2) &= Q(X_2, X_1, X_3, X_2) - Q(X_3, X_1, X_2, X_2) \\ &= 72t^2 X_2 - 72t^3 X_3, \\ Q(X_2, X_2, X_3, X_1) &= Q(X_2, X_1, X_3, X_2) - Q(X_2, X_1, X_2, X_3) \\ &= -12t^2(3t^2 - 2)X_2 - 66t^3 X_3. \end{aligned}$$

In addition, we have

$$\begin{aligned} &Q(X_1, X_1, X_2, X_3) = Q(X_1, X_1, X_3, X_2) - Q(X_1, X_2, X_3, X_1) \\ &= -IQ(X_1, X_1, X_2, X_3) - Q(X_2, X_1, X_3, X_1) - IQ(X_2, X_1, X_3, X_1), \end{aligned}$$

from which we have $Q(X_2, X_1, X_3, X_1) = -Q(X_1, X_1, X_2, X_3)$ because $Q(X_1, X_1, X_2, X_3)$, $Q(X_2, X_1, X_3, X_1) \in \langle X_1 \rangle$ and $IX_1 = X_1$. Finally, for the remaining components, from the property (3.4), we have

$$\begin{aligned} &Q(X_1, X_2, X_3, X_3) = IQ(X_1, X_2, X_3, X_2), \\ &Q(X_3, X_1, X_2, X_1) = -IQ(X_2, X_1, X_3, X_1), \\ &Q(X_3, X_2, X_3, X_1) = IQ(X_2, X_2, X_3, X_1), \\ &Q(X_1, X_2, X_3, X_1) = Q(X_2, X_1, X_3, X_1) - Q(X_3, X_1, X_2, X_1) = 48t^2 X_1. \end{aligned}$$

From these calculations, we know that all components of R and Q are divisible by the powers of t, and hence we have $\|R\|$, $\|\nabla R\| \to 0$ as $t \to 0$, which proves that S^3 is almost affinely 1-flat.

Next, we prove $\|\nabla^2 R\| \to 0$ as $t \to 0$. From the definition, we have

$$\begin{aligned} &(\nabla_{X_i} Q)(X_j, X_k, X_l, X_m) = \\ &\nabla_{X_i}(Q(X_j, X_k, X_l, X_m)) - Q(\nabla_{X_i} X_j, X_k, X_l, X_m) - Q(X_j, \nabla_{X_i} X_k, X_l, X_m) \\ &-Q(X_j, X_k, \nabla_{X_i} X_l, X_m) - Q(X_j, X_k, X_l, \nabla_{X_i} X_m) \end{aligned}$$

for $i \sim m = 1, 2, 3$. From this equality, we know that all components of ∇Q are expressed in the form $\Sigma \pm \Gamma_{ij}^k Q_{pqrst}$, where $\nabla_{X_i} X_j = \Sigma\ \Gamma_{ij}^k X_k$ and $Q(X_p, X_q, X_r, X_s) = \Sigma\ Q_{pqrst} X_t$. From (3.1), Γ_{ij}^k is a polynomial of t except the cases $\Gamma_{22}^1 = \Gamma_{33}^1 = -1/t$, and from the above calculations of Q, the components Q_{pqrst} are all divisible by t^2 except the cases $Q(X_2, X_2, X_3, X_3) = -Q(X_3, X_2, X_3, X_2) = 12tX_1$. Hence, we have only to check that the terms of the form

$$\Gamma_{ii}^1 Q_{22331}, \ \Gamma_{ii}^1 Q_{32321} \ (i = 2,\ 3) \tag{3.5}$$

do not appear in $\nabla Q = \Sigma \pm \Gamma_{ij}^{k} Q_{pqrst}$.

In the above definition of ∇Q, the latter terms of the form $-Q(\cdots, \nabla_{X_p} X_q, \cdots)$ do not contain (3.5) because $\nabla_{X_2} X_2$ and $\nabla_{X_3} X_3$ are $\langle X_1 \rangle$-valued. As for the first term $\nabla_{X_i}(Q(X_j, X_k, X_l, X_m))$ in ∇Q, we have only to consider the case

$$\nabla_{X_i}(Q(X_2, X_2, X_3, X_3)), \quad \nabla_{X_i}(Q(X_3, X_2, X_3, X_2))$$

($i = 2, 3$). But both $Q(X_2, X_2, X_3, X_3)$ and $Q(X_3, X_2, X_3, X_2)$ take values in $\langle X_1 \rangle$, and hence the components Γ_{ii}^{1} ($i = 2, 3$) do not appear. As a result, all components of ∇Q are divisible by t, which shows that S^3 is almost affinely 2-flat. □

4. Remarks

Finally, we state some remarks concerning the Theorem.

(A) In the definition of the connection (3.1), if we put $s = 0$ instead of $s = 3t^2$, then we obtain the connection essentially equal to the one constructed in [2]. We remark that for the connection corresponding to $s = 0$, the norm $\|\nabla R\|$ is not almost zero because the equalities

$$(\nabla_{X_2} R)(X_2, X_3)X_2 = (\nabla_{X_3} R)(X_2, X_3)X_3 = 12X_1$$

hold in this case. (Note that the parameter t disappears in these equalities.)

(B) As in the example in [2], the left invariant connection (3.1) is also S^1-right invariant, where S^1 is the subgroup of S^3 generated by the vector field X_1. It is easy to check that this class of connections on S^3 just coincides with the invariant affine connections on the homogeneous space $U(2)/U(1) \simeq S^3$ (cf. [1]). By some calculations, we can show that this class does not contain a connection such that $\|R\| \sim \|\nabla^3 R\|$ are simultaneously almost zero. (As for the connection (3.1), we have for example

$$(\nabla_{X_2} P)(X_2, X_2, X_2, X_3, X_2) = -72(2t^4 + 10t^2 + 1)X_1,$$

where $P(X_i, X_j, X_k, X_l, X_m) = (\nabla_{X_i} Q)(X_j, X_k, X_l, X_m)$, and hence (3.1) does not satisfy $\|\nabla^3 R\| \to 0$ as $t \to 0$.) As stated in section 1, it is an open question whether S^3 is almost affinely 3-flat or not.

(C) It is an important and interesting problem to find another example of compact almost affinely flat manifolds. As such an example, we consider a three-dimensional compact quotient manifold $M = \Gamma\backslash\widetilde{SL}(2,\mathbf{R})$, where $\widetilde{SL}(2,\mathbf{R})$ is the universal covering group of $SL(2,\mathbf{R})$ and Γ is a discrete subgroup of $\widetilde{SL}(2,\mathbf{R})$. By using the orthonormal left invariant vector fields X_1, X_2, X_3 on $\widetilde{SL}(2,\mathbf{R})$ satisfying $[X_1, X_2] = 2X_2$, $[X_1, X_3] = -2X_3$, $[X_2, X_3] = 2X_1$, we define the left invariant torsion free affine connection on $\widetilde{SL}(2,\mathbf{R})$ by

$$\begin{array}{lll} \nabla_{X_1}X_1 = tX_1, & \nabla_{X_2}X_1 = tX_2, & \nabla_{X_3}X_1 = tX_3, \\ \nabla_{X_1}X_2 = (t+2)X_2, & \nabla_{X_2}X_2 = 0, & \nabla_{X_3}X_2 = -(1-4/t)X_1, \\ \nabla_{X_1}X_3 = (t-2)X_3, & \nabla_{X_2}X_3 = (1+4/t)X_1, & \nabla_{X_3}X_3 = 0, \end{array}$$

where $t \in \mathbf{R} \setminus \{0\}$. Then, we can easily show that $\|R\| \to 0$ as $t \to 0$, and by projecting ∇ to M, we obtain a new example of three-dimensional compact almost affinely 0-flat manifold M. (Note that $\widetilde{SL}(2,\mathbf{R})$ does not admit a left invariant torsion free flat affine connection because it is semi-simple. See [8, p.31] or [9].)

From this example combined with the previous result, we know that the three-dimensional Brieskorn manifold $M(p,q,r)$ ($p,\ q,\ r \in \mathbf{Z}$ and $p,\ q,\ r \geq 2$) obtained by intersecting the complex algebraic surface ${z_1}^p + {z_2}^q + {z_3}^r = 0$ with the unit sphere $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ is almost affinely 0-flat. In fact, from the result of Milnor [11], the manifold $M(p,q,r)$ is diffeomorphic to a coset space of $SU(2)$ ($\simeq S^3$), $\widetilde{SL}(2,\mathbf{R})$ or H_3 (= 3-dimensional Heisenberg Lie group) by some discrete subgroup, and it is easy to see that the bi-invariant connection on H_3 defined by $\nabla_X Y = 1/2 \cdot [X,Y]$ is torsion free and flat. Clearly, the products of these manifolds are also almost affinely 0-flat.

Note that these examples are essentially all obtained from three-dimensional Lie groups with left invariant affine connections. It is an interesting problem to find another new example, or to show the non-existence of such a structure on other higher dimensional simple Lie groups. (It should be remarked that the assumption "torsion free" in the definition of k-flatness is essential in finding such an example because every non-abelian Lie group admits a left invariant "flat" affine connection with "non-vanishing" torsion defined by $\nabla_X Y = 0$ for left invariant vector fields X and Y.)

References

[1] Y. Agaoka, *On the curvature of the homogeneous space $U(n+1)/U(n)$*, Mem. Fac. Integrated Arts and Sci., Hiroshima Univ., Ser.IV, **19** (1993), 1-17.

[2] Y. Agaoka, *An example of almost flat affine connections on the three-dimensional sphere*, Proc. Amer. Math. Soc. **123** (1995), 3519-3521.
[3] L. Auslander, *Examples of locally affine spaces*, Ann. of Math. **64** (1956), 255-259.
[4] L. Auslander and L. Markus, *Holonomy of flat affinely connected manifolds*, Ann. of Math. **62** (1955), 139-151.
[5] Y. Benoist, *Une nilvariété non affine*, J. Diff. Geom. **41** (1995), 21-52.
[6] P. Buser and H. Karcher, *Gromov's almost flat manifolds*, Astérisque **81** (1981).
[7] M. Gromov, *Almost flat manifolds*, J. Diff. Geom. **13** (1978), 231-241.
[8] J. Helmstetter, *Radical d'une algèbre symétrique à gauche*, Ann. Inst. Fourier, Grenoble, **29**, 4 (1979), 17-35.
[9] H. Matsushima and K. Okamoto, *Non-existence of torsion free flat connections on a real semisimple Lie group*, Hiroshima Math. J. **9** (1979), 59-60.
[10] J. Milnor, *On the existence of a connection with curvature zero*, Comm. Math. Helv. **32** (1958), 215-223.
[11] J. Milnor, *On the 3-dimensional Brieskorn manifolds $M(p,q,r)$*, in "Knots, Groups and 3-manifolds" (L.P.Neuwirth, ed.), Ann. of Math. Studies **84**, Princeton Univ. Press (1975), 175-225.
[12] E.A. Ruh, *Almost flat manifolds*, J. Diff. Geom. **17** (1982), 1-14.

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