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PERIODS FOR TRANSVERSAL MAPS ON COMPACT MANIFOLDS WITH A GIVEN HOMOLOGY

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ABSTRACT. Let M be a compact C^1 differentiable manifold such that its rational homology is $H_j(M;\mathbf{Q}) \approx \mathbf{Q}$ if $j \in J \cup \{0\}$, and $H_j(M;\mathbf{Q}) \approx \{0\}$ otherwise. Here J is a subset of the set of natural numbers $\mathbf{N}$ with cardinal 1, 2 or 3. A C^1 map $f : M \to M$ is called transversal if for all $m \in \mathbf{N}$ the graph of f^m intersects transversally the diagonal of $M \times M$ at each point (x, x) such that x is a fixed point of f^m. We study the set of periods of f by using the Lefschetz numbers for periodic points.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In dynamical systems it is often the case that differentiable topological information can be used to study qualitative and quantitative properties of the system. This paper deals with the problem of determining the periods of the periodic points of a class of C^1 self-maps given the homology class of the map. Similar problems have been studied in [6] for compact manifolds with homology $H_0(M;\mathbf{Q}) \approx \mathbf{Q}$, $H_1(M;\mathbf{Q}) \approx \mathbf{Q} \oplus \mathbf{Q}$, $H_j(M;\mathbf{Q}) \approx \{0\}$ for $j \neq 0, 1$. From other point of view periodic points for transversal maps have been studied by Franks in [4], [5] Matsuoka in [9], see also [2] and [8].

The preliminary notation and definitions which are necessary to state our main results are those of [6]. We include them here for completeness.

Let $f : X \to X$ be a continuous map. A *fixed point* of f is a point x of X such that $f(x) = x$. Denote the totality of fixed points by $\mathrm{Fix}(f)$. The point $x \in X$ is *periodic with period* m if $x \in \mathrm{Fix}(f^m)$ but $x \notin \mathrm{Fix}(f^k)$ for all $k = 1, \ldots, m-1$. Let $\mathrm{Per}(f)$ denote the set of all periods of periodic point of f.

Let M be a compact manifold of dimension n. A continuous map $f : M \to M$ induces endomorphisms $f_{*j} : H_j(M;\mathbf{Q}) \to H_j(M;\mathbf{Q})$ (for $j = 0, 1, \ldots, n$) on the rational homology groups of M. The *Lefschetz number* of f is defined by

$$L(f) = \sum_{k=0}^{n} (-1)^k \text{trace}(f_{*k}).$$

By the renowned *Lefschetz fixed point theorem*: if $L(f) \neq 0$ then f has fixed points (see, for instance [1]). Of course, we can consider the Lefschetz number of f^m but (in general) it is not true that if $L(f^m) \neq 0$ then f has periodic points of period m. It could have periodic points with period some proper division of m. Therefore we will use the Lefschetz numbers for periodic points introduced in [3] (see also [7]) for analysing if a given period belongs to the set of periods of a self-map. More precisely, for every $m \in \mathbf{N}$ we define the *Lefschetz number of period* m, $l(f^m)$, as follows

$$l(f^m) = \sum_{r|m} \mu(r) L(f^{\frac{m}{r}}),$$

where $\sum_{r|m}$ denotes the sum over all positive divisors r of m, and μ is the *Moebius function* defined by

$$\mu(m) = \begin{cases} 1 \text{ if } m = 1, \\ 0 \text{ if } k^2 | m \text{ for some } k \in \mathbf{N}, \\ (-1)^r \text{ if } m = p_1 \cdots p_r \text{ distinct prime factors.} \end{cases}$$

According to the inversion formula (see for instance [11])

$$L(f^m) = \sum_{r|m} l(f^r).$$

The Lefschetz number of period m will become interesting after showing that for many classes of maps we have: if $l(f^m) \neq 0$ then $m \in \text{Per}(f)$. This is almost the case when f is a transversal map.

A C^1 map $f : M \to M$ defined on a compact C^1 differentiable manifold is called *transversal* if $f(M) \subset \text{Int}(M)$ and if for all $m \in \mathbf{N}$ at each point $x \in \text{Fix}(f^m)$ we have $\det(I - df^m(x)) \neq 0$; i.e. 1 is not an eigenvalue of $df^m(x)$. We note that if f is transversal then for all $m \in \mathbf{N}$ the graph of f^m intersects transversally the diagonal $\{(y, y) : y \in M\}$ at each point (x, x) such that $x \in \text{Fix}(f^m)$. Since for a transversal map f the fixed points of f^m are isolated and M is compact, the cardinal of $\text{Fix}(f^m)$ is finite for every $m \in \mathbf{N}$. Dold [3] showed that m divides $l(f^m)$ for any $m \in \mathbf{N}$. The following result will play a key role in this paper and it was proved in [7] (see another proof in [6]).

Theorem 1.1. *Let f be a transversal map. Suppose that $l(f^m) \neq 0$ for some $m \in \mathbf{N}$.*

(a) *If m is odd then $m \in Per(f)$.*
(b) *If m is even then $\{m/2, m\} \cap Per(f) \neq \emptyset$.*

The results on transversal maps on arbitrary compact manifolds given in Theorem 1.1 are in general difficult to apply because of the computation of $l(f^m)$. Of course, if the rational homology groups are simple then these computations become easier.

In this paper we deal with transversal maps on a compact manifold M with rational homology

$$(1) \qquad H_j(M;\mathbf{Q}) \approx \mathbf{Q} \text{ if } j \in J \cup \{0\}, \quad H_j(M;\mathbf{Q}) \approx \{0\} \text{ otherwise.}$$

Here J is a subset of the set of natural numbers $\mathbf{N}$ with cardinality 1, 2, or 3. Transversal maps on compact manifolds with such homology are among the easiest nontrivial maps for which we can compute the numbers $l(f^m)$ and apply Theorem 1.1 to obtain information about their sets of periods.

Now we present few examples of manifolds having the homology given by (1). For instance, if the cardinal of J is equal to 1, then the p-dimensional sphere S^p satisfies that $H_j(S^p;\mathbf{Q}) \approx \mathbf{Q}$ if $j \in J \cup \{0\}$ with $J = \{p\}$, and $H_j(S^p;\mathbf{Q}) \approx \{0\}$ otherwise. If the cardinal of J is equal to n there is the complex projective space $\mathbf{C}P^n$ whose rational homology groups are (see, for instance, [10]) $H_j(\mathbf{C}P^n;\mathbf{Q}) \approx \mathbf{Q}$ if $j \in J \cup \{0\}$ with $J = \{2, 4, \dots, n\}$, and $H_j(\mathbf{C}P^n;\mathbf{Q}) \approx \{0\}$ otherwise. If the cardinal of J is equal to 3 there is the product of two spheres of different dimensions $S^p \times S^q$ with $p \neq q$, p and q positive, then from Kunneth's formula (see again [10]) we have $H_j(S^p \times S^q;\mathbf{Q}) \approx \mathbf{Q}$ if $j \in J \cup \{0\}$ with $J = \{p, q, p+q\}$, and $H_j(S^p \times S^q;\mathbf{Q}) \approx \{0\}$ otherwise. The easiest higher dimensional examples are the products of these spaces with acyclic manifold, but also there are other spaces with these homologies.

In Section 2 we will give an easy proof of the following result.

Theorem 1.2. *Let $f : M \to M$ be a transversal map. Suppose that the rational homology of M satisfies (1) with $J = \{p\}$. We denote by (a) the 1×1 integer matrix defined by the induced homology endomorphism $f_{*p} : H_p(M;\mathbf{Q}) \to H_p(M;\mathbf{Q})$ (a is called the degree of f). Then the following statements hold.*

(a) $l(f) = L(f) = 1 + (-1)^p a$.
(b) $l(f^2) = 0$ *if and only if* $a \in \{0, 1\}$.
(c) *If $m > 2$ then $l(f^m) = 0$ if and only if $a \in \{-1, 0, 1\}$.*

From Theorem 1.1 and Theorem 1.2 it follows easily the following result of Casasayas, Llibre and Nunes [2].

Corollary 1.3. *In the assumptions of Theorem 1.2 if we assume that* $a \notin \{-1,0,1\}$ *then the following statements hold.*

(a) $Per(f) \supset \{1,3,5,7,\dots\}$.
(b) *If* m *is even and* $m \notin Per(f)$, *then* $\{m/2, 2m\} \subset Per(f)$.

Our main results on the set of periods of transversal maps follows from the next two theorems and Theorem 1.1.

Theorem 1.4. *Let* $f : M \to M$ *be a transversal map. Suppose that the rational homology of* M *satisfies (1) with* $J = \{p,q\}$. *If we denote by* (a_j) *the* 1×1 *integer matrix defined by the induced homology endomorphism* $f_{*j} : H_j(M;\mathbf{Q}) \to H_j(M;\mathbf{Q})$ *for each* $j \in J$, *then the following statements hold.*

(a) $l(f) = L(f) = 1 + (-1)^p a_p + (-1)^q a_q$.
(b) $l(f^2) = 0$ *if and only if* $q-p$ *is even and* $\{a_p, a_q\} \subset \{0,1\}$, *or* $q-p$ *is odd and* $a_p = a_q$ *or* $a_p + a_q = 1$.
(c) *If* $\{a_p, a_q\} \subset \{-1,0,1\}$, *then* $l(f^m) = 0$ *for every natural number* $m > 2$.
(d) *If* $\{a_p, a_q\} \not\subset \{-1,0,1\}$ *and* $m > 1$ *is odd, then* $l(f^m) = 0$ *if and only if* $a_p + (-1)^{q-p} a_q = 0$.
(e) *If* $\{a_p, a_q\} \not\subset \{-1,0,1\}$ *and* $4|m$, *then* $l(f^m) = 0$ *if and only if* $q-p$ *is odd and* $a_p = \pm a_q$.
(f) *If* $\{a_p, a_q\} \not\subset \{-1,0,1\}$, $m > 2$ *is even and* $4 \not| m$, *then* $l(f^m) = 0$ *if and only if* $q-p$ *is odd and* $a_p = a_q$.

Theorem 1.4 will be proved in Section 2. From Theorem 1.1 and Theorem 1.4 it follows easily the following corollary.

Corollary 1.5. *In the assumptions of Theorem 1.4 the following statements hold.*

(a) *If* $(-1)^p a_p + (-1)^q a_q \neq -1$, *then* $1 \in Per(f)$.
(b) *If neither* $q-p$ *is even and* $\{a_p, a_q\} \subset \{0,1\}$, *nor* $q-p$ *is odd and* $a_p = a_q$ *or* $a_p + a_q = 1$, *then* $\{1,2\} \cap Per(f) \neq \emptyset$.
(c) *If* $\{a_p, a_q\} \subset \{-1,0,1\}$, *then the unique periods* m *that can be forced from the numbers* $l(f^m)$ *are* 1 *and* 2.
(d) *If* $\{a_p, a_q\} \not\subset \{-1,0,1\}$ *and* $a_p + (-1)^{q-p} a_q \neq 0$, *then* $\{3,5,7,\dots\} \subset Per(f)$.
(e) *If* $\{a_p, a_q\} \not\subset \{-1,0,1\}$, $q-p$ *and* $m > 2$ *are even, and* $m \notin Per(f)$, *then* $\{m/2, 2m\} \subset Per(f)$.
(f) *If* $\{a_p, a_q\} \not\subset \{-1,0,1\}$, $q-p$ *is odd,* $a_p \neq \pm a_q$, $4|m$ *and* $m \notin Per(f)$, *then* $\{m/2, 2m\} \subset Per(f)$.

(g) *If* $\{a_p, a_q\} \not\subset \{-1, 0, 1\}$, $q - p$ *is odd*, $a_p \neq a_q$, $m > 2$ *is even and* $4 \not| m$, *then* $\{m/2, m\} \cap Per(f) \neq \emptyset$.

Note that in Theorem 1.2 and Theorem 1.4 we have described completely the zero set of $l(f^m)$ for all $m \in \mathbf{N}$. So, in Corollaries C and E we give all the information on the set of periods that can be obtained through Theorem 1.1.

Let S be the set of $(z_1, z_2, z_3) \in \mathbf{Z}_0^3$ with $\mathbf{Z}_0 = Z \setminus \{-1, 0, 1\}$ satisfying at least one of the following conditions:

(i) All the components have the same sign.

(ii) $|z_i| < \max_{j \neq i}\{|z_j|\}$ if z_i is the component that has different sign.

(iii) $|z_i| > \sum_{j \neq i} |z_j|$.

Theorem 1.6. *Let* $f : M \to M$ *be a transversal map. Suppose that the rational homology of* M *satisfies (1) with* $J = \{p, q, r\}$. *For each* $j \in J$ *we denote by* (a_j) *the* 1×1 *integer matrix defined by the induced homology endomorphism* $f_{*j} : H_j(M; \mathbf{Q}) \to H_j(M; \mathbf{Q})$. *We assume that* p *and* q *are even (respectively odd). Then the following statements hold.*

(a) *Let* $m > 1$ *be odd. If* r *is even (respectively odd) and* $(a_p, a_q, a_r) \in S$, *or* r *is odd (respectively even) and* $(a_p, a_q, -a_r) \in S$, *then* $l(f^m) \neq 0$.

(b) *Let* $m > 1$ *be even. If* r *is even (respectively odd), or* r *is odd (respectively even) and* $(|a_p|, |a_q|, -|a_r|) \in S$, *then* $l(f^m) \neq 0$.

Theorem 1.6 will be proved in Section 2. From Theorem 1.1 and Theorem 1.6, it follows immediately the following corollary.

Corollary 1.7. *In the assumptions of Theorem 1.6 the following statements hold.*

(a) *Let* $m > 1$ *be odd. If* r *is even (respectively odd) and* $(a_p, a_q, a_r) \in S$, *or* r *is odd (respectively even) and* $(a_p, a_q, -a_r) \in S$, *then* $\{3, 5, 7, \dots\} \subset Per(f)$.

(b) *Let* $m > 1$ *be even. If* r *is even (respectively odd), or* r *is odd (respectively even) and* $(|a_p|, |a_q|, -|a_r|) \in S$, *then* $\{m/2, m\} \cap Per(f) \neq \emptyset$.

While Theorem 1.2 and 1.4 characterize completely the zeros of $l(f^m)$, this is not the case of Theorem 1.6. This is due to the fact that for knowing all the zeros of $l(f^m)$ in the assumptions of Theorem 1.6, we must know the solutions of some diophantine equations that in general are more difficult to solve than the last theorem of Fermat.

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2. Proofs of the results

Let $f : M \to M$ be a transversal map and suppose that the rational homology of M satisfies (1). Let (a_j) be the 1×1 integer matrix defined by the induced homology endomorphism $f_{*j} : H_j(M; \mathbf{Q}) \to H_j(M; \mathbf{Q})$ for each $j \in J$. Since $H_0(M; \mathbf{Q}) \approx \mathbf{Q}$, M is connected, and consequently f_{*0} is the identity (see [10] for more details). Then $L(f^m) = 1 + \sum_{j \in J}(-1)^j a_j^m$ for all $m \in \mathbf{N}$. We note that

$$\begin{aligned}\sum_{r|m} \mu(r) &= 1 - \Big(\sum_{1 \le i \le n} 1\Big) + \Big(\sum_{1 \le i < j \le n} 1\Big) - \dots + (-1)^n \\ &= 1 - \tbinom{n}{1} + \tbinom{n}{2} - \dots + (-1)^n \tbinom{n}{n} \\ &= (1-1)^n = 0,\end{aligned}$$

where $m = p_1^{\alpha_1} \cdots p_n^{\alpha_n} > 1$ with $p_1, \dots, p_n$ distinct primes. Therefore, if $m > 1$ the Lefschetz number of period m will be

$$l(f^m) = \sum_{j \in J} (-1)^j \sum_{r|m} \mu(r) a_j^{\frac{m}{r}}.$$

For each $m \in \mathbf{N}$ we define the polynomial

$$Q_m(x) = \sum_{r|m} \mu(r) x^{\frac{m}{r}}.$$

Then, if $m > 1$ we can write

$$l(f^m) = \sum_{j \in J} (-1)^j Q_m(a_j). \tag{2}$$

Hence we will study when $l(f^m)$ is zero or not by analysing the polynomials $Q_m(x)$ and evaluating them at a_j.

Set $m = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ where $p_1, \dots, p_n$ are distinct primes. The next proposition is proved in [6].

Proposition 2.1. *Let $m \in \mathbf{N}$.*

(a) *If m is odd, then Q_m is an odd function.*
(b) *If $4|m$, then Q_m is an even function.*
(c) *If $2|m$ and $4 \not| m$, then $Q_m(x) = Q_{\frac{m}{2}}(x^2) - Q_{\frac{m}{2}}(x)$.*
(d) *$Q_m(0) = 0$.*
(e) *If $m > 1$, then $Q_m(1) = 0$.*
(f) *If $m > 2$, then $Q_m(-1) = 0$.*
(g) *For all $i \in \mathbf{N}$ we have $Q_m^{(i)}(1) \ge 0$, where $Q_m^{(i)}(x)$ denotes the i-th derivative of $Q_m(x)$ with respect to the variable x.*

(h) *$Q_m(x)$ is positive and increasing in $(1,\infty)$.*

(i) *If m is even, then the function $Q_m(x)$ is positive and decreasing in $(-\infty,-1)$. Furthermore, if $4 \not| m$ we have that $Q_m(x) \le Q_m(-x)$ for all $x \in [1,\infty)$.*

(j) *If $m > 2$, then $Q_m(1.6) > 2$.*

PROOF OF THEOREM 1.2: Clearly $L(f) = 1 + (-1)^p a$. This proves (a).

From (2) we get that $l(f^m) = (-1)^p Q_m(a)$. From Proposition 1(c) we get that $Q_2(x) = x^2 - x$. So $Q_2(a) = a(a-1)$, and consequently $l^2(f) \neq 0$ if and only if $a \in \{0,1\}$. So (b) is proved.

If $m > 2$ then from statements (d), (e), and (f) of Proposition 1 we obtain that $l(f^m) = 0$ if $a \in \{-1,0,1\}$. Now we assume that $a \notin \{-1,0,1\}$. From statements (a), (h) and (i) of Proposition 1, it follows that $l(f^m) = (-1)^p Q_m(a) \neq 0$. Hence (c) follows. □

PROOF OF THEOREM 1.4: From the definitions of $L(f)$ and $l(f)$ it follows immediately (a).

From (2) and Proposition 1(c) we get that $l(f^2) = (-1)^p Q_2(a_p^2) + (-1)^q Q_2(a_q^2)$ and $Q_2(x) = x^2 - x$. Therefore $l(f^2) = (-1)^p(a_p^2 - a_p) + (-1)^q(a_q^2 - a_q)$. We assume that $q-p$ is odd. Then $l(f^2) = 0$ if and only if $(a_p - a_q)(a_p + a_q) = a_p - a_q$; or equivalently either $a_p = a_q$, or $a_p + a_q = 1$. Now we assume that $q - p$ is even. Then $l(f^2) = 0$ if and only if $a_p^2 + a_q^2 = a_p + a_q$, or equivalently $\{a_p, a_q\} \subset \{0,1\}$. Therefore (b) is proved.

From (2) and statements (d), (e) and (f) of Proposition 1 it follows (c).

Let m be odd. From (a) and (h) of Proposition 1, we have that $Q_m(x)$ is an increasing odd function in $(-\infty,-1) \cup (1,\infty)$. Assume that $\{a_p, a_q\} \not\subset \{-1,0,1\}$. Then, from (2) $l(f^m) = 0$ if and only if $Q_m(a_p) + (-1)^{q-p} Q_m(a_q) = 0$, or equivalently $a_p + (-1)^{q-p} a_q = 0$. Hence (d) is proved.

Assume that $4|m$. From (b), (h) and (i) of Proposition 1, we have that $Q_m(x)$ is an even function, increasing in $(1,\infty)$, and $Q_m(x) = Q_m(-x)$ for all $x \in [1,\infty)$. Assume that $\{a_p, a_q\} \not\subset \{-1,0,1\}$. Then, from (2) we get that $l(f^m) = 0$ if and only if $Q_m(a_p) + (-1)^{q-p} Q_m(a_q) = 0$, or equivalently $q - p$ is odd and $a_p = \pm a_q$. Hence (e) is proved.

Now we prove (f). Assume that $m > 2$ is even and $4 \not| m$. From (2) we get that $l(f^m) = 0$ if and only if $Q_m(a_p) + (-1)^{q-p} Q_m(a_q) = 0$. Assume that $\{a_p, a_q\} \not\subset \{-1,0,1\}$. From statements (h) and (i) of Proposition 1 we have that $Q_m(a_p)$ and $Q_m(a_q)$ are positive. So, if $q - p$ is even then $l(f^m) \neq 0$.

In the rest of the proof of statement (f) we suppose that $q - p$ is odd. Then $l(f^m) = 0$ if and only if $Q_m(a_p) = Q_m(a_q)$. By Proposition 1(h), $Q_m(x)$ is positive and increasing in $(1,\infty)$. So, if $a_p, a_q > 0$ then $l(f^m) = 0$ if and only if $a_p = a_q$. Now we assume that $a_p, a_q < 0$. By Proposition 1(c) we have that $Q_m(x) = Q_{\frac{m}{2}}(x^2) - Q_{\frac{m}{2}}(x)$ with $m/2$ odd. Then, $l(f^m) = 0$ if and only if $Q_{\frac{m}{2}}(a_p^2) - Q_{\frac{m}{2}}(a_p) = Q_{\frac{m}{2}}(a_q^2) - Q_{\frac{m}{2}}(a_q)$, or equivalently $Q_{\frac{m}{2}}(a_p^2) + Q_{\frac{m}{2}}(|a_p|) = Q_{\frac{m}{2}}(a_q^2) + Q_{\frac{m}{2}}(|a_q|)$ (see Proposition 1(a)). In short, if $a_p, a_q < 0$ then from Proposition 1(h) $l(f^m) = 0$ if and only if $a_p = a_q$. So in the rest of the proof of (f) we can assume that $a_p < 0 < a_q$.

First we consider $0 < a_q \leq -a_p$. Since $l(f^m) = 0$ if and only if $Q_{\frac{m}{2}}(a_p^2) + Q_{\frac{m}{2}}(|a_p|) = Q_{\frac{m}{2}}(a_q^2) - Q_{\frac{m}{2}}(a_q)$, and $0 < a_q \leq |a_p|$, by statements (a) and (h) of Proposition 1 we obtain that $Q_{\frac{m}{2}}(a_p^2) + Q_{\frac{m}{2}}(|a_p|) > Q_{\frac{m}{2}}(a_q^2) - Q_{\frac{m}{2}}(a_q)$, hence $l(f^m) \neq 0$. So we can assume that $0 < -a_p < a_q$.

By Proposition 1(g) we have that $Q_m(x) = \sum_{k=1}^{m} A_k(x-1)^k$ with $A_k = Q_m^{(k)}(1)/k! \geq 0$. Therefore, since $l(f^m) = 0$ if and only if $Q_m(a_q) = \sum_{k=1}^{m} A_k(a_q - 1)^k = \sum_{k=1}^{m} A_k(a_p - 1)^k = Q_m(a_p)$, it follows that if $|a_p| + 1 < a_q - 1$, then $Q_m(a_q) > Q_m(a_p)$, and consequently $l(f^m) \neq 0$. Hence, since $0 < -a_p < a_q$ the unique cases that remain to consider are $|a_p| + 1 = a_q - 1$ and $|a_p| + 1 = a_q$.

We assume that $|a_p| + 1 = a_q - 1 = b$. So $a_q = b + 1$ and $a_p = 1 - b$. Therefore, since $\{a_p, a_q\} \not\subset \{-1, 0, 1\}$ we get that $b \geq 1$. Now $l(f^m) = 0$ if and only if $Q_m(b+1) = Q_m(-b+1)$, or equivalently $Q_{\frac{m}{2}}((b+1)^2) - Q_{\frac{m}{2}}(b+1) = Q_{\frac{m}{2}}((b-1)^2) - Q_{\frac{m}{2}}(1-b)$ (see Proposition 1(c)). Since $m/2$ is odd, from Proposition 1(a) $l(f^m) = 0$ if and only if $Q_{\frac{m}{2}}((b+1)^2) = Q_{\frac{m}{2}}(b+1) + Q_{\frac{m}{2}}((b-1)^2) + Q_{\frac{m}{2}}(b-1)$, or equivalently

$$\sum_{k=1}^{\frac{m}{2}} B_k \left[(b+1)^2 - 1\right]^k = \sum_{k=1}^{\frac{m}{2}} B_k \left[b^k + ((b-1)^2 - 1)^k + (b-2)^k\right],$$

because $Q_{\frac{m}{2}}(x) = \sum_{k=1}^{\frac{m}{2}} B_k(x-1)^k$, with $B_k = Q_{\frac{m}{2}}^{(k)}(1)/k! \geq 0$. Therefore, since $\left[(b+1)^2 - 1\right]^k = b^k(b+2)^k > b^k\left[2 + (b-2)^k\right] > b^k\left[1 + (b-2)^k\right] + (b-2)^k = b^k + ((b-1)^2 - 1)^k + (b-2)^k$ if $b \geq 2$, and $\left[(b+1)^2 - 1\right]^k > b^k + ((b-1)^2 - 1)^k + (b-2)^k$ if $b = 1$, it follows that $l(f^m) \neq 0$.

Finally we assume that $|a_p| + 1 = a_q = b + 1$. So $a_p = -b$. Therefore, since $\{a_p, a_q\} \not\subset \{-1, 0, 1\}$ we get that $b \geq 1$. Now $l(f^m) = 0$ if and only if $Q_m(b+1) = Q_m(-b)$, or equivalently $Q_{\frac{m}{2}}((b+1)^2) - Q_{\frac{m}{2}}(b+1) = Q_{\frac{m}{2}}(b^2) - Q_{\frac{m}{2}}(-b)$ (see Proposition 1(c)). Since $m/2$ is odd, from Proposition 1(a) $l(f^m) = 0$ if and only if $Q_{\frac{m}{2}}((b+1)^2) = Q_{\frac{m}{2}}(b+1) + Q_{\frac{m}{2}}(b^2) + Q_{\frac{m}{2}}(b)$, or equivalently

$$\sum_{k=1}^{\frac{m}{2}} B_k \left[(b+1)^2 - 1\right]^k = \sum_{k=1}^{\frac{m}{2}} B_k \left[b^k + (b^2 - 1)^k + (b-1)^k\right].$$

Therefore, since $\left[(b+1)^2-1\right]^k = b^k(b+2)^k > \left[(b-1)^k+1\right](b+2)^k > (b-1)^k(b+2)+b^k = b^k+(b^2-1)^k+(b-1)^k$ if $k \geq 2$ and $b \geq 1$, and $\left[(b+1)^2-1\right]^k > b^k+(b^2-1)^k+(b-1)^k$ if $k=1$ and $b \geq 1$, it follows that $l(f^m) \neq 0$. Hence (f) is proved. □

PROOF OF THEOREM 1.6: Without loss of generality we can assume that p and q are both even. The case that both are odd follows in a similar way.

We suppose that $m > 1$ is odd and r is even. If a_p, a_q and a_r have the same sign, then from (2) and statements (a) and (h) of Proposition 1 it follows that $l(f^m) = Q_m(a_p)+Q_m(a_q)+Q_m(a_r) \neq 0$. Otherwise, since $(a_p, a_q, a_r) \in S$, without loss of generality we can assume that $a_r < -1$ and $1 < a_p \leq a_q$. Then, if $|a_r| < a_q$ it follows that $l(f^m) = Q_m(|a_p|)+Q_m(|a_q|)-Q_m(|a_r|) > Q_m(|a_q|)-Q_m(|a_r|) > 0$. On the other hand, if $|a_r| > |a_p|+|a_q||$ we have that

$$Q_m(|a_r|) > Q_m(|a_p|+|a_q|) = \sum_{k=1}^{m} A_k(|a_p|+|a_q|-1)^k,$$

because by Proposition 1(g) we have that $Q_m(x) = \sum_{k=1}^m A_k(x-1)^k$ with $A_k = Q_m^{(k)}(1)/k! \geq 0$. Since $(|a_p|+|a_q|-1)^k > [(|a_p|-1)+(|a_q|-1)]^k \geq (|a_p|-1)^k + (|a_q|-1)^k$, we obtain that

$$\begin{aligned}\textstyle\sum_{k=1}^m A_k(|a_p| &+ |a_q|-1)^k > \textstyle\sum_{k=1}^m A_k(|a_p|-1)^k \\ &+ \textstyle\sum_{k=1}^m A_k(|a_q|-1)^k = Q_m(|a_p|)+Q_m(|a_q|).\end{aligned}$$

Hence

$$Q_m(|a_r|) > Q_m(|a_p|+|a_q|) > Q_m(|a_p|)+Q_m(|a_q|), \tag{3}$$

and consequently $l(f^m) \neq 0$.

Now we assume that r is odd. Then $l(f^m) = Q_m(a_p)+Q_m(a_q)-Q_m(a_r) = Q_m(a_p)+Q_m(a_q)+Q_m(-a_r)$, and since $(a_p, a_q, -a_r) \in S$ the arguments of the above case can be applied again to obtain that $l(f^m) \neq 0$. So we have proved (a).

We assume that $m \geq 2$ is even. By statements (b), (h) and (i) of Proposition 1, the function $Q_m(x)$ is positive in $(-\infty,-1)\cup(1,\infty)$. Therefore, if r is even, then $l(f^m) = Q_m(a_p)+Q_m(a_q)+Q_m(a_r) > 0$. We assume that r is odd. Then $l(f^m) = Q_m(a_p)+Q_m(a_q)-Q_m(a_r)$ and we consider two cases.

Case 1: $4|m$. By Proposition 1(b) the function $Q_m(x)$ is even. Therefore $l(f^m) = Q_m(|a_p|)+Q_m(|a_q|)-Q_m(-|a_r|)$. From the assumptions we have that $(|a_p|, |a_q|, -|a_r|) \in S$. So we can repeat the arguments of the proof of statement (a), and we obtain $l(f^m) \neq 0$.

Case 2: $4 \not| m$. We have that $|a_r| < \max\{|a_p|, |a_q|\}$ or $|a_r| > |a_p| + |a_q|$, because $(|a_p|, |a_q|, -|a_r|) \in S$. We assume that the first inequality holds. Then, from statements (h) and (i) of Proposition 1 $l(f^m) = Q_m(a_p) + Q_m(a_q) - Q_m(a_r) \geq Q_m(|a_p|) + Q_m(|a_q|) - Q_m(-|a_r|)$. Since $(|a_p|, |a_q|, -|a_r|) \in S$ we can repeat the arguments of the proof of statement (a), and we obtain that $l(f^m) \neq 0$. Now we assume that the second inequality holds. Then, by statements (c) and (a) of Proposition 1, and (3) we get that $Q_m(-(|a_p| + |a_q|)) = Q_{\frac{m}{2}}((|a_p| + |a_q|)^2) + Q_{\frac{m}{2}}(|a_p| + |a_q|) \geq Q_{\frac{m}{2}}((|a_p|^2 + |a_q|^2) + Q_{\frac{m}{2}}(|a_p| + |a_q|) > Q_{\frac{m}{2}}(|a_p|^2) + Q_{\frac{m}{2}}(|a_q|^2) + Q_{\frac{m}{2}}(|a_p|) + Q_{\frac{m}{2}}(|a_q|) = Q_m(-|a_p|) + Q_m(-|a_q|)$. Therefore, from statements (h) and (i) of Proposition 1 and (3), we have that $l(f^m) = Q_m(a_p) + Q_m(a_q) - Q_m(a_r) \leq Q_m(-|a_p|) + Q_m(-|a_q|) - Q_m(|a_r|) < Q_m(-(|a_p| + |a_q|)) - Q_m(|a_r|) \leq Q_m(|a_p| + |a_q|) - Q_m(a_r) < 0$. Hence (b) is proved. □

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