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ON WEIGHTED GENERALIZED LOGARITHMIC MEANS

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ABSTRACT. An integral representation of Neuman is extended and used to suggest a multidimensional weighted generalized logarithmic mean. Some inequalities are established for such means. A number of known results appear as special cases.

1. INTRODUCTION

The logarithmic mean $L(x,y)$ of a pair of positive numbers x and y, defined by

$$L(x,y) = \begin{cases} \dfrac{x-y}{\ln x - \ln y} & , x \neq y \\ x & , x = y, \end{cases}$$

has proved a seminal concept (see, for example, Bullen, Mitrinović and Vasić [3], Carlson [5]). It has been given the integral representation

$$L(x,y) = \left[\int_0^1 \frac{dt}{tx + (1-t)y} \right]^{-1} \tag{1.1}$$

(Carlson [4]). Neuman [3] found a further representation

$$L(x,y) = \int_0^1 x^t y^{1-t} dt \tag{1.2}$$

and made extensive use of it to develop a variety of extensions of known results. These include a weighted logarithmic mean of several numbers.

Alzer [1, 2] has considered an interesting form of generalized logarithmic mean that is a special case of the Stolarsky mean. Define

$$F_r(x,y) = \begin{cases} \dfrac{r}{r+1}\dfrac{x^{r+1}-y^{r+1}}{x^r-y^r} & ,r \neq 0,-1, x \neq y \\ \dfrac{x-y}{\ln x - \ln y} & ,r=0,\ x \neq y \\ xy\dfrac{\ln x - \ln y}{x-y} & ,r=-1,\ x\neq y \\ x & ,\ x=y, \end{cases}$$

so that $L(x,y) = F_0(x,y)$.

In this article we present an integral representation for Alzer's generalized logarithmic mean that includes (1.3) in the case $r = 0$. This is used to develop a multidimensional weighted generalized logarithmic mean that subsumes Neuman's multidimensional weighted logarithmic mean as the case $r = 0$. This turns out to be a unifying concept from which a number of known results fall out as special cases.

Our starting point is as follows. The argument of the integral in (1.3) is a classical weighted geometric mean. Now the power mean of order r and weights t and $1-t$ (for $t \in [0,1]$) of positive numbers x, y is defined generally by

$$M_r(x,y;t) = \begin{cases} (tx^r + (1-t)y^r)^{1/r} & ,r \neq 0 \\ x^t y^{1-t} & ,r=0. \end{cases}$$

Set $M_r(t) := M_r(x,y;t)$. Then one can verify readily that

$$F_r(x,y) = \int_0^1 M_r(t)dt. \tag{1.3}$$

We proceed from this convenient integral representation.

2. Multidimensional Weighted Generalized Logarithmic Means

Define

$$E_{n-1} = \left\{ (u_1,\dots,u_{n-1}) : u_i \geq 0\ (1 \leq i \leq n-1),\ \sum_{j=1}^{n-1} u_j \leq 1 \right\}$$

and put $u_n = 1 - u_1 - \dots - u_{n-1}$. Let μ be a probability measure on E_{n-1}. We write x to represent an n-tuple $(x_1,\dots,x_n)$ of positive real numbers.

The power mean of order r of positive numbers $x_1, \dots, x_n$ with weights $u_1, \dots, u_n$ is defined by

$$M_r(u) = \quad M_r(x;u) = \begin{cases} \left(\sum_{i=1}^{n} u_i x_i^r\right)^{1/r} & , r \neq 0 \\ \prod_{i=1}^{n} x_i^{u_i} & , r = 0. \end{cases}$$

We shall define the weighted generalized logarithmic mean of positive numbers $x_1, \dots, x_n$ by

$$\mathcal{F}_r(\mu; x) = \int_{E_{n-1}} M_r(u) d\mu(u). \tag{2.1}$$

For $r = 0$ this reduces to the generalized weighted logarithmic mean

$$\mathcal{L}(\mu; x) = \int_{E_{n-1}} \prod_{i=1}^{n} x_i^{u_i} d\mu(u)$$

defined in [6].

The close correspondence between (1.3) and (1.4) enables the following results to be deduced as simple extensions of results from [6].

$$\min\{x_i; 1 \leq i \leq n\} \leq \mathcal{F}_r(\mu; x) \leq \max\{x_i; 1 \leq i \leq n\},$$

$$\mathcal{F}_r(\mu; x, \dots, x) = x \ (x > 0)$$

and

$$\mathcal{F}_r(\mu; \alpha x) = \alpha \mathcal{F}_r(\mu; x) \ (\alpha > 0), \tag{2.2}$$

where $\alpha x = (\alpha x_1, \dots, \alpha x_n)$. In association with this we have also

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \mathcal{F}_r(\mu; x) = \mathcal{F}_r(\mu; x),$$

which is Euler's equation for a homogenous function with order of homogeneity equal to unity.

The following result generalizes a result of Yang and Cao [7] that $F_r(x, y)$ is nondecreasing in r.

Theorem 2.1. *The means $\mathcal{F}_r(\mu; x)$ are nondecreasing in r.*

PROOF. If is well known that the power mean $M_r(x;\mu)$ is nondecreasing in r. By (2.1) the same is valid for $\mathcal{F}_r(\mu; x)$. □

Remark. Denote by $w_i := \int_{E_{n-1}} u_i d\mu(u)$ $(1 \le i \le n)$ the ith weight associated with the probability measure μ on E_{n-1}. Clearly $w_i > 0$ $(1 \le i \le n)$ and $w_1 + \cdots + w_n = 1$. From the inequality

$$\mathcal{F}_0(\mu; x) \le \mathcal{F}_1(\mu; x)$$

we have the result

$$\mathcal{L}(\mu; x) \le \sum_{i=1}^{n} w_i x_i$$

given in [6].

3. Additive And Multiplicative Properties

Theorem 3.1. *Let α, β be positive numbers with $\alpha + \beta = 1$ and suppose that $r \ge 0$. Then*

$$\mathcal{F}_r(\mu; x^\alpha y^\beta) \le \mathcal{F}_r(\mu; x)^\alpha \mathcal{F}_r(\mu; y)^\beta,$$

where $x^\alpha y^\beta = (x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta)$.

PROOF. We have for $r > 0$ from the integral Hölder inequality that

$$\begin{aligned}
\mathcal{F}_r(\mu; x^\alpha y^\beta) &= \int_{E_{n-1}} M_r(x^\alpha y^\beta; u) d\mu(u) \\
&= \int_{E_{n-1}} \left(\sum_{i=1}^{n} u_i \left(x_i^\alpha y_i^\beta \right)^r \right)^{1/r} d\mu(u) \\
&\le \int_{E_{n-1}} \left(\sum_{i=1}^{n} u_i x_i^r \right)^{\alpha/r} \left(\sum_{i=1}^{n} u_i y_i^r \right)^{\beta/r} d\mu(u) \\
&\le \left(\int_{E_{n-1}} \left(\sum_{i=1}^{n} u_i x_i^r \right)^{1/r} d\mu \right)^{\alpha} \left(\int_{E_{n-1}} \left(\sum_{i=1}^{n} u_i y_i^r \right)^{1/r} d\mu(u) \right)^{\beta} \\
&= \mathcal{F}_r(\mu; x)^\alpha \mathcal{F}_r(\mu; y)^\beta.
\end{aligned}$$

For $r \to 0$ this gives the result

$$\mathcal{L}(\mu; x^\alpha y^\beta) \le \mathcal{L}(\mu; x)^\alpha \mathcal{L}(\mu; y)^\beta$$

in [3]. □

Theorem 3.2. *For each real number r we have*

$$\mathcal{F}_r(\mu; x^\alpha y^\beta) \leq \mathcal{F}_r(\mu; \alpha x + \beta y),$$

where $\alpha x + \beta y = (\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n)$.

PROOF. We have for $r \neq 0$ by the arithmetic–geometric inequality that

$$\begin{aligned}
\mathcal{F}_r(\mu; x^\alpha y^\beta) &= \int_{E_{n-1}} M_r(x^\alpha y^\beta; u) d\mu(u) \\
&= \int_{E_{n-1}} \left(\sum_{i=1}^n u_i \left(x_i^\alpha y_i^\beta \right)^r \right)^{1/r} d\mu(u) \\
&\leq \int_{E_{n-1}} \left(\sum_{i=1}^n u_i \left(\alpha x_i + \beta y_i \right)^r \right)^{1/r} d\mu(u) \\
&= \mathcal{F}_r(\mu; \alpha x + \beta y).
\end{aligned}$$

Letting $r \to 0$ gives the result

$$\mathcal{L}(\mu; x^\alpha y^\beta) \leq \mathcal{L}(\mu; \alpha x + \beta y)$$

for $r = 0$. □

Alzer [2] has shown that

$$\begin{aligned}
F_r(x_1 + y_1, x_2 + y_2) &\leq F_r(x_1, x_2) + F_r(y_1, y_2) \quad \text{if} \quad r \geq 1, \\
F_r(x_1 + y_1, x_2 + y_2) &\geq F_r(x_1, x_2) + F_r(y_1, y_2) \quad \text{if} \quad r \leq 1.
\end{aligned} \tag{3.1}$$

We give a generalization of this result. In the case of classical means F_r our proof in fact provides a shorter derivation of (3.1).

Theorem 3.3. *We have for* $r \geq 1$ *that*

$$\mathcal{F}_r(\mu; x + y) \leq \mathcal{F}_r(\mu; x) + \mathcal{F}_r(\mu; y), \tag{3.2}$$

while for $r \leq 1$ *the inequality is reversed.*

PROOF. For $r \geq 1$, we have by the discrete Minkowski inequality that

$$\begin{aligned}
\mathcal{F}_r(\mu; x + y) &= \int_{E_n-1} \left(\sum_{i=1}^n u_i \left(x_i + y_i \right)^r \right)^{1/r} d\mu(u) \\
&\leq \int_{E_n-1} \left(\sum_{i=1}^n u_i x_i^r \right)^{1/r} d\mu(u) + \int_{E_{n-1}} \left(\sum_{i=1}^n u_i x_i^r \right)^{1/r} d\mu(u) \\
&= \mathcal{F}_r(\mu; x) + \mathcal{F}(\mu; y).
\end{aligned}$$

For $r < 0$ the reverse result applies by virtue of the corresponding Minkowski result. □

Remark. The case $r = 0$ gives the interesting result

$$\mathcal{L}(\mu; x+y) \geq \mathcal{L}(\mu; x) + \mathcal{L}(\mu; y). \tag{3.3}$$

4. Means Using Dirichlet Measure

Neuman devotes considerable attention to the case where the measure μ is Dirichlet measure μ_b, which for $n \geq 2$ is given by

$$\mu_b(u) = \prod_{i=1}^{n} u_i^{b_i - 1}/B(b),$$

where $b = (b_1, \dots, b_n)$ is an n–tuple of positive numbers and B stands for the multivariate beta function. In particular he noted that

$$\mathcal{L}(\mu_b; x) = S(b; \ln x) := S(b; \ln x_1, \ln x_2, \dots, \ln x_n),$$

where S is the confluent hypergeometric function. For $z = (z_1, \dots, z_n) \in C^n$, this function has an integral representation

$$S(b; z) = \int_{E_{n-1}} \exp\left(\sum_{i=1}^{n} u_i z_i\right) d\mu_b(u).$$

If $b = (1, \dots, 1)$ and if $x_i \neq x_j$ for all $i \neq j$, then

$$\mathcal{L}(\mu_b; x) = (n-1)! \sum_{i=1}^{n} \left[x_i \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{n} \ln(x_i/x_j) \right].$$

So in this case, (3.3) becomes

$$\begin{aligned} &\sum_{i=1}^{n} \left[(x_i + y_i) \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{n} \ln\Big((x_i + y_i)\Big/(x_j + y_j)\Big) \right] \\ &\qquad \geq \sum_{i=1}^{n} \left[x_i \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{n} \ln(x_i/x_j) \right] + \sum_{i=1}^{n} \left[y_i \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{n} \ln(y_i/y_j) \right]. \end{aligned}$$

Remark. The map: $x \to \mathcal{F}_r(\mu; x)$ is convex for $r \geq 1$ and concave for $r \leq 1$. Indeed, let $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then by (3.2) and (2.2) for $r \geq 1$, we have

$$\begin{aligned} \mathcal{F}_r(\mu; \alpha x + \beta y) &\leq \mathcal{F}_r(\mu; \alpha x) + \mathcal{F}_r(\mu; \beta y) \\ &= \alpha \mathcal{F}_r(\mu; x) + \beta \mathcal{F}_r(\mu; y). \end{aligned}$$

The reverse inequality applies for $r \leq 1$.

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