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A CHARACTERIZATION OF FLAT CONTACT METRIC GEOMETRY

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ABSTRACT. We prove that a closed flat contact manifold carries a nontrivial parallel vector field. As a consequence, there are closed, orientable 3-manifolds which admit no flat contact metrics.

1. INTRODUCTION

A tangent plane field $\mathcal{F}$ on a Riemannian manifold (M, g) is said to be *geodesic* if any geodesic tangent at some point is everywhere tangent to $\mathcal{F}$. The *plane field* $\mathcal{F}$ is said to be *Riemannian* if its orthogonal is geodesic. In codimension 1, the duality between Riemannian and totally geodesic foliations translates into the following: *A codimension 1 foliation $\mathcal{F}$ is geodesic if and only if its orthogonal is a 1-dimensional Riemannian foliation.*

A contact form on a $2n+1$-dimensional manifold M is a 1-form α such that the identity $\alpha \wedge (d\alpha)^n \neq 0$ holds everywhere on M. Given such a 1-form α, there is always a unique vector field ξ satisfying $\alpha(\xi) = 1$ and $i_\xi d\alpha = 0$. The vector field ξ is called the *characteristic vector field* of the contact manifold (M, α) and the corresponding 1-dimensional foliation is called a *contact flow.*

The $2n$-dimensional distribution $D(x) = \{v \in T_xM / \alpha(x)(v) = 0\}$ is called the *contact distribution.* It carries a (1,1) tensor field J such that $J^2 = -I_{2n}$, where I_{2n} is the identity $2n$ by $2n$ matrix. The tensor field J extends to all of TM by requiring $J\xi = 0$.

Also, the contact manifold (M, α) carries a non-unique Riemannian metric g adapted to J and α in the sense that the identities

$$d\alpha(X, Y) = g(X, JY) \text{ and } \alpha(X) = g(\xi, X)$$

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are satisfied for any vector fields X and Y on M. Such a metric g is called a *contact metric.*

On a compact $(2n+1)$-dimensional contact metric manifold (M, g, α), $n > 1$, the contact metric g cannot be flat. In dimension 3 however, flat contact metrics do exist, as the following example shows ([1]):

In local coordinates x_1, x_2, x_3, the standard contact form α on $\mathbf{T}^3$ is given by

$$\alpha = \frac{1}{2}(\cos x_3 dx_1 + \sin x_3 dx_2).$$

Its characteristic vector field is $\xi = 2(\cos x_3 \frac{\partial}{\partial x_1} + \sin x_3 \frac{\partial}{\partial x_2})$ and its flat contact metric is $g_{ij} = \frac{1}{4}\delta_{ij}$.

The contact form α is invariant under all translations with vector of the type $(a, b, k2\pi)$. This form α is also invariant under any screw motion (R_θ, t_μ) where R_θ is a rotation of angle θ in the (x_1, x_2) plane and t_μ is a translation of vector $\mu = (0, 0, 2\pi - \theta)$. If R_θ is a rotation which preserves a lattice containing $(a, b, 2\pi)$, then α induces a contact form on the quotient $\mathbf{T}^3 / < R_\theta, t_\mu >$, where $< R_\theta, t_\mu >$ is the cyclic group generated by (R_θ, t_μ).

Letting $X = \sin x_3 \frac{\partial}{\partial x_1} - \cos x_3 \frac{\partial}{\partial x_2}$ and $Y = \frac{\partial}{\partial x_3}$, one can check that X is the characteristic vector field of another contact form on $\mathbf{T}^3$ and $[X] \oplus [\xi]$ is a totally geodesic foliation which is preserved by the Killing vector field Y. We will show that this situation is characteristic to most, but not all closed, 3-dimensional, flat contact manifolds.

It is shown in [4] that there are six affine diffeomorphism classes of closed flat 3-manifolds. In this note, we prove that one of the six classes cannot admit a flat contact metric structure.

Theorem A. *If M is a closed flat contact manifold, then M carries a nontrivial parallel vector field. Furthermore, M is isometric to the quotient of a flat 3-torus by a finite cyclic group of isometries of order $m = 1, 2, 3, 4$ or 6.*

2. Flat 3-Dimensional Contact Geometry

On a contact metric manifold (M, α, ξ, g, J), one has the following identities involving the symmetric tensor field $h = \frac{1}{2} L_\xi J$, the covariant derivative operator ∇ and the curvature tensor R.

$$\nabla_X \xi = -JX - JhX \tag{1}$$

$$\frac{1}{2}(R(\xi, X)\xi - JR(\xi, JX)\xi) = h^2 X + J^2 X \tag{2}$$

If the curvature tensor is identically zero, then identity (2) with X a unit tangent vector leads to:

$$0 = g(hX, hX) - g(X, X) = g(hX, hX) - 1.$$

Therefore, the eigenvalues of h are ± 1 and the contact distribution D decomposes into the positive and negative eigenbundles as $D = [+1] \oplus [-1]$.

From now on, α, g, J, and ξ will denote structure tensors of a flat contact metric 3-manifold M. Let $\{X, JX, \xi\}$ be a local orthonormal basis of eigenvector fields for h, where $hX = -X$ and $hJX = JX$.

Lemma 2.1. *The distribution $[-1] \oplus [\xi] = [X] \oplus [\xi]$ is a totally geodesic Riemannian foliation, and $[JX]$ is parallel.*

PROOF. From identity (1), one has:

$$\nabla_X \xi = -JX + JX = 0. \tag{3}$$

Also

$$\begin{aligned} 0 = g(R(\xi, X)\xi, X) &= -g(\nabla_\xi \nabla_X \xi - \nabla_X \nabla_\xi \xi - \nabla_{[X,\xi]}\xi, X) && (4)\\ &= 2g([X, \xi], JX) && (5) \end{aligned}$$

So, since $g([X, \xi], \xi) = 0$, one has $[X, \xi] \in [-1]$, which means that the distribution $[-1] \oplus [\xi]$ is integrable. But then, using the fact that X is a unit vector field,

$$g([\xi, X], X) = g(\nabla_\xi X - \nabla_X \xi, X) = -g(\nabla_X \xi, X) = g(JX + JhX, X) = 0$$

so that $[\xi, X] = 0$. Thus obtaining the identity:

$$\nabla_\xi X = \nabla_X \xi = 0. \tag{6}$$

To proceed with the proof of our lemma, we need to prove that $\nabla_X X$ belongs to $[-1] \oplus [\xi]$. Actually we will prove the identity $\nabla_X X = 0$.

Since $[-1] \oplus [\xi] = [X] \oplus [\xi]$ is integrable and $[X, \xi] = 0$, we can choose local coordinates x, t, y such that $X = \frac{\partial}{\partial x}$, $\xi = \frac{\partial}{\partial t}$, and we define a local vector field $Y = \frac{\partial}{\partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial t} = \frac{\partial}{\partial y} + aX + b\xi$, where a and b are functions chosen so that $Y \in [+1] = [JX]$, that is, $Y = fJX$ for some function f.

Clearly, $[X, Y]$ and $[\xi, Y]$ are in $[X] \oplus [\xi]$, and hence due to the identities (3) and $\nabla_\xi \xi = 0$, ξ is parallel along $[X, Y]$ and along $[\xi, Y]$. We will need just the identity $\nabla_{[X,Y]}\xi = 0$ in our proof.

Using the identities (1), (3) and $R(X,Y)\xi = 0$, we obtain:

$$(7) \quad 0 = \nabla_{[X,Y]}\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi = -2\nabla_X fX = -2(f\nabla_X X + df(X)X).$$

Inner product (7) with JX leads to $g(\nabla_X X, JX) = 0$. Since $g(\nabla_X X, \xi) = 0$ and $g(\nabla_X X, X) = 0$, we deduce that $\nabla_X X = 0$. To continue with the proof of the proposition, we are going to prove that $\nabla_{JX} JX = 0$, then JX will define a Riemannian geodesic, hence Killing flow.

Inner product of (7) with X leads to $df(X) = 0$ and

$$(8) \qquad \nabla_{JY} JY = f df(X)X + f^2\nabla_X X = 0.$$

Using identities (1), (6), $Y = fJX$, (8) and the vanishing of the curvature tensor, one has

$$(9) \qquad 0 = R(Y, JY)\xi = -\nabla_{[Y,JY]}\xi = J[Y, JY] + Jh[Y, JY].$$

Inner product of (9) with JY leads to $g([Y, JY], Y) = 0$ and therefore:

$$0 = g(-f^3[JX, X] - f^2 df(JX)X + f^2 df(X)JX, JX) = -f^3 g([JX, X], JX).$$

Hence $g([JX, X], JX) = 0$ and

$$-g(X, \nabla_{JX} JX) = g(\nabla_{JX} X, JX) = g([JX, X], JX) = 0.$$

Since $g(\nabla_{JX} JX, \xi) = 0 = g(\nabla_{JX} JX, JX)$, we deduce that $\nabla_{JX} JX = 0$. In addition to that, $\nabla_\xi JX = (\nabla_\xi J)X + J\nabla_\xi X = 0$ and direct computations show that $g(\nabla_X JX, JX) = 0$, $g(\nabla_X JX, \xi) = 0$, $g(\nabla_X JX, X) = 0$. All of these identities show that JX is parallel. $\square$

Remark 1. A simple calculation shows that the identity $[\xi, JX] = -2X$ is valid. This implies that, locally, X is the characteristic vector field of another contact form, the metric dual 1-form of X, with flat contact metric g.

Also, another simple calculation shows that $[X, JX] = 2\xi$. This combined with the previous identity means that JX is a local foliate vector field for the geodesic foliation defined by ξ and X. This fact could also have been deduced from the earlier identities $[X, Y] \in [X] \oplus [\xi]$ and $[\xi, Y] \in [X] \oplus [\xi]$.

The classification of flat 3-manifolds ([4]) leads to the the following globalization of Lemma 1:

Proposition 2.2. *Let (M, α, g, J, ξ) be a closed flat contact metric manifold. Then M carries a parallel vector field preserving an orthogonal, totally geodesic, parallelizable, 2-dimensional foliation.*

PROOF. We need to prove that the vector field JX, and hence $X = -J(JX)$, is indeed global. To that end, observe that JX is well defined up to sign.

Recall from [4] that $M = \mathbf{R}^3/\Gamma = \mathbf{T}^3/\Psi$ where Γ is a uniform subgroup of the group $E(3)$ of Euclidean motions and the linear holonomy group Ψ of M consists of all A_γ where $\gamma = (A_\gamma, t_{a_\gamma}) \in \Gamma$. Now, JX lifts to $\mathbf{R}^3$ into a parallel vector field, hence JX is holonomy invariant. This implies that JX is a global vector field on M. $\square$

3. Proof of Theorem A and Remarks About Other Geometries

By Proposition 1, the vector field JX is parallel. This establishes the first statement of Theorem A. If $\psi = (A, t_a)$ is an element of Γ where $A \in SO(3)$ and t_a is a translation along the axis of symmetry of A, then A is a rotation of angle $0, \pm\frac{\pi}{3}, \pm\frac{\pi}{2}, \pm\frac{2\pi}{3}$ or $\pm\pi$. These correspond to the first five affine diffeomorphism classes of flat compact 3-manifolds in [4]. The sixth class containing flat manifolds with holonomy $\mathbf{Z}_2 \times \mathbf{Z}_2$ doesn't occur here, since its members have trivial first De Rham cohomology ([4], page 122).

Remark 2. The geodesic fields ξ, X and JX lift into 3 oriented orthogonal geodesic fields (i.e. straight lines) in $\mathbf{R}^3$. Since ξ is not parallel, it has integral lines of both rational and irrational slopes. One deduces that any closed, flat contact metric manifold admits infinitely many closed and nonclosed characteristics.

It is known that there is no closed hyperbolic contact geometry. Indeed, it is proven in [5] that no geodesic plane field can exist on a Riemannian manifold with negative curvature.

A closed contact metric manifold with constant curvature 1 is K-contact, hence Sasakian in dimension 3. Any such manifold is finitely covered by the sphere. However, we don't know of any contact metric manifold with constant positive curvature $\epsilon < 1$. We don't even know if such a manifold would necessarily admit a contact metric with constant curvature 1.

Theorem B. *Let (M, g, α) be a contact metric structure with constant positive vertical sectional curvature $\epsilon < 1$. Then M admits a contact metric structure with constant vertical sectional curvature $1 + \frac{\epsilon - 1}{\epsilon^2}$. In particular, if $0 < \epsilon < \frac{-1+\sqrt{5}}{2}$, then M admits a contact metric structure with constant negative vertical sectional curvature.*

PROOF. Let $\{E_i, \xi\}$ be an orthonormal basis of eigenvectors for the tensor field $h = \frac{1}{2}L_\xi J$. Let V_ϵ denote the vertical sectional curvature of the contact metric structure whose tensors are $g_\epsilon = \epsilon g + (\epsilon^2 - \epsilon)\alpha \otimes \alpha$, $\alpha_\epsilon = \epsilon\alpha$, $\xi_\epsilon = \frac{1}{\epsilon}\xi$ and $J_\epsilon = J$.

Notice that $h_\epsilon = \frac{1}{\epsilon}h$. Let $\pm\lambda$ be the eigenvalues of h. The inner product of identity (2) with E_i leads to:

$$-\epsilon = \lambda^2 - 1 \tag{10}$$

Similarly,

$$-V_\epsilon = g_\epsilon(h_\epsilon(\frac{1}{\sqrt{\epsilon}}E_i), h_\epsilon(\frac{1}{\sqrt{\epsilon}}E_i)) - 1 = \frac{1}{\epsilon^2}g(h(E_i), h(E_i)) - 1 = \frac{1}{\epsilon^2}\lambda^2 - 1. \tag{11}$$

Hence, combining identities (10) and (11), one obtains $V_\epsilon = 1 + \frac{\epsilon - 1}{\epsilon^2}$. It is clear that if $0 < \epsilon < \frac{-1+\sqrt{5}}{2}$, then $V_\epsilon < 0$. $\square$

4. Contact Metric Manifolds With $QJ = JQ$

Proposition 1 leads to an improvement of the classification theorem for closed contact 3-manifolds whose Ricci tensor Q commutes with the tensor J ([2]).

Taking into account our Theorem A, we may state a version of the main theorem in [2] for closed manifolds.

Theorem C. *Let M be a closed contact metric 3-dimensional manifold on which $QJ = JQ$. Then M is either Sasakian, the quotient of a flat torus by a cyclic group of isometries of order $m = 1, 2, 3, 4$ or 6, or locally isometric to a left invariant metric on the Lie group $SU(2)$. In the latter case, M has constant vertical sectional curvature $k < 1$ and constant J-sectional curvature $-k$, $k > 0$.*

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