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AN ANALOGUE OF BERNSTEIN'S THEOREM

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ABSTRACT. We prove that for a function $f(x_1, x_2)$ defined on $\mathbf{R}^2$, the graph of ∇f is a minimal surface if and only if f is harmonic or a quadratic polynomial. Using this result we prove the following classical result of Jögens: if f satisfies the Monge-Ampère equation $f_{x_1x_1}f_{x_2x_2} - f_{x_1x_2}^2 = 1$, then f must be a quadratic polynomial.

1. MAIN RESULTS

The classical Bernstein theorem says that if $f(x_1, x_2)$ is a function defined on $\mathbf{R}^2$ satisfying the minimal surface equation, then f is a linear function. In this paper we prove the following analogue of Bernstein's theorem:

Theorem 1.1. *Let $f(x_1, x_2)$ be a function defined on $\mathbf{R}^2$. Then the graph of ∇f is a minimal surface in $\mathbf{R}^2 \times \mathbf{R}^2$ if and only if f is harmonic or a quadratic polynomial.*

Recall the following result of Harvey and Lawson ([1]):

Proposition 1.2. *Let $f(x_1, \dots, x_n)$ be a real valued function defined in a connected open subset in $\mathbf{R}^n$. The graph of ∇f is a minimal submanifold of $\mathbf{R}^n \times \mathbf{R}^n$ if and only if there exists a constant θ such that f satisfies*

$$\operatorname{Im}\det(e^{i\theta}(I + i\mathrm{Hess}(f))) = 0,$$

where Im denotes the imaginary part, I is the identity matrix, and Hess(f) is the hessian matrix of f. In particular if $f(x_1, x_2)$ is a function defined on $\mathbf{R}^2$, then the graph of ∇f is a minimal surface in $\mathbf{R}^2 \times \mathbf{R}^2$ if and only if

$$\sin\theta(1 - f_{x_1x_1}f_{x_2x_2} + f_{x_1x_2}^2) + \cos\theta(f_{x_1x_1} + f_{x_2x_2}) = 0.$$

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This proposition follows from Lemma III 2.2, Theorem III 2.3 and Proposition III 2.17 in [1].

By Proposition 1.2, Theorem 1.1 is equivalent to the following theorem:

Theorem 1.3. *Let $f(x_1, x_2)$ be a function defined on $\mathbf{R^2}$. Assume f satisfies*

$$\sin\theta(1 - f_{x_1x_1}f_{x_2x_2} + f^2_{x_1x_2}) + \cos\theta(f_{x_1x_1} + f_{x_2x_2}) = 0$$

for some constant θ. If $\sin\theta \neq 0$, then f must be a quadratic polynomial; if $\sin\theta = 0$, then f is harmonic.

As a corollary we obtain the following result of Jörgens ([2]):

Corollary 1.4. *Let $f(x_1, x_2)$ be a function defined on $\mathbf{R^2}$ satisfying the Monge-Ampère equation*

$$f_{x_1x_1}f_{x_2x_2} - f^2_{x_1x_2} = 1.$$

Then f must be a quadratic polynomial.

I don't know whether we have a similar classification theorem in higher dimensional cases. Let's ask the following:

Question. Let $f(x_1, \dots, x_n)$ be a convex function defined on $\mathbf{R^n}$. Assume the graph of ∇f is a minimal submanifold in $\mathbf{R^n} \times \mathbf{R^n}$, that is, there exists a constant θ such that

$$\mathrm{Im}\ \det(e^{i\theta}(I + i\mathrm{Hess}(f))) = 0.$$

Is it true that f must be a quadratic polynomial.

We have the following results of Calabi and Flanders:

Let $f(x_1, \dots, x_n)$ be a function defined on $\mathbf{R^n}$. Assume its hessian matrix $\mathrm{Hess}(f)$ is positive definite. Then f is a quadratic polynomial if one of the following conditions holds:

(1) $\det \mathrm{Hess}(f) = 1$ and $1 \leq n \leq 5$ ([3]).

(2) $\mathrm{tr}\ (I + \mathrm{Hess}(f))^{-1}$ =constant ([4]).

(3) $\mathrm{tr}\ (\mathrm{Hess}(f))^{-1}$ =constant ([4]).

Using these results we can solve some very special cases of our question.

2. Proof of Theorem 1.1

We start with some lemmas:

Lemma 2.1. *Let $f(x_1, x_2) = (f_3(x_1, x_2), \dots, f_n(x_1, x_2))$ be a function defined on $\mathbf{R^2}$. If the graph of f is a minimal surface in $\mathbf{R^n}$, then there exists a linear transformation*

$$x_1 = u_1, x_2 = au_1 + bu_2,\ \ (b \neq 0)$$

such that (u_1, u_2) *are global isothermal parameters for the graph of* f.

This lemma is Theorem 5.1 of [5].

We also need the following obvious fact:

Lemma 2.2. *Two different conics in the* x_1x_2*-plane have at most four common points.*

Now let's prove Theorem 1.3, which is equivalent to Theorem 1.1. The second part of Theorem 1.3 is obvious. So let's assume $\sin\theta \neq 0$ and prove that f is a quadratic polynomial. It is enough to show that $f_{x_1x_1}$, $f_{x_1x_2}$ and $f_{x_2x_2}$ are constants.

By assumption the graph of ∇f is minimal. By Lemma 2.1, there exists constants a and $b \neq 0$ such that

$$(u_1, u_2) \mapsto (u_1, au_1 + bu_2, f_{x_1}, f_{x_2})$$

is an isothermal parametrization.

We have

$$\frac{\partial}{\partial u_1}(u_1, au_1 + bu_2, f_{x_1}, f_{x_2}) = (1, a, f_{x_1x_1} + af_{x_1x_2}, f_{x_1x_2} + af_{x_2x_2}),$$

$$\frac{\partial}{\partial u_2}(u_1, au_1 + bu_2, f_{x_1}, f_{x_2}) = (0, b, bf_{x_1x_2}, bf_{x_2x_2}).$$

That (u_1, u_2) are isothermal parameters is equivalent to that we have

$$1 + a^2 + (f_{x_1x_1} + af_{x_1x_2})^2 + (f_{x_1x_2} + af_{x_2x_2})^2 = b^2 + b^2 f_{x_1x_2}^2 + b^2 f_{x_2x_2}^2,$$

$$ab + (f_{x_1x_1} + af_{x_1x_2})bf_{x_1x_2} + (f_{x_1x_2} + af_{x_2x_2})bf_{x_2x_2} = 0.$$

For convenience we let $X = f_{x_1x_1}$, $Y = f_{x_1x_2}$ and $Z = f_{x_2x_2}$. The above equations can be rewritten as

$$1 + a^2 + (X + aY)^2 + (Y + aZ)^2 = b^2 + b^2Y^2 + b^2Z^2,$$

$$ab + (X + aY)bY + (Y + aZ)bZ = 0.$$

Simplifying these equations we get

$$(1 + a^2 - b^2) + (1 + a^2 - b^2)Y^2 + (1 + a^2 - b^2)Z^2 + (X^2 - Z^2) + 2aY(X + Z) = 0,$$

$$ab(1 + Y^2 + Z^2) + bY(X + Z) = 0.$$

So we have

$$(1) \qquad (1 + a^2 - b^2)(1 + Y^2 + Z^2) + (X - Z + 2aY)(X + Z) = 0$$

$$(2) \qquad a(1 + Y^2 + Z^2) + Y(X + Z) = 0$$

where in the second equation we omit the factor b since $b \neq 0$. From these equations and the fact that $1 + Y^2 + Z^2 \neq 0$, we get

$$\det \begin{pmatrix} 1 + a^2 - b^2 & X - Z + 2aY \\ a & Y \end{pmatrix} = 0,$$

that is

$$-aX + (1 - a^2 - b^2)Y + aZ = 0 \tag{3}$$

By assumption there exists a constant θ such that

$$(1 - XZ + Y^2)\sin\theta + (X + Z)\cos\theta = 0 \tag{4}$$

Note that $X + Z$ is everywhere nonzero. Indeed, if $X + Z = 0$ at some point, then $Z = -X$ at that point. Substituting this into (4) we get

$$(1 + X^2 + Y^2)\sin\theta = 0.$$

But this cannot happen since $1 + X^2 + Y^2 \neq 0$ and $\sin\theta \neq 0$.

We have the following two cases:

Case A. $a = 0$.

From equations (1) and (2), we get

$$(1 - b^2)(1 + Y^2 + Z^2) + (X - Z)(X + Z) = 0 \tag{5}$$

$$Y(X + Z) = 0 \tag{6}$$

Since $X + Z$ is everywhere nonzero, we get $Y = 0$ from (6). Substituting this into (5), we get

$$X^2 - b^2 Z^2 + (1 - b^2) = 0 \tag{7}$$

Substituting $Y = 0$ into (4) we get

$$(1 - XZ)\sin\theta + (X + Z)\cos\theta = 0 \tag{8}$$

Equations (7) and (8) define two different conics. By Lemma 2.2 they have only finitely many common solutions. So there are only finitely many possible values for X, Y and Z.

Case B. $a \neq 0$.

From (3) we get

$$X = \frac{1 - a^2 - b^2}{a} Y + Z.$$

Substituting this into (2) and (4) we get

$$(9) \qquad \frac{1-b^2}{a}Y^2 + aZ^2 + 2YZ + a = 0$$

$$(10) \quad (1+Y^2-Z^2-\frac{1-a^2-b^2}{a}YZ)\sin\theta + (\frac{1-a^2-b^2}{a}Y+2Z)\cos\theta = 0$$

Equation (9) and (10) define two different conics. By Lemma 2.2 they have only finitely many common solutions. So there are only finitely many possible values for X, Y and Z.

In any case, there are only finitely many possible values for X, Y and Z, that is, there are only finitely many possible values for $f_{x_1x_1}$, $f_{x_1x_2}$ and $f_{x_2x_2}$. Since the domain of f is connected, $f_{x_1x_1}$, $f_{x_1x_2}$ and $f_{x_2x_2}$ must be constants. Therefore f is a quadratic polynomial.

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