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SOLVING OPERATOR EQUATIONS IN NEST ALGEBRAS

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ABSTRACT. Let X and Y be operators on Hilbert space, and let $\mathcal{L}$ be a nest of projections on the space. We consider the problem of finding an operator A in Alg $\mathcal{L}$ such that A is Hilbert-Schmidt and such that $AX = Y$. A necessary and sufficient condition involving X, Y, and the projections in the lattice is found. We also indicate how the statements of the results can be modified so that the main theorem is true for any commutative subspace lattice $\mathcal{L}$.

A number of authors have considered the equation $Ax = y$, where x and y represent given vectors in Hilbert space and the (bounded) operator A is to be found subject to certain criteria. For instance, suppose that $\mathcal{N}$ is a nest of projections acting on the Hilbert space; what conditions on x and y guarantee the existence of an operator $A \in \operatorname{Alg}\mathcal{N}$ so that $Ax = y$? This question was discussed by Lance [7]. Several subsequent articles have generalized and expanded on Lance's original result, including ones by Munch ([8]), Hopenwasser ([3], [4]), and the authors of this paper, in conjunction with Anoussis and Katsoulis ([1], [2], and [6]). Of particular interest for this article is Munch's discussion of the problem of finding a Hilbert-Schmidt operator A in $\operatorname{Alg}\mathcal{N}$ so that $Ax = y$, a problem motivated by general systems theory. Munch's characterization and construction depend on the Arveson model, which represents commutative subspace lattices as lattices of increasing subsets of a partially ordered measure space. In this article, we adopt a point of view that proved useful in [6]; namely, we consider an operator equation $AX = Y$ instead of the vector equation $Ax = y$. This change allows us to investigate both single-vector and multiple-vector interpolation simultaneously. At the same time, we want to use a "coordinate-free" approach, so we have avoided the Arveson model. We first consider the case in which $\mathcal{N}$ is a nest, and then we indicate how certain definitions can be changed so that the main theorem remains true for other CSL algebras.

The issue, then, is this: Let $\mathcal{N}$ be a (strongly closed) nest of (self-adjoint) projections on a Hilbert space $\mathcal{H}$, and let Alg $\mathcal{N}$ represent the collection of operators that leave invariant all the projections in the nest $\mathcal{N}$. Let X and Y be bounded operators on $\mathcal{H}$; we don't require that X and Y be in Alg $\mathcal{N}$. Question: Under what conditions on X and Y can we be sure that there will exist a Hilbert-Schmidt operator A in Alg $\mathcal{N}$ so that $AX = Y$? We first consider the simplest possible version of the problem, namely, the case where $\mathcal{N}$ is the trivial nest $\{0, I\}$. In case, Alg $\mathcal{N}$ consists of all bounded operators on the Hilbert space and the problem becomes one of finding a Hilbert-Schmidt operator A that solves the equation $AX = Y$. The celebrated range inclusion theorem of Douglas [5] tells us when there is an operator (not necessarily Hilbert-Schmidt) such that $AX = Y$; namely, such an A exists precisely when the range of Y^* is contained in the range of X^*, or, equivalently, there exists some number K so that $Y^*Y \leq K^2X^*X$; in this case, the norm of the solution A can be chosen to be no greater than K. Note that the inequality $Y^*Y \leq K^2X^*X$ implies that the set of numbers $\{\frac{\|Ye\|^2}{\|Xe\|^2} : e$ is any vector in $\mathcal{H}\}$ is uniformly bounded by $\mathcal{K}$, provided we understand the expression "$\frac{0}{0}$" to mean "0." For the Hilbert-Schmidt result, we obviously need a stronger condition.

Given a pair of operators X and Y, an orthonormal basis $\{f_n\}$, and any sequence of vectors $\{e_n\}$, we define the following quantities:

$$L_1(X, Y, \{f_n\}, \{e_n\}) = \sum \frac{|\langle Ye_n, f_n\rangle|^2}{\|Xe_n\|^2};$$

$$L_2(X, Y, \{f_n\}) = \sup\{L_1(X, Y, \{f_n\}, \{e_n\}) : e_n \in \mathcal{H}\};$$

$$L(X, Y) = \sup\{L_2(X, Y, \{f_n\}) : \{f_n\} \text{ is an orthonormal basis of } \mathcal{H}.\};$$

In performing these computations, we use the conventions that $\frac{0}{0} = 0$ and that $\frac{a}{0} = \infty$ if $a > 0$. Our first theorem will show that these quantities are not as impractical to compute as the definitions make them seem.

Theorem 1. *Suppose that $L(X, Y) < \infty$. Then*

(a) *$ranY^* \subseteq ranX^*$.*

(b) *If $\{f_n\}$ and $\{g_n\}$ are any two orthonormal bases, then*

$$L_2(X, Y, \{f_n\}) = L_2(X, Y, \{g_n\}).$$

(c) *Y is a Hilbert-Schmidt operator.*

(d) *If $\{f_n\}$ is any orthonormal basis, and if, for each n, we choose a vector*

$$q_n \in \ker^{\perp} X^* \text{ such that } X^*q_n = Y^*f_n, \text{ then } L(X, Y) = \sum \|q_n\|^2.$$

(e) *There exists a Hilbert Schmidt operator A such that $AX = Y$ and such that*

$$\|A\|_2^2 \leq L(X, Y).$$

PROOF. (a) Let e be any vector in $\mathcal{H}$ and choose $e_1 = e,\ e_j = 0$ for $j \geq 2$. Since any unit vector f can be considered to be the first element of an orthonormal basis, we have

$$\frac{|\langle Ye, f\rangle|^2}{\|Xe\|^2} \leq L(X, Y) < \infty.$$

By choosing f to be the vector $Ye/\|Ye\|$, we have $\|Ye\|^2 \leq L(X, Y)\|Xe\|^2$ for any unit vector e. Consequently, $Y^*Y \leq L(X, Y)X^*X$, and by Douglas' range inclusion theorem, we have that ran $Y^* \subseteq$ ran X^*. The same theorem guarantees the existence of an operator A such that $\|A\| \leq (L(X, Y))^{1/2}, \operatorname{ran}^\perp X \subseteq \ker A$, and such that $AX = Y$.

(b) Let $\{f_n\}$ be a fixed orthonormal basis of $\mathcal{H}$. We have

$$\begin{aligned} L_2(X, Y, \{f_n\}) &= \sup\{\textstyle\sum_{n=1}^\infty \frac{|\langle Ye_n, f_n\rangle|^2}{\|Xe_n\|^2} : e_n \in \mathcal{H}\} \\ &= \sup\{\textstyle\sum_{n=1}^\infty \frac{|\langle AXe_n, f_n\rangle|^2}{\|Xe_n\|^2} ; e_n \in \mathcal{H}\} \\ &= \sup\{\textstyle\sum_{n=1}^\infty \frac{|\langle Xe_n, A^*f_n\rangle|^2}{\|Xe_n\|^2} : e_n \in \mathcal{H}\} \\ &= \sup\{\textstyle\sum_{n=1}^\infty |\langle \frac{Xe_n}{\|Xe_n\|}, A^*f_n\rangle|^2 : e_n \in \mathcal{H}\} \\ &= \sup\{\textstyle\sum_{n=1}^\infty |\langle u_n, A^*f_n\rangle|^2 : u_n \in \operatorname{ran}X, \|u_n\| = 1\} \\ &\leq \textstyle\sum \|A^*f_n\|^2. \end{aligned}$$

Now, since $\operatorname{ran}^\perp X \subseteq \ker A = \operatorname{ran}^\perp A^*$, it follows that $(\operatorname{ran}X)^- \supseteq (\operatorname{ran}A^*)^-$. Consequently, given a positive number ϵ, we can find unit vectors $\{u_n\}$ in $\operatorname{ran}X$ that are sufficiently close to the vectors A^*f_n that $|\sum \|A^*f_n\|^2 - \sum |\langle u_n, A^*f_n\rangle|^2| < \epsilon$. Therefore, $L_2(X, Y, \{f_n\}) = \sum \|A^*f_n\|^2$. Since this sum is finite, we know that A is a Hilbert-Schmidt operator, and (since $AX = Y$) so is Y. Furthermore, the sum $\sum \|A^*f_n\|^2$ is equal to the square of the Hilbert-Schmidt norm of the operator A, and it doesn't matter which orthonormal basis $\{f_n\}$ is used to compute the sum. Thus, parts (b) and (c) are proved. For any orthonormal basis $\{f_n\}$, we have $X^*A^*f_n = Y^*f_n$. Letting $q_n = A^*f_n$, we have $X^*q_n = Y^*f_n$ and

$$L(X, Y) = L_2(X, Y, \{f_n\}) = \sum \|q_n\|^2.$$

Moreover, $q_n \in \operatorname{ran}A^* \subseteq (\operatorname{ran}X)^- = \ker^\perp X^*$. The proof is complete. □

Most of the authors who have considered interpolation problems in nest algebras or CSL algebras have found it convenient to reduce first to the case of finite lattices, and we will follow their lead in this respect. Suppose that $\mathcal{F} = \{E_k\}_{k=1}^m$

is a finite nest of projections on $\mathcal{H}$, where $0 = E_0 < E_1 < \cdots < E_{m-1} < E_m = I$. We define

$$L_0(X, Y, \mathcal{F}) = \sum_{k=1}^{m} L(E_{k-1}^{\perp} X, (E_k - E_{k-1})Y).$$

The result we wish to prove is this:

Theorem 2. *Let $\mathcal{N}$ be a nest of projections on a Hilbert space $\mathcal{H}$ and let X and Y be operators defined on $\mathcal{H}$. Let $K = \sup\{L_0(X, Y, \mathcal{F}) : \mathcal{F}$ is a finite subnest of $\mathcal{N}\}$, and suppose that $K < \infty$. Then there exists a Hilbert-Schmidt operator A in Alg $\mathcal{N}$ such that $AX = Y$; furthermore, A can be chosen so that $\|A\|_2^2 \leq K$.*

Suppose, first, that the theorem can be proved whenever the nest involved is finite. Now suppose that $\mathcal{N}$ is any nest, and the conditions of the theorem hold. Then, to any finite subnest $\mathcal{F}$ of $\mathcal{N}$, there corresponds an operator A in Alg $\mathcal{F}$ such that $AX = Y$, and $\|A\|_2^2 \leq K$. Note that Alg $\mathcal{F} \supseteq$ Alg $\mathcal{N}$, and that Alg $\mathcal{N} = \cap\{\text{Alg } \mathcal{F} : \mathcal{F}$ is a finite subnest of $\mathcal{N}\}$. Order the collection of finite subnests of $\mathcal{N}$ by inclusion, and choose any increasing net $\{F_\alpha\}$ such that $\mathcal{F}_\alpha \uparrow \mathcal{N}$; let the corresponding operators be denoted A_α. The operator norms of the operators A_α are bounded (being smaller than the Hilbert-Schmidt norms), so there is a weakly convergent subnet; without harm, we assume that the whole net $\{A_\alpha\}$ converges weakly, say to the operator A. Since any projection $E \in \mathcal{N}$ is eventually in the subnest $\mathcal{F}_\alpha$, it is easy to see that A lies in Alg $\mathcal{N}$. Likewise, the equation $AX = Y$ holds because $A_\alpha X = Y$ for each α. Thus, the operator A would satisfy our requirements, provided that it were Hilbert-Schmidt. We include a proof of the following fact for the convenience of the reader.

Lemma 3. *Suppose that $\{A_\alpha\}$ is a net of Hilbert-Schmidt operators, with uniformly bounded Hilbert-Schmidt norms, and that $A_\alpha \to A$ in the weak operator topology. Then the operator A is Hilbert-Schmidt, with norm no larger than the uniform bound of the Hilbert-Schmidt norms of the A_α.*

PROOF. Suppose that $\{A_\alpha\}$ converges in the weak operator topology to the bounded operator A and suppose that $\sup_\alpha \|A_\alpha\|_2 \leq K < \infty$. Let $\{e_i\}$ be an orthonormal basis for the Hilbert space. Fix a number $N \geq 1$ and choose vectors $\{u_i\}_1^N$ such that $\sum_{i=1}^N \|u_i\|^2 = 1$. Then

$$\begin{aligned} |\textstyle\sum_{i=1}^N \langle Ae_i, u_i\rangle| &= \lim_\alpha |\textstyle\sum_{i=1}^N \langle A_\alpha e_i, u_i\rangle| \\ &\leq \lim_\alpha \textstyle\sum_{i=1}^N \|A_\alpha e_i\| \|u_i\| \\ &\leq \lim_\alpha (\textstyle\sum_{i=1}^N \|A_\alpha e_i\|^2)^{1/2} \cdot 1 \\ &\leq K, \end{aligned}$$

where the next-to-last line is a result of Schwartz's inequality. But

$$(\sum_{i=1}^{N} \|A_\alpha e_i\|^2)^{1/2} = \sup\{|\sum_{i=1}^{N}\langle A_\alpha e_i, u_i\rangle| : \sum_{i=1}^{N} \|u_i\|^2 = 1\}.$$

Consequently, A is Hilbert-Schmidt and $\|A\| \leq K$. □

PROOF OF THEOREM 2. We have already seen that we may restrict attention to the case that the nest $\mathcal{N}$ is finite, say $\mathcal{N} = \{E_k\}_{k=1}^{m}$ with $0 = E_0 < E_1 < \cdots < E_{m-1} < E_m = I$. By Theorem 1, there is an operator A_k such that $A_k E_{k=1}^{\perp} X = E_k E_{k-1}^{\perp} Y$, and we may obviously assume that $E_k E_{k-1}^{\perp} A_k E_{k-1}^{\perp} = A_k$. Then, if E_ℓ is any projection in $\mathcal{N}$, we have

$$E_\ell^{\perp} A_k E_\ell = E_\ell^{\perp} E_k E_{k-1}^{\perp} A_k E_{k-1}^{\perp} E_\ell.$$

Now, if $\ell \leq k-1$, then $E_\ell \leq E_{k-1}$ and $E_\ell E_{k-1}^{\perp} = 0$; on the other hand, if $\ell \geq k$, then $E_k \leq E_\ell$ and $E_\ell^{\perp} E_k = 0$. In either case, the quantity $E_\ell^{\perp} A_k E_\ell$ is zero, and it follows that $A_k \in \operatorname{Alg} \mathcal{N}$.

Put $A = \sum_{k=1}^{n} A_k$. The operator A lies in Alg $\mathcal{N}$ and we have

$$\begin{aligned}
AX &= (\sum E_k E_{k-1}^{\perp}) AX \\
&= \sum E_k E_{k-1}^{\perp} A E_{k-1}^{\perp} X \qquad (\text{since } A \in \operatorname{Alg} \mathcal{N}) \\
&= \sum E_k E_{k-1}^{\perp} A_k E_{k-1}^{\perp} X \qquad (\text{since } E_k E_{k-1}^{\perp} A_\ell = 0 \text{ if } \ell \neq k) \\
&= \sum E_k E_{k-1}^{\perp} Y \\
&= Y.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|A\|_2^2 &= \|\sum A_k\|_2^2 \\
&= \|\sum E_k E_{k-1}^{\perp} A_k\|_2^2 \\
&= \sum \|A_k\|_2^2 \qquad (\text{since the intervals } E_k E_{k-1}^{\perp} \text{ are mutually orthogonal.}) \\
&= \sum L(E_{k-1}^{\perp} X, E_k E_{k-1}^{\perp} Y) \\
&= L_0(X, Y, \mathcal{N}) \\
&\leq K.
\end{aligned}$$

This complete the proof. □

As promised, we now indicate how the theorem can be extended to other CSL-algebras. Recall that, if a lattice $\mathcal{L}$ of projections on Hilbert space commutes pairwise, the lattice is called a *commutative subspace lattice* (CSL), and the associated algebra is called a CSL algebra. If E and F are projections in $\mathcal{L}$, and if $E \leq F$, then the projection $P = F - E$ is an *interval* drawn from $\mathcal{L}$. If the interval P is minimal among the collection of intervals $\mathcal{L}$, then P is called an *atomic* interval; it is not hard to show that, if P is an atomic interval from

$\mathcal{L}$, then, for every projection G in $\mathcal{L}$, either $P \leq G$ or $P \leq G^\perp$. If P is such an interval, let $F(P)$ be defined by $F(P) = \vee\{G \in \mathcal{L} : P \leq G^\perp\}$. [Note that, if $\mathcal{N}$ is a nest, then atomic intervals from $\mathcal{N}$ have the form $F - F_-$ where there is no projection G that lies strictly between F_- and F; if the nest is finite, then every projection in $\mathcal{N}$ is the span of a finite number of atomic intervals. In this case, if $P = F - F_-$, then $F(P) = F_-$.] Now let $\mathcal{L}$ be a CSL, let $\mathcal{J}$ be a finite sublattice of $\mathcal{L}$, and let $\mathcal{F}$ be a subnest of $\mathcal{J}$ of maximal length; then every atomic interval from $\mathcal{F}$ is also an atomic interval from $\mathcal{J}$. If X and Y are operators, we define

$$L_{\mathcal{J}}(X, Y, \mathcal{F}) = \sum_{k=1}^{m} L(F(P_k)^\perp X, P_k Y),$$

where $F(P_k)$ is computed inside the lattice $\mathcal{J}$. Let $K_{\mathcal{J}} = \sup\{L_{\mathcal{J}}(X, Y, \mathcal{F}) : \mathcal{F}$ is a subnest of $\mathcal{J}\}$, and let $K = \sup\{L_{\mathcal{J}} : \mathcal{J}$ is a finite sublattice of $\mathcal{L}\}$. With these changes, Theorem 2 becomes true for any CSL $\mathcal{L}$.

References

[1] Anoussis, M., Katsoulis, E.G., Moore, R.L., Trent, T.T., *Interpolation problems for ideals in nest algebras*, Math. Proc. Camb. Phil. Soc., **111** (1991), 151-160.

[2] ———, *Interpolation problems for Hilbert-Schmidt operators in reflexive algebras*, Houston J. Math. (to appear).

[3] Hopenwasser, A., *The equation $Tx = y$ in a reflexive operator algebra*, Indiana University Math. J., **49** (1980), 121-126.

[4] ———, *Hilbert-Schmidt interpolation in CSL algebras*, Illinois J. Math., **33** (1989), 657-672.

[5] Douglas, R.G., *On majorization, factorization, and range inclusion of operators on Hilbert Space*, Proc. Amer. Math. Soc., **17** (1966), 413-415.

[6] Katsoulis, E.G., Moore, R.L., Trent, T.t., *Interpolation in nest algebras and applications to operator corona theorems,* J. Operator Theory, **29** (1993), 115-123.

[7] Lance, E.C., *Some properties of nest algebras*, Proc. London Math. Soc. (3), **19** (1969), 45-68.

[8] Munch, N., *Compact causal data interpolation*, Aarhus University reprint series 1986/1986, no. 11; J. Math. Anal. Appl., **140**(1989) no.2, 407-418.

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