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SUMS OF STRONGLY IRREDUCIBLE OPERATORS

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1. INTRODUCTION

A bounded linear operator T on a complex Hilbert space H is *irreducible* if it has no reducing subspace other than the trivial ones $\{0\}$ and H; it is *strongly irreducible* if every operator similar to T is irreducible. Equivalently, T is irreducible (resp. strongly irreducible) if the only projections (resp. idempotent operators) commuting with T are 0 and I. Since (strongly) irreducible operators can act only on a separable space, in the following we will restrict ourselves to operators on such spaces.

Strongly irreducible operators were first considered by Gilfeather [16]. Jordan blocks

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix},$$

the simple unilateral shift,

$$\begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{bmatrix} \text{ on } \ell^2,$$

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Cowen-Douglas operators of index one (to be specified below) and their adjoints are some examples of them.

For a nonempty, bounded, open and connected subset Ω of C and $n \geq 1$, $\mathcal{B}_n(\Omega)$ denotes the class of operators T on H, called *Cowen-Douglas operators* of index n, satisfying $(T-\lambda)H = H$ and dim $\ker(T-\lambda) = n$ for all λ in Ω, and $\vee\{\ker(T-\lambda) : \lambda \in \Omega\} = H$. Such operators were first defined and studied in [6]. A typical operator in the class $\mathcal{B}_1(D)$ ($D = \{\lambda \in C : |\lambda| < 1\}$) is the backward shift S^*. These operators relate to our present situation in that operators in any $\mathcal{B}_1(\Omega)$ are always strongly irreducible (cf. [13, Theorem 2.2]). Note that in the definition of $\mathcal{B}_n(\Omega)$, the condition $\vee\{\ker(T-\lambda) : \lambda \in \Omega\} = H$ can be replaced by the weaker one $\vee\{\ker(T-\lambda)^k : \lambda \in \Omega, k \geq 1\} = H$. More generally, an operator T (on an infinite-dimensional space) satisfying $\vee\{\ker(T-\lambda)^k : \lambda \in C, k \geq 1\} = H$ is said to be *triangular*. This is equivalent to requiring that it be unitarily equivalent to an operator of the form

$$\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & * \\ & 0 & \ddots \end{bmatrix}.$$

An interesting result proved by Radjavi [20] is that every operator is the sum of two irreducible operators. In this paper, we consider the more refined problem whether every operator is even the sum of two strongly irreducible ones. This latter problem is much more intricate. So far we are only able to show that for certain special-class operators it does have a positive answer. These operators include operators on finite-dimensional spaces, triangular operators, multicyclic operators and compact operators. In general, we prove that every operator is the sum of three strongly irreducible operators. Many of these results are proved via exploiting special matrix representations of the operators under consideration, expressing them as sums of affine functions of perturbations of the backward shift and showing such operators are in the Cowen-Douglas class of index one. The result on the sum of three strongly irreducible operators will be proved in Section 2 below; those on the sum of two will be discussed in Section 3.

For Hilbert spaces H and $K, \mathcal{B}(H,K)$ denotes the algebra of all operators from H to K and $\mathcal{B}(H,H)$ is abbreviated as $\mathcal{B}(H)$. If $T \in \mathcal{B}(H)$, then $\sigma(T), \sigma_p(T), \sigma_e(T)$, $\sigma_{\ell e}(T)$ and $\sigma_{re}(T)$ denote the spectrum, point spectrum (=set of eigenvalues), essential spectrum, left essential spectrum and right essential spectrum of T, respectively.

2. Sum of three

The main result of this section is the following

Theorem 2.1. *Every operator on an infinite-dimensional space is the sum of three strongly irreducible operators.*

This we prove through a series of lemmas, the first of which was noted before in [1,p.131] (cf. also [14, Lemma 2.2]).

Lemma 2.2. *Every operator is unitarily equivalent to an operator of the form*

$$(*)\qquad \begin{bmatrix} A_1 & B_1 & & \\ C_1 & A_2 & B_2 & 0 \\ & C_2 & A_3 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}$$

on $H_1 \oplus H_2 \oplus \cdots$, where the H_n's are all finite-dimensional.

It is known that if $[t_{ij}]_{i,j=1}^{\infty}$ is an arbitrary matrix representation of an operator T, then its upper triangular part, the matrix $[t'_{ij}]$, where

$$t'_{ij} = \begin{cases} t_{ij} & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases},$$

in general may not define a (bounded) operator even when T is Hermitian and compact. One example is the operator $T = \sum_{n=2}^{\infty} \oplus T_n$, where

$$T_n = i\frac{\log\log n}{\log n} \begin{bmatrix} 0 & 1 & \frac{1}{2} & \cdots & \frac{1}{n-1} \\ -1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & 1 \\ -\frac{1}{n-1} & \cdots & & -1 & 0 \end{bmatrix}$$

(cf. [7, pp.39-40]). This should be contrasted with the next result.

Lemma 2.3. *For any operator T, there is a matrix representation $[t_{ij}]$ whose upper triangular part defines a (bounded) operator.*

PROOF. Let T be represented in the form $(*)$. It is obvious that

$$\begin{bmatrix} 0 & & & \\ C_1 & 0 & 0 & \\ & C_2 & \ddots & \\ & 0 & \ddots & \end{bmatrix}$$

is bounded, and hence the same is true for

$$T' = \begin{bmatrix} A_1 & B_1 & & \\ & A_2 & B_2 & 0 \\ & & A_3 & \ddots \\ & 0 & & \end{bmatrix}.$$

Since each A_n is acting on a finite-dimensional space H_n, there is a unitary operator U_n on H_n such that $U_n^* A_n U_n$ is triangular. Let U be the unitary operator $\sum_n \oplus U_n$. Since the (bounded) operator $U^*T'U$ is exactly the upper triangular part of U^*TU, our assertion follows. □

Since operators in $\mathcal{B}_1(\Omega)$ are strongly irreducible, our strategy in proving Theorem 2.1 is to write, via Lemma 2.3, an arbitrary operator as the sum of three operators each of which is a multiple of an operator very close to S^* or S, and then to show that these latter operators are in $\mathcal{B}_1(D)$ or $\mathcal{B}_1(D)^*$ (the set of adjoints of operators in $\mathcal{B}_1(D)$).

Lemma 2.4. *Let $0 < r < 1$. If $T = S^* + X$, where S^* is the backward shift and X is an operator with $\| X \| \leq 1 - r$, then $T - \lambda$ is surjective and dim ker $(T - \lambda) = 1$ for any $\lambda, |\lambda| < r$.*

PROOF. Since

$$\| S(X - \lambda) \| \leq \| X \| + |\lambda| < (1 - r) + r = 1$$

for $|\lambda| < r$, we infer that $1 + S(X - \lambda)$ is invertible. Hence $T - \lambda = S^*(1 + S(X - \lambda))$ is surjective.

To prove the assertion on the kernel, we first show that $\gamma(S^* - \lambda)$, the reduced minimum modulus, and $m_e(S^* - \lambda)$, the essential minimum modulus, of $S^* - \lambda$ are both equal to $1 - |\lambda|$ for any $|\lambda| < r$. Recall that $\gamma(A) = \inf(\sigma(|A|) \setminus \{0\})$ and $m_e(A) = \inf \sigma_e(|A|)$ for any operator A, where $|A| = (A^*A)^{\frac{1}{2}}$ (cf. [5, Section XI.6] and [2] for their basic properties). Since

$$\begin{aligned} \sigma((S-\bar{\lambda})^*(S-\bar{\lambda})) &= \sigma(1-2\ \mathrm{Re}(\lambda S)+|\lambda|^2) \\ =1+|\lambda|^2-2|\lambda|\sigma(\mathrm{Re}\ S) &= [(1-|\lambda|)^2,(1+|\lambda|)^2], \end{aligned}$$

we have indeed $\gamma(S^*-\lambda)=m_e(S^*-\lambda)=1-|\lambda|$. Thus if $\| X \| \leq 1-r < 1-|\lambda|=\gamma(S^*-\lambda)=m_e(S^*-\lambda)$, then dim $\ker(T-\lambda)$ =dim $\ker(S^*-\lambda)=1$ (cf. [5, Proposition XI.6.6] and [21, Corollary 4.5]), completing the proof. □

Note that in the situation of the preceding lemma, S^*+X is in general not a Cowen-Douglas operator (cf. [17, p.214]). We need some extra condition on X to guarantee this. This is what the next two lemmas do.

Lemma 2.5. *If X is a block triangular operator of the form*

$$X=\begin{bmatrix} X_{11} & X_{12} & & * \\ & X_{22} & X_{23} & \\ 0 & & \ddots & \ddots \end{bmatrix}$$

on $\ell^2=H_1\oplus H_2\oplus\cdots$, where the H_j's are all finite-dimensional with dimensions bounded, then for any $r, 0<r<1$, there is $\delta>0$ such that $S^*+\delta_1 X$ is in $\mathcal{B}_1(D_r)$ for any $\delta_1, 0<\delta_1\leq\delta$. Here $D_r=\{\lambda\in C:|\lambda|<r\}$.

Proof. For $n\geq 1$, let J_n denote the $n\times n$ Jordan block

$$\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & 1 \\ 0 & & & 0 \end{bmatrix},$$

and for $j\geq 1$, let n_j =dim H_j. Since the mapping taking a finite-dimensional operator to its spectrum is continuous and the n_j's are bounded, for $0<r<1$ we can choose $\delta, 0<\delta<(1-r)/\parallel X \parallel$, such that the spectrum $\sigma(J_{n_j}+\delta_1 X_{jj})$ is contained in D_r for every $j\geq 1$ and $\delta_1, 0<\delta_1\leq\delta$. Since

$$T\equiv S^*+\delta_1 X=\begin{bmatrix} J_{n_1}+\delta_1 X_{11} & & * \\ & J_{n_2}+\delta_1 X_{22} & \\ 0 & & \ddots \end{bmatrix}$$

is unitarily equivalent to a triangular matrix with all diagonal entries in D_r, we have $\vee\{\ker(T-\lambda)^k:\lambda\in D_r, k\geq 1\}=\ell^2$. This together with the assertions of Lemma 2.4 implies that T is of class $\mathcal{B}_1(D_r)$. □

Corollary 2.6. *If*

$$X = \begin{bmatrix} x_{11} & x_{12} & & * \\ & x_{22} & x_{23} & \\ 0 & & \ddots & \ddots \end{bmatrix}$$

on ℓ^2, *then for any* $r, 0 < r < 1$, *there is* $\delta > 0$ *such that* $S^* + \delta_1 X$ *is in* $\mathcal{B}_1(D_r)$ *for any* δ_1, $0 < \delta_1 \leq \delta$.

In the next lemma, we consider the unilateral shift S acting on $H^2 : (Sf)(z) = zf(z)$ for $f \in H^2$.

Lemma 2.7. *If* ϕ *is a function analytic on a neighborhood of* $\overline{D}$, *then for any* $r, 0 < r < 1$, *there is* $\delta > 0$ *such that* $S^* + \delta_1\phi(S)$ *is in* $\mathcal{B}_1(D_r)$ *for any* $0 < \delta_1 \leq \delta$.

PROOF. Assume that ϕ is analytic on D_{r_1}, where $r_1 > 1$. Fix $r_2, 1 < r_2 < r_1$, and let $M = \sup\{|\phi(z)| : |z| \leq r_2\}$. We tentatively require that $0 < \delta < (1-r)/M$. If $T = S^* + \delta_1\phi(S)$, where $0 < \delta_1 \leq \delta$, then, by Lemma 2.4, $T - \lambda$ is surjective and dim ker $(T-\lambda) = 1$ for any λ in D_r. Hence to complete the proof, we need only show that for sufficiently small δ, the condition $\vee\{\ker T^k : k \geq 1\} = H^2$ holds. Let $Y = 1 + \delta_1 S\phi(S)$. Then, as shown in the proof of Lemma 2.4, Y is invertible, and also

$$\begin{aligned} & T(Y^{-1}S) = (S^* + \delta_1\phi(S))Y^{-1}S \\ = \ & S^*(1 + \delta_1 S\phi(S))Y^{-1}S = 1. \end{aligned}$$

If e denotes the function in H^2 which is constant 1, then

$$\begin{aligned} & T^kY^{-1}(SY^{-1})^{k-1}e = T^k(Y^{-1}S)^{k-1}Y^{-1}e \\ = \ & TY^{-1}e = S^*e = 0 \end{aligned}$$

and hence $Y^{-1}(SY^{-1})^{k-1}e$ is in ker Y^k for any $k \geq 1$. Thus for our purpose it suffices to show that $\vee\{(SY^{-1})^k e : k \geq 0\} = H^2$. Since $SY^{-1} = \psi(S)$, where $\psi(z) = z/(1 + \delta_1 z\phi(z))$ for $z \in D_{r_2}$, this is the same as showing that $\vee\{\psi^k : k \geq 0\} = H^2$.

To prove this, we check that for very small $\delta > 0, \psi$ is univalent on $\bar{D}$. Indeed, assume that $\psi(z_1) = \psi(z_2)$ for z_1 and z_2 in $\overline{D}$. Then

$$z_1 + \delta_1 z_1 z_2 \phi(z_2) = z_2 + \delta_1 z_1 z_2 \phi(z_1),$$

which implies that

$$z_1 - z_2 = \delta_1 z_1 z_2(\phi(z_1) - \phi(z_2)).$$

Let

$$L = \sup\{|\phi(w_1) - \phi(w_2)|/|w_1 - w_2| : w_1, w_2 \in D_{r_2}, w_1 \neq w_2\} < \infty.$$

If $z_1 \neq z_2$, then

$$\frac{1}{\delta_1} = |z_1| \cdot |z_2| \; |\phi(z_1) - \phi(z_2)|/|z_1 - z_2| \leq L.$$

Hence if we choose δ_1 smaller than $1/L$, then ψ must be univalent on $\overline{D}$. We conclude from Theorem 4 and Proposition 1 of [8] that $\vee\{\psi^k : k \geq 0\} = H^2$, completing the proof. □

Proof of Theorem 2.1. By Lemma 2.3, every operator can be expressed as a sum $A + B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & & * \\ & a_{22} & a_{23} & \\ 0 & & \ddots & \ddots \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & & & \\ b_{21} & 0 & & 0 \\ b_{31} & b_{32} & 0 & \\ * & \ddots & \ddots & \ddots \end{bmatrix}$$

are bounded operators. Fix some r, $0 < r < 1$. Corollary 2.6 and Lemma 2.7 imply that there is $\delta > 0$ such that $S^* + \delta^2 A, S^* + \delta B^*$ and $S^* + \delta S$ are all in $\mathcal{B}_1(D_r)$. Hence

$$T = (1/\delta^2)(S^* + \delta^2 A) + (1/\delta)(S + \delta B) + (-1/\delta^2)(S^* + \delta S)$$

expresses T as the sum of three strongly irreducible operators by [13, Theorem 2.2].

3. Sum of two

In this section, we prove that operators in certain special classes can be written as the sum of two strongly irreducible operators. We start with the finite-dimensional case.

Proposition 3.1. *Every operator on a finite-dimensional space is the sum of two strongly irreducible operators.*

Proof. Let T be any operator on an n-dimensional space, and let $A = T - \left(\frac{1}{n} \operatorname{tr} T\right) I$. Then A has trace 0. We consider two separate cases.

If rank $A \leq 1$, then A is unitarily equivalent to a matrix of the form

$$\begin{bmatrix} 0 & a & 0 & \cdots & 0 \\ & 0 & & & \vdots \\ & & \ddots & & \\ 0 & & & & 0 \end{bmatrix}.$$

Let $b = \frac{1}{2n}$ tr T and c be any scalar $\neq 0, a$. Then T is unitarily equivalent to the sum of

$$\begin{bmatrix} b & c & & 0 \\ & b & \ddots & \\ & & \ddots & c \\ 0 & & & b \end{bmatrix} \text{ and } \begin{bmatrix} b & a-c & & 0 \\ & b & -c & \\ & & \ddots & -c \\ 0 & & & b \end{bmatrix}.$$

Since these latter two matrices are both similar to a Jordan block, they are strongly irreducible. This shows that in this case, T is the sum of two strongly irreducible operators.

On the other hand, if rank $A > 1$, then, by [4, Theorem 2.f], A is similar to a matrix $[a_{ij}]$, whose zero entries are exactly its diagonal elements. Hence T is similar to the sum of

$$\begin{bmatrix} b & a_{12} & \cdots & a_{1n} \\ & b & \ddots & \vdots \\ & & \ddots & a_{n-1n} \\ 0 & & & b \end{bmatrix} \text{ and } \begin{bmatrix} b & & & 0 \\ a_{21} & b & & \\ \vdots & \ddots & \ddots & \\ a_{n1} & \cdots & a_{n\,n-1} & b \end{bmatrix}.$$

As above, these latter two matrices are strongly irreducible. This completes the proof. □

For the remainder of this paper, we consider only operators on an infinite-dimensional separable space. Making use of some of the tools developed in Section 2, we can prove the following sum-of-two-strongly-irreducible results.

Theorem 3.2. *If an operator T has the matrix representation $[a_{ij}]_{i,j=1}^{\infty}$ with $a_{ij} = 0$ for all pairs (i, j) satisfying $i - j > n$, where $n \geq 0$, then T is the sum of two strongly irreducible operators. Moreover, if T is compact, then for any z_0 in C and arbitrarily large $R > 0$, the two strongly irreducible operators can be chosen to have spectrum equal to $\{z \in C : |z - z_0| \leq R\}$.*

PROOF. We may assume that $n \geq 1$. By the hypothesis, T can be expressed as the sum of operators

$$A = \begin{bmatrix} A_1 & & \\ & A_2 & * \\ 0 & & \ddots \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & & & \\ & B_1 & & 0 \\ & & B_2 & \\ & 0 & & \ddots \end{bmatrix},$$

where the $A_j's$ and $B_j's$ are all of size $2n$ and the diagonal 0 of B is of size n. Lemma 2.5 implies that for any fixed $r, 0 < r < 1$, there is $\delta > 0$ such that $S^* + \delta A$ and $S^* - \delta B$ are both in $\mathcal{B}_1(D_r)$. Hence

$$T = (1/\delta)(S^* + \delta A) + (-1/\delta)(S^* - \delta B)$$

is the sum of two strongly irreducible operators.

If T is compact, then so are A and B. Since S has no eigenvalue, $\sigma(S + \delta A^*)$ consists of points of $\overline{D}$ together with at most countably many isolated points (cf. [12, Corollary of Theorem 3.3]). But the strong irreducibility of $S + \delta A^*$ implies that its spectrum is connected. We conclude that $\sigma(S + \delta A^*) = \overline{D}$ and hence $\sigma((1/\delta)(S^* + \delta A)) = \overline{D_{1/\delta}}$ for arbitrarily small δ. The same arguments apply to $\sigma((-1/\delta)(S^* - \delta B))$, and our assertion follows. □

Recall that operator T on H is *multicyclic* if there are finitely many vectors $x_1, \cdots, x_n$ in H such that $\vee\{T^k x_j : k \geq 0, 1 \leq j \leq n\} = H$. If T is multicyclic, then T has a matrix representation of the form given in Theorem 3.2. Thus follows the next corollary.

Corollary 3.3. *Any multicyclic operator is the sum of two strongly irreducible operators.*

Corollary 3.4. *If T has the matrix representation $[a_{ij}]_{i,j=-\infty}^{\infty}$ with $a_{ij} = 0$ for $|i \pm j| > n$, where $n \geq 0$, then T is the sum of two strongly irreducible operators.*

PROOF. Let $\{e_k\}_{k=-\infty}^{\infty}$ be the orthonormal basis of the underlying space with respect to which T has the asserted matrix. Rearranging $\{e_k\}$ into the (ordered) basis $\{e_0, e_1, e_{-1}, e_2, e_{-2}, \cdots\}$ results in a matrix representation $[b_{ij}]_{i,j=1}^{\infty}$ for T with $b_{ij} = 0$ for $i - j > 2n + 1$. Our assertion then follows from Theorem 3.2. □

Corollary 3.5. *Any bilateral weighted shift is the sum of two strongly irreducible operators.*

PROOF. This follows from the case $n = 1$ in Corollary 3.4. □

If T is a triangular or an analytic Toeplitz operator, then, by Theorem 3.2, T is the sum of two strongly irreducible operators. The next proposition says something more about these two latter operators.

Proposition 3.6. *Any triangular (resp. analytic Toeplitz) operator is the sum of two strongly irreducible triangular (resp. analytic Toeplitz) operators.*

PROOF. If $T = [a_{ij}]_{i,j=1}^{\infty}$ is such that $a_{ij} = 0$ for $i > j$, then for a fixed $r, 0 < r < 1$, let $\delta > 0$ be such that $S^* + \delta T$ is in $\mathcal{B}_1(D_r)$ by Lemma 2.6. Then $T = (1/\delta)(S^* + \delta T) + (-1/\delta)S^*$ is the sum of two strongly irreducible triangular operators.

The same arguments above applied to the adjoint of an analytic Toeplitz operator yield our second assertion. □

Proposition 3.7. *If $T = A + B$, where $A = [a_{ij}]_{i,j=1}^{\infty}$ and $B = [b_{ij}]_{i,j=1}^{\infty}$ are such that $a_{ij} = 0$ for $i > j$ and $\sum_{i,j} |b_{ij}| < \infty$, then T is the sum of two strongly irreducible operators.*

PROOF. As before, for a fixed $r, 0 < r < 1$, the operator $S^* + \delta A$ is in $\mathcal{B}_1(D_r)$ for all sufficiently small $\delta > 0$. Since $\sum_{i,j} |b_{ij}| < \infty$, the operator $S - \delta B^*$ is similar to S for very small δ (cf. [19, Corollary 2.7] or [3, Theorem A.4]; the idea of the similarity of S with its small perturbation originates from a paper of Freeman [15]). Hence $T = (1/\delta)(S^* + \delta A) + (-1/\delta)(S^* - \delta B)$ is the sum of two strongly irreducible operators. □

It follows from above that any operator having a matrix representation $[t_{ij}]_{i,j=1}^{\infty}$ with summable entries is the sum of two strongly irreducible operators. Note that such an operator must be of trace class. Indeed, in this case we have

$$\sum_j \Big(\sum_i |t_{ij}|^2\Big)^{\frac{1}{2}} \le \sum_j \Big(\sum_i |t_{ij}|\Big) < \infty$$

and our assertion follows from [10, Lemma XI.9.32]. The next result, our final one, is much more general than this.

Theorem 3.8. *Any compact operator is the sum of two strongly irreducible operators.*

For its proof, we need the following two lemmas. If A and B are operators on H and K, respectively, denote by $\tau_{A,B}$ the operator

$$\tau_{A,B}(X) = AX - XB \text{ for } X \text{ in } \mathcal{B}(K,H).$$

Lemma 3.9. *If*

$$T = \begin{bmatrix} T_1 & T_{12} & & \\ & T_2 & T_{23} & * \\ 0 & & \ddots & \ddots \end{bmatrix}$$

satisfies (1) T_n is strongly irreducible, (2) $\tau_{T_n,T_{n+1}}$ is one-to-one, (3) $T_{n,n+1} \notin$ ran $\tau_{T_n,T_{n+1}}$ and (4) τ_{B_n,A_n} is one-to-one, where

$$A_n = \begin{bmatrix} T_1 & T_{12} & \cdots & T_{1n} \\ & \ddots & \ddots & \vdots \\ & & & T_{n-1\ n} \\ 0 & & & T_n \end{bmatrix} \text{ and } B_n = \begin{bmatrix} T_{n+1} & T_{n+1\ n+2} & \ddots \\ & T_{n+2} & \ddots \\ 0 & & \ddots \end{bmatrix},$$

for every $n \geq 1$, then T is strongly irreducible.

PROOF. Let $E = [E_{ij}]_{i,j=1}^{\infty}$ be an idempotent operator commuting with T. From (4), we derive that $E_{ij} = 0$ for all $i > j$. Hence each E_{ii} is an idempotent commuting with T_i. Since the latter is strongly irreducible, we obtain $E_{ii} = 0$ or 1. Assume that $E_{11} = 0$. From $ET = TE$, we derive that $E_{12}T_2 = T_1E_{12} + T_{12}E_{22}$. Hence we have $E_{22} = 0$ from (3) and thus $E_{12} = 0$ from (2). In a similar fashion, we obtain $E_{ii} = 0$ for all i and $E_{ij} = 0$ for all $i < j$. In this case, $E = 0$. On the other hand, if $E_{11} = 1$, apply the above arguments to $1 - E$ to obtain $E = 1$. This proves the strong irreducibility of T. □

The next lemma from [18, Lemma 2] is useful in verifying the injectivity of operators of the form $\tau_{A,B}$.

Lemma 3.10. *Let A and B be operators on H and K, respectively. If there is a subset Γ of C with the properties $\Gamma \cap \sigma_p(A) = \phi$ and $\vee\{ker(B - \lambda)^k\colon \lambda \in \Gamma, k \geq 1\} = K$, then $\tau_{A,B}$ is one-to-one.*

PROOF OF THEOREM 3.8. Let T be a compact operator. By Corollary 3.3, we may assume that T is of multicyclic. Hence T can be expressed in the form

$$\begin{bmatrix} T_1 & T_{12} & & \\ & T_2 & T_{23} & * \\ 0 & & \ddots & \ddots \end{bmatrix} \text{ on } H_1 \oplus H_2 \oplus \cdots,$$

where the $T_n's$ are all cyclic (cf. [9, Theorem 5]). By "absorbing" the finite-dimensional $T_n's$ into the infinite-dimensional ones, we may further assume, in view of Theorem 3.2, that each T_n has a matrix representation of the form $[a_{ij}]_{i,j=1}^{\infty}$ with $a_{ij} = 0$ for $i - j > 1$. Apply Theorem 3.2 to each of such $T_n's$ to obtain $T_n = A_n + B_n$ with A_n and B_n in $\mathcal{B}_1(D^{(n)})$ and $\sigma(A_n) = \sigma(B_n) = \overline{D^{(n)}}$, where $D^{(n)} = \{z \in C : |z - \frac{1}{2^n}| < R\}$ with R a sufficiently large number.

We next define $A_{n\ n+1}$ and $B_{n\ n+1}$ for $n \geq 1$. Let M_n=ran $\tau_{A_n,A_{n+1}}$ and N_n =ran $\tau_{B_n,B_{n+1}}$. Since $\sigma_{\ell e}(A_n) \cap \sigma_{re}(A_n) = \partial D^{(n)}$ (cf. [5, Theorem XI, 6.8]), we have $\sigma_{re}(A_n) \cap \sigma_{\ell e}(A_{n+1}) \neq \phi$ for any $n \geq 1$. By [11, Theorem 1.1], this implies that M_n is not dense in $\mathcal{B}(H_n)$. Similarly, N_n is not dense in $\mathcal{B}(H_n)$. Hence by the Baire category theorem, we have $M_n \cup N_n \neq \mathcal{B}(H_n)$. Let $C_{n\ n+1} \in \mathcal{B}(H_n)$ be an operator not in $M_n \cup N_n$ with $\| C_{n\ n+1} \| < \varepsilon_n$, where ε_n is a small positive number to be specified later, and let

$$
\begin{aligned}
A_{n\ n+1} &= B_{n\ n+1} = \frac{1}{2}T_{n\ n+1} \text{ if } T_{n\ n+1} \notin M_n \cup N_n, \\
A_{n\ n+1} &= T_{n\ n+1} + C_{n\ n+1} \text{ and } B_{n\ n+1} = -C_{n\ n+1} \text{ if } T_{n\ n+1} \in M_n \backslash N_n, \\
A_{n\ n+1} &= -C_{n\ n+1} \text{ and } B_{n\ n+1} = T_{n\ n+1} + C_{n\ n+1} \text{ if } T_{n\ n+1} \in N_n \backslash M_n,
\end{aligned}
$$

and

$$A_{n\ n+1} = \frac{1}{2}T_{n\ n+1} + C_{n\ n+1} \text{ and } B_{n\ n+1} = \frac{1}{2}T_{n\ n+1} - C_{n\ n+1} \text{ if } T_{n\ n+1} \in M_n \cap N_n.$$

Then $A_{n\ n+1} \notin M_n, B_{n\ n+1} \notin N_n$ and $A_{n\ n+1} + B_{n\ n+1} = T_{n\ n+1}$.

Finally, let

$$A = \begin{bmatrix} A_1 & A_{12} & T_{13} & * & \\ & A_2 & A_{23} & T_{23} & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix} \text{ and } B = \begin{bmatrix} B_1 & B_{12} & & 0 \\ & B_2 & B_{23} & \\ & 0 & \ddots & \ddots \end{bmatrix}.$$

Then $T = A + B$. It remains to show that both A and B are strongly irreducible. We prove this for A by verifying the conditions in Lemma 3.9; that for B is analogous. By our construction, we need only check conditions (2) and (4) there.

To prove (2), note that $D^{(n+1)} \not\subseteq \overline{D^{(n)}}$ for any $n \geq 1$. Hence the set $\Gamma = D^{(n+1)} \backslash \overline{D^{(n)}}$ is nonempty. We have $\Gamma \cap \sigma(A_n) = \phi$ and $\vee\{\ker(A_{n+1} - \lambda)^k : \lambda \in \Gamma, k \geq 1\} = H_{n+1}$. Lemma 3.10 implies that $\tau_{A_n,A_{n+1}}$ is one-to-one.

Now we verify (4). First note that $\sigma\left(\sum_{j=n+1}^{\infty}\oplus A_j\right) = \overline{\bigcup_{j=n+1}^{\infty} D^{(j)}}$ for any $n \geq 0$. Indeed, since by our construction $\sum_j \oplus A_j$ is a compact perturbation of $\sum_j \oplus \left(RS^* + \frac{1}{2^j}\right)$, its spectrum is contained in $\sigma\left(\sum_j \oplus \left(RS^* + \frac{1}{2^j}\right)\right) \cup \{\lambda_k\} = \overline{\bigcup_j D^{(j)}} \cup \{\lambda_k\}$, where $\{\lambda_k\}$ is a set of at most countably many isolated eigenvalues of $\sum_j \oplus A_j$ disjoint from $\overline{\bigcup_j D^{(j)}}$. In particular, each $\bar{\lambda}_k$ is an eigenvalue of $\sum_j \oplus A_j^*$ and hence an eigenvalue of some A_j^*. Since A_j, being in $\mathcal{B}_1(D^{(j)})$, has a triangular matrix representation

$$\begin{bmatrix} x_1 & & * \\ & x_2 & \\ 0 & & \ddots \end{bmatrix}$$

with the diagonal entries x_i in $D^{(j)}, \lambda_k$ must be equal to some $\overline{x_i}$ and hence in $D^{(j)}$, which contradicts our assumption. This proves $\sigma(\sum_j \oplus A_j) \subseteq \overline{\bigcup_j D^{(j)}}$. Since the converse containment is trivial, our assertion is proved. Let

$$X_n = \begin{bmatrix} A_1 & A_{12} & \cdots & A_{1\ n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & A_{n-1\ n} \\ 0 & & & A_n \end{bmatrix} \text{ and } Y_n = \begin{bmatrix} A_{n+1} & A_{n+1\ n+2} & * \\ & A_{n+2} & \ddots \\ 0 & & \ddots \end{bmatrix},$$

and let $\Omega = \bigcap_{j=1}^{n} D^{(j)}$. Since A_j is in $\mathcal{B}_1(D^{(j)})$ for $1 \leq j \leq n$, we have $X_n \in \mathcal{B}_n(\Omega)$. By the upper semicontinuity of the mapping $X \mapsto \sigma(X)$, we may choose very small $\varepsilon_n > 0$ such that $\sigma(Y_n)$ is contained in

$$\{z \in C : \text{dist}\left(z, \overline{\bigcup_{j=n+1}^{\infty} D^{(j)}}\right) < 2^{-n-2}\} \cup E,$$

where E is a countable set of isolated points. Hence $\Gamma \equiv \Omega \backslash \sigma(Y_n)$ is a nonempty subset of Ω. From Lemma 3.10, we infer that τ_{Y_n, X_n} is one-to-one. This proves (4) and hence the strong irreducibility of A.

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