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BSE BANACH MODULES AND BUNDLES OF BANACH SPACES

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Abstract. In a recent paper, S.-E. Takahasi defined the notion of a *BSE* Banach module over a commutative Banach algebra A with bounded approximate identity. We show that the multiplier space $M(X)$ of X can be represented as a space of sections in a bundle of Banach spaces, and we use bundle techniques to obtain shorter proofs of various of Takahasi's results on C^*-algebra modules and to answer several questions which he raised.

1. Introduction

In this paper, A will denote a commutative Banach algebra with bounded approximate identity $\{u_j\}$. Denote by $\Delta = \Delta_A$ the space of multiplicative functionals on A, and, for $h \in \Delta$, let $K_h = \ker h \subset A$ be the corresponding maximal ideal. We give Δ its weak-* topology. Let X be a Banach A-module, and, for $h \in \Delta$, let K_hX be the closure in X of $span\{ax : a \in K_h, x \in X\}$. As usual, $C_0(\Delta)$ is the space of continuous complex-valued functions on Δ which vanish at infinity, and $\widehat{\ }: A \to C_0(\Delta)$ is the Gelfand representation of A. If A is a C^*-algebra, we will identify A and $C_0(\Delta_A)$. Following the notation of Takahasi [7], if $h \in \Delta$, we choose $e_h \in A$ such that $\widehat{e_h}(h) = h(e_h) = 1$, and we let X^h be the closure in X of $K_hX + (1 - e_h)X$. Then X^h is independent of the choice of e_h. We set $X_h = X/X^h$. If A is a C^*-algebra, for each $h \in \Delta$ we will choose $\|e_h\| = 1$.

Let $\mathcal{E} = \dot{\bigcup}\{X_h : h \in \Delta\}$be the disjoint union of the X_h. (We can, if we like, identify – confuse! – $\mathcal{E}$ with $\bigcup\{\{h\} \times X_h : h \in \Delta\}$; the h's are useful for book-keeping purposes.) We give an element $x + X^h \in \mathcal{E}$ $(x \in X, h \in \Delta)$ its coset

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norm $\|x + X^h\|$. Let $\pi : \mathcal{E} \to \Delta$ be the obvious projection map. Given this data, let $\mathcal{C}(\mathcal{E})$ denote the linear space of selections (= choice functions) $\sigma : \Delta \to \mathcal{E}$, and let $\mathcal{C}^b(\mathcal{E})$ denote the subspace of bounded selections. Since each X^h is a Banach A-module, so is each X_h. In particular, for each $a \in A$ and $x \in X$ we have $ax - \widehat{a}(h)x \in X^h$. (Proof: $ax - \widehat{a}(h)x = (ax - ae_h x) + (ae_h x - \widehat{a}(h)e_h x) + (\widehat{a}(h)e_h x - \widehat{a}(h)x)$; the first two terms of the sum are in $K_h X$, and the last term is in $(1 - e_h)X$.) Hence, $a(x + X^h) = \widehat{a}(h)x + X^h$. It follows that $\mathcal{C}(\mathcal{E})$ and $\mathcal{C}^b(\mathcal{E})$ are both A- and $\widehat{A}$-modules under the operation $(a \cdot \sigma)(h) = \widehat{a}(h)\sigma(h) = (\widehat{a} \cdot \sigma)(h)$.

Consider the space $M(X) = Hom_A(A, X)$ of multipliers of X. Again, following [7], for $T \in M(X)$, we define a selection $\widetilde{T} : \Delta \to \mathcal{E}$ by $\widetilde{T}(h) = T(e_h) + X^h \in X_h$, where $\widehat{e_h}(h) = 1$. We see that this definition is also independent of the choice of e_h. The map $\widetilde{\ } : M(X) \to \mathcal{C}^b(\mathcal{E})$ is an A-module homomorphism of $M(X)$ into $\mathcal{C}(\mathcal{E})$, as noted in [7]. (In fact, $\widetilde{M(X)}$ is an A- and $\widehat{A}$-submodule of $\mathcal{C}^b(\mathcal{E})$.) We also note that $T(A) \subset X_e$, the essential part of X. Further, if $x \in X$, then the map $T_x : A \to X, a \mapsto ax$, is an element of $M(X)$. If X is an A-module, then so is X^*, with the multiplication given by $\langle x, f \cdot a\rangle = \langle ax, f\rangle$ $(a \in A, x \in X, f \in X^*)$.

We associate two topologies with the fibered space $\mathcal{E}$, and study the properties of the spaces of continuous and bounded selections from Δ to $\mathcal{E}$ under these topologies. In [7], Takahasi explores some of the consequences of endowing $\mathcal{E}$ with the quotient topology induced by the product topology on $\Delta \times X$ and the projection map $p : \Delta \times X \to \mathcal{E}, (h, x) \mapsto x + X^h = \widetilde{T_x}(h) = \widetilde{x}(h)$. We denote $\mathcal{E}$ with this topology by $\mathcal{E}_1$.

In this paper, we show that $\mathcal{E}$ can be endowed with a topology which makes $\pi : \mathcal{E} \to \Delta$ a bundle of Banach spaces; we denote $\mathcal{E}$ with this topology by $\mathcal{E}_2$, and call it the multiplier bundle of X. In particular, we show that if A has a bounded approximate identity, then the bundle and quotient topologies coincide; i.e. $\mathcal{E}_1 = \mathcal{E}_2$. We then show that the multiplier bundles for X and for X_e are homeomorphic, and we use this result and the machinery of section spaces of Banach bundles to subsume several of the examples concerning C^*-algebra modules adduced in [7]. We also answer several questions posed in [7]. The reader may also wish to consult [8] for more recent developments and applications of this construction of a field of quotient modules.

2. The bundle topology

We refer the reader to [1], [2], or [4] for fundamental notions regarding bundles of Banach spaces and Banach modules, and we especially draw upon the following results, which are key to our investigations.

Proposition 2.1. *([2, Corollary 3.7]) Suppose that U is a topological space, and that $\{Y_p : p \in U\}$ is a collection of Banach spaces. Let $\mathcal{E}$ be the disjoint union of the Y_p, and let $\gamma : \mathcal{E} \to U$ be the obvious projection. Suppose that Y is a vector space of bounded selections $\sigma : U \to \mathcal{E}$ such that 1) $\mathcal{E} = \dot{\bigcup}\{\sigma(u) : u \in U, \sigma \in Y\}$ ("Y is a full space of selections"); and 2) for each $\sigma \in Y$, the map $u \mapsto \|\sigma(u)\|$ is upper semicontinuous. Then there is a unique topology on $\mathcal{E}$ making $\gamma : \mathcal{E} \to U$ into a full bundle of Banach spaces, and such that each $\sigma \in Y$ is continuous.*

As a special case of the above, we obtain

Proposition 2.2. *([4, Proposition 1.3]) Let U be a topological space, and let $\{X^p : p \in U\}$ be a collection of closed subspaces of the Banach space X. Let $\mathcal{E} = \dot{\bigcup}\{X/X^p : p \in U\}$ be the disjoint union of the quotient spaces X/X^p. Then $\mathcal{E}$ can be topologized in such a way that 1) $\pi : \mathcal{E} \to U$ is a bundle of Banach spaces; and 2) for each $x \in X$, the selection $\widetilde{x} : U \to \mathcal{E}, \widetilde{x}(p) = x + X^p$, is a bounded section of the bundle $\pi : \mathcal{E} \to U$, iff the function $p \mapsto \|\widetilde{x}(p)\|$ is upper semicontinuous on U for each $x \in X$.*

Lemma 2.3. *Let $h \in \Delta$, and let C be the bound on the approximate identity for A. Then $\|h\| \geq \frac{1}{C}$. Moreover, we may choose our collection $\{e_h : h \in \Delta\}$ so that $\{\|e_h\| : h \in \Delta\}$ is bounded.*

Proof. For a given $\delta > 0$, we may choose $u = u_{j_0}$ in the approximate identity so that $\widehat{u}(h) > 1 - \delta$. Hence, $\|h\| \geq \frac{1}{C}\widehat{u}(h) > \frac{1-\delta}{C}$, and since δ was arbitrary, it follows that $\|h\| \geq \frac{1}{C}$. Now, for a fixed but arbitrary h, if we choose u in the approximate identity so that $\widehat{u}(h) > 1 - \delta$ and if we set $e_h = \frac{1}{\widehat{u}(h)}u$, we see that $\|e_h\| \leq \frac{C}{1-\delta}$. □

Corollary 2.4. *For any $T \in M(X)$, the selection $\widetilde{T} : \Delta \to \mathcal{E}$ is bounded.*

Proof. For $h \in \Delta$, we have $\left\|\widetilde{T}(h)\right\| = \|T(e_h) + X^h\| \leq \|T(e_h)\| \leq \|T\|\,\|e_h\|$; and the collection of e_h's can be chosen to be bounded, by the preceding. □

Proposition 2.5. *Given the data above, let $T \in M(X)$. Then the map $h \mapsto \left\|\widetilde{T}(h)\right\| = \|T(e_h) + X^h\|$ is upper semicontinuous on Δ.*

PROOF. Suppose that $\varepsilon > 0$ is given, and that $\left\|\widetilde{T}(h)\right\| < \varepsilon$. Choose $a_i \in K_h, y_i \in X$ $(i = 1, ..., n), z \in X$, such that

$$\left\|\widetilde{T}(h)\right\| \leq \left\|T(e_h) + \sum a_i y_i + (1 - e_h)z\right\| < \varepsilon,$$

and set

$$\varepsilon' = \varepsilon - \left\|T(e_h) + \sum a_i y_i + (1 - e_h)z\right\|.$$

From the upper semicontinuity of the map $h' \mapsto \|a + K_{h'}\|$ (see [5]) and the continuity of the function $\widehat{e_h}$ on Δ, we can choose a neighborhood V of h such that when $h' \in V$ all of the following hold:

$$\sum \|a_i + K_{h'}\| < \frac{\varepsilon'}{3\sum \|y_i\|};$$

$$\left|\frac{1}{\widehat{e_h}(h)} - \frac{1}{\widehat{e_h}(h')}\right| = \left|1 - \frac{1}{\widehat{e_h}(h')}\right| < \frac{\varepsilon'}{3\,\|T(e_h)\|};$$

and

$$|1 - \widehat{e_h}(h')| < \frac{\varepsilon'}{3C\,\|z\|}$$

(where C is the bound on the approximate identity). Since the definition of $X^{h'}$ is independent of the choice of $e_{h'}$, for $h' \in V$ we may just as well take $e_{h'} = \dfrac{1}{\widehat{e_h}(h')} e_h$. We also make use of the fact that, for $a \in A$ and $h \in \Delta$, we have $\|a + K_h\| = \dfrac{1}{\|h\|} |\widehat{a}(h)|$; see [5, Lemma 1.3].

Then, for $h' \in V$, we have the following:

$$\begin{aligned}
&\left\|\widetilde{T}(h')\right\| = \left\|T(e_{h'}) + X^{h'}\right\| \\
\leq\ & \left\|T(e_{h'}) - T(e_h) + X^{h'}\right\| + \left\|T(e_h) + \textstyle\sum a_i y_i + (1-e_h)z + X^{h'}\right\| \\
&\quad + \left\|\textstyle\sum a_i y_i + (1-e_h)z + X^{h'}\right\| \\
\leq\ & \|T(e_{h'}) - T(e_h)\| + \|T(e_h) + \textstyle\sum a_i y_i + (1-e_h)z\| + \left\|\textstyle\sum a_i y_i + X^{h'}\right\| \\
&\quad + \left\|(e_{h'} - e_h)z + X^{h'}\right\| \\
\leq\ & \left|1 - \frac{1}{\widehat{e_h}(h')}\right| \|T(e_h)\| + \|T(e_h) + \textstyle\sum a_i y_i + (1-e_h)z\| + \|\textstyle\sum a_i y_i + K_{h'}X\| \\
&\quad + \left\|(e_{h'} - e_h)z + X^{h'}\right\| \\
<\ & \varepsilon'/3 + \|T(e_h) + \textstyle\sum a_i y_i + (1-e_h)z\| + \|\textstyle\sum a_i y_i + K_{h'}X\| \\
&\quad + \|(e_h - e_{h'}) + K_{h'}\|\,\|z + K_{h'}X\| \\
\leq\ & \varepsilon'/3 + \|T(e_h) + \textstyle\sum a_i y_i + (1-e_h)z\| + \textstyle\sum \|a_i + K_{h'}\|\,\|y_i\| \\
&\quad + \frac{|\widehat{e_{h'}}(h') - \widehat{e_h}(h')|}{\|h'\|}\|z\| \\
<\ & \varepsilon'/3 + \|T(e_h) + \textstyle\sum a_i y_i + (1-e_h)z\| + \varepsilon'/3 + C\,|1 - \widehat{e_h}(h')|\,\|z\| \\
<\ & \varepsilon.
\end{aligned}$$

$\square$

The space $\widetilde{X} = \{\widetilde{T_x} : x \in X\}$ is a full space of selections in $\mathcal{C}^b(\mathcal{E})$, since, for each each $x \in X$ and $h \in \Delta$ we have $\widetilde{T_x}(h) = T_x(e_h) + X^h = e_h x + X^h = x + X^h \in X_h$. The space $\widetilde{M(X)} = \{\widetilde{T} : T \in M(X)\} \supset \widetilde{X}$ is therefore also full. It follows by Proposition 1 or Proposition 2, above, that there is a unique topology on $\mathcal{E}$ which turns $\pi : \mathcal{E} \to \Delta$ into a bundle of Banach spaces, such that each $\widetilde{T}$ ($T \in M(X)$) is a member of the Banach $C_0(\Delta)$-module $\Gamma^b(\pi)$ of all continuous and bounded sections of the bundle $\pi : \mathcal{E} \to \Delta$. Moreover, we may regard $\Gamma^b(\pi)$ as a Banach A-module under the operation $(a \cdot \sigma)(h) = \widehat{a}(h)\sigma(h)$, as described above. In the language of [4], the map $\widetilde{\ }: M(X) \to \Gamma^b(\pi)$ is a sectional representation of $M(X)$ of Gelfand type.

We recall that, in this bundle topology on $\mathcal{E}$, neighborhoods of a point $x + X^h$ are described by tubes: let $\sigma \in \Gamma^b(\pi)$ be such that $\sigma(h) = x + X^h$, let V be a neighborhood in Δ of h, and let $\varepsilon > 0$. Then $\mathcal{T} = \mathcal{T}(V, \sigma, \varepsilon) = \{z + X^{h'} : h' \in V, \left\|\sigma(h') - (z + X^{h'})\right\| < \varepsilon\}$ is a neighborhood of $\sigma(h)$, and in fact sets of this form, as V ranges over all neighborhoods of h and $\varepsilon > 0$ varies, form a fundamental system of neighborhoods of $\sigma(h)$. Denote by $\mathcal{E}_2$ the set $\mathcal{E}$ with its bundle topology generated by the $\widetilde{T}$ ($T \in M(X)$), and let $p : \Delta \times X \to \mathcal{E}_2$ be the natural map $(h, x) \mapsto \widetilde{x}(h)$.

Proposition 2.6. *Let A be a commutative Banach algebra with bounded approximate identity, and let X be a Banach A-module. Then the spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ are homeomorphic.*

PROOF. Since the topology on $\mathcal{E}_1$ is the quotient topology on $\mathcal{E}$ induced by the map $p : \Delta \times X \to \mathcal{E}$, it suffices to show that the topology $\mathcal{E}_2$ is also the quotient topology. We will show that the map $p : \Delta \times X \to \mathcal{E}_2$ is continuous and open, and the desired result then follows from a standard topological argument.

a) p is continuous: Let $\mathcal{T} = \mathcal{T}(V, \widetilde{x}, \varepsilon)$ be a bundle neighborhood of $\widetilde{x}(h) = x + X^h = p(h, x)$ in $\mathcal{E}_2$, as described above. If $B(x, \varepsilon)$ denotes the open ball around $x \in X$ of radius $\varepsilon > 0$, then $V \times B(x, \varepsilon)$ is a neighborhood of (h, x) in the product topology on $\Delta \times X$. If $(h', y) \in V \times B(x, \varepsilon)$, then $h' \in V$, and $\|\widetilde{x}(h') - \widetilde{y}(h')\| \leq \|x - y\| < \varepsilon$; i.e. $\widetilde{y}(h') = p(h', y) \in \mathcal{T}$.

b) p is open: Consider a set of form $V \times B(0, \varepsilon)$, which is nearly a typical open set in the product topology on $\Delta \times X$. We claim that $p(V \times B(0, \varepsilon))$ is open in $\mathcal{E}_2$. Let $h \in V$ and $y \in B(0, \varepsilon)$, and set $\varepsilon' = \varepsilon - \|y\|$.

Consider the tube $\mathcal{T} = \mathcal{T}(V, \widetilde{y}, \varepsilon')$ around $\widetilde{y}(h)$, and let $\widetilde{x}(h') \in \mathcal{T}$. Then $h' \in V$, and $\|\widetilde{x}(h') - \widetilde{y}(h')\| < \varepsilon'$. From the definition of the coset norm, we may choose $q = \sum a_i w_i + (1 - e_{h'})z \in K_{h'}X + (1 - e_{h'})X \subset X^{h'}$ such that

$$\|\widetilde{x}(h') - \widetilde{y}(h')\| \leq \|(x - y) + q\| < \varepsilon'.$$

Then

$$\|x + q\| \leq \|x - y + q\| + \|y\| < \varepsilon$$

and

$$\widetilde{(x + q)}(h') = \widetilde{x}(h') + \widetilde{q}(h') = \widetilde{x}(h').$$

That is, $\widetilde{x}(h') \in p(V \times B(0, \varepsilon))$. Hence, $\mathcal{T} \subset p(V \times B(0, \varepsilon))$, and thus $p(V \times B(0, \varepsilon))$ is an open set. The final result follows by translation. $\square$

The space $\widetilde{M(X)} = \{\widetilde{T} : T \in M(X)\}$ is a subspace of $\Gamma^b(\pi)$ for the bundle $\pi : \mathcal{E}_1 = \mathcal{E}_2 \to \Delta$. Given the homeomorphism of $\mathcal{E}_1$ and $\mathcal{E}_2$, for the remainder of the paper we will denote by $\mathcal{E}$ the fibered set with its bundle topology, and we will speak of the section space $\Gamma(\pi)$ $(\Gamma^b(\pi))$ of all continuous (bounded) sections of the bundle $\pi : \mathcal{E} \to \Delta$. We will call this the *multiplier bundle* for X.

We now examine for a moment the module $Y = X_e =$ closure in X of $span\{ax : a \in A, x \in X\}$. We note that, due to the Hewitt-Cohen factorization theorem, we actually have $X_e = AX$.

As an A-module, Y has a representation (its sectional Gelfand representation, in the language of [4]) as a space $\widehat{Y} \subset \Gamma(\pi')$ of a "canonical bundle" of Banach spaces $\pi' : \mathcal{F} \to \Delta$; the fibers of this bundle are the spaces $Y_h = Y/Y^h$, where $Y^h =$ closure in Y of $span\{ay : a \in K_h, y \in Y\}$. It can be easily checked that, because Y is an essential A-module, we actually have $Y^h = K_hY + (1 - e_h)Y$, as defined earlier. We note that we can also write $Y^h = K_hX$, since A has a bounded approximate identity and is therefore factorable. The mapping $\widehat{\ } : Y \to \Gamma(\pi')$ is given by $\widehat{y}(h) = y + Y^h$.

Consider the space $M(Y) = Hom_A(A, Y)$ of multipliers from A to Y. Let $y \in Y$, and let $T_y \in M(Y)$ be given by $T_y(a) = ay$. Then $\widetilde{T_y}(h) = T_y(e_h) + Y^h = e_hy + Y^h = y + Y^h = \widehat{y}(h)$. In other words, the representation of T_y as a section in the multiplier bundle for Y can be identified with the representation of y as a section in the canonical bundle for Y. It turns out that the fibers of the multiplier bundle for $Y = X_e$ and the multiplier bundle for X are related in general.

Proposition 2.7. *There is a topological linear isomorphism of the fiber $X_h = X/X^h$ of the multiplier bundle for X and of the fiber $Y_h = Y/Y^h = X_e/(X_e)^h$ of the multiplier (canonical) bundle for $Y = X_e$. If $\|e_h\| \leq 1$, then these fibers are isometrically isomorphic.*

PROOF. Let $i : X_e \to X$ be the inclusion map, and let $\rho_h : X \to X_h = X/X^h$ be the quotient map. Then $\rho_h \circ i : X_e \to X_h$ is norm-decreasing and surjective (because, for each $x \in X$, we have $(\rho_h \circ i)(e_hx) = e_hx + X^h = x + X^h$). If $ay \in K_hX = (X_e)^h = Y^h$, then $ay \in X^h$, and so $(\rho_h \circ i)(ay) = ay + X^h = X^h$; i.e. $K_hX \subset \ker(\rho_h \circ i)$. Thus, there is a norm-decreasing map $\phi_h : X_e/(X_e)^h \to X/X^h$ which carries $ax + K_hX = ax + (X_e)^h$ to $ax + X^h$.

On the other hand, consider the map of $X \to X_e$ given by $x \mapsto e_hx$; this map clearly has norm $\leq \|e_h\|$. We compose this with the quotient map $\rho'_h : X_e \to X_e/K_hX = X_e/(X_e)^h = (X_e)_h\rho'_h : X_e \to X_e/K_hX = X_e/(X_e)^h = (X_e)_h$ and obtain a map $\psi'_h : X \to (X_e)_h$ of norm $\leq \|e_h\|$. If $w = ay + (1 - e_h)z \in X^h$, then

$\psi'_h(w) = e_h w + K_h X = K_h X$, so that we obtain a map $\psi_h : X/X^h \to X_e/K_h X$, with $\|\psi_h\| \le \|e_h\|$. Note also that ψ_h is surjective, since, for $ax \in X_e, \psi_h(ax + X^h) = e_h ax + K_h X = \widehat{e_h}(h)(ax + K_h X) = ax + K_h X$.

It is now easily checked that $\phi_h \circ \psi_h$ and $\psi_h \circ \phi_h$ are the identities on the appropriate spaces. This establishes the desired topological linear isomorphism. That the isomorphism is isometric if $\|e_h\| \le 1$ is evident. □

The preceding establishes a bijection between the fiber space $\mathcal{E}$ of $\pi : \mathcal{E} \to \Delta$, the multiplier bundle for X, and the fiber space $\mathcal{F}$ of the multiplier bundle $\pi' : \mathcal{F} \to \Delta$ for X_e. We define the maps $\Phi : \mathcal{F} \to \mathcal{E}$ and $\Psi : \mathcal{E} \to \mathcal{F}$ fiberwise; e.g. $\Phi(x + (X_e)^h) = \phi_h(x + (X_e)^h)$.

Proposition 2.8. *The spaces $\mathcal{E}$ and $\mathcal{F}$, with the bundle topologies generated by X and X_e, respectively, are homeomorphic.*

PROOF. We will show that $\Psi : \mathcal{E} \to \mathcal{F}$ is continuous; the proof of the continuity of Φ will be similar. Let $D = \sup\{\|e_h\| : h \in \Delta\}$, let $x + X^h \in \mathcal{E}$, and consider the tube $\mathcal{T}_1 = \mathcal{T}_1(V, \widetilde{e_h x}, \varepsilon)$ around $e_h x + (X_e)^h = \Psi(x + X^h) = \psi_h(x + X^h) \in \mathcal{F}$. Then $\mathcal{T}_2 = \mathcal{T}_2(V, \widetilde{x}, \varepsilon/D)$ is a neighborhood of $x + X^h = \widetilde{x}(h)$ in $\mathcal{E}$. Let $y + X^{h'} \in \mathcal{T}_2$. Then $h' \in V$, and

$$\left\|(y + X^{h'}) - \widetilde{x}(h')\right\| = \left\|(y + X^{h'}) - (x + X^{h'})\right\| < \varepsilon/D.$$

Then

$$\begin{aligned}
\left\|\Psi(y + X^{h'}) - \Psi(x + X^{h'})\right\| &= \left\|\psi_{h'}(y + X^{h'}) - \psi_{h'}(x + X^{h'})\right\| \\
&\le D\left\|(y + X^{h'}) - (x + X^{h'})\right\| \\
&< \varepsilon,
\end{aligned}$$

so that $\Psi(y + X^{h'}) \in \mathcal{T}_1$. □

We offer the following without proof.

Corollary 2.9. *Let $\pi : \mathcal{E} \to \Delta$ and $\pi' : \mathcal{F} \to \Delta$ be the multiplier bundle of X and the multiplier bundle for X_e, respectively. Then there is a topological linear isomorphism $\psi : \Gamma^b(\pi) \to \Gamma^b(\pi')$, which is an isometry if the approximate identity for A is bounded by 1. Moreover, ψ is a $C_0(\Delta)$-linear map. For $\sigma \in \Gamma^b(\pi)$, we have $\psi(\sigma) = \Psi \circ \sigma$. The inverse map $\phi : \Gamma^b(\pi') \to \Gamma^b(\pi)$ is given by $\phi(\tau) = \Phi \circ \tau$.*

The following diagram illustrates the relationship among the maps constructed in this section. Here, ρ_h and ρ_h' will denote the quotient maps, ev_h will denote the evaluation maps, and $\widetilde{\ }$ will denote the section maps.

$$\begin{tikzcd}
X \arrow[r, "\sim"] \arrow[dr, "\rho_h"'] & \Gamma^b(\pi) \arrow[r, shift left, "\psi"] \arrow[d, "ev_h"] & \Gamma^b(\pi') \arrow[l, shift left, "\phi"] \arrow[d, "ev_h"'] & X_e \arrow[l, "\sim"'] \arrow[dl, "\rho_h'"] \\
& X_h \arrow[r, shift left, "\psi_h"] & (X_e)_h \arrow[l, shift left, "\phi_h"] &
\end{tikzcd}$$

3. The *BSE* condition

An element $\sigma \in \mathcal{C}(\mathcal{E})$ is said to be *BSE* (this refers to Bochner-Schoenberg-Eberlein; see [7] for an etymology of the term) if there exists some $\beta = \beta_\sigma > 0$ such that, for any choice of $h_i \in \Delta$, $f_i \in (X_{h_i})^*$ $(i = 1, ..., n)$ we have

$$\left| \sum_{i=1}^{n} \langle \sigma, f_i \circ ev_{h_i} \rangle \right| = \left| \sum_{i=1}^{n} \langle \sigma(h_i), f_i \rangle \right| \leq \beta \left\| \sum_{i=1}^{n} f_i \circ \rho_{h_i} \right\|_{X^*},$$

where $\rho_h : X \to X_h$ is the quotient map and $ev_h : \mathcal{C}(\mathcal{E}) \to X_h$, $\sigma \mapsto \sigma(h)$, is the evaluation map. Takahasi [7] shows that if $x \in X$, then $\widetilde{T_x}$ is *BSE*. If in addition A is a regular Banach algebra, then $\widetilde{T}$ is *BSE* for each $T \in M(X)$. The fundamental question explored in [7] is, When is $\widetilde{M(X)}$ equal to the space of all continuous $\mathcal{E}$-valued *BSE* selections on Δ, with $\mathcal{E}$ given its quotient topology? From the work done in the previous section, this is equivalent to the question of when $\widetilde{M(X)}$ is equal to $\Gamma_{BSE}(\pi)$, the space of *BSE* sections of the multiplier bundle $\pi : \mathcal{E} \to \Delta$. If $\widetilde{M(X)} = \Gamma_{BSE}(\pi)$, then the A-module X is said to be *BSE*.

We first make an elementary observation, noted without proof in [7].

Lemma 3.1. *Suppose that $\sigma \in \mathcal{C}(\mathcal{E})$ is BSE. Then σ is bounded.*

PROOF. From the definition of the BSE property, there exists $\beta = \beta_\sigma > 0$ such that for each $h \in \Delta$ and $f \in (X/X^h)^*$, we have

$$|\langle \sigma(h), f\rangle| \leq \beta \|f \circ \rho_h\|_{X^*} \leq \beta \|f\|,$$

since $\|\rho_h\| \leq 1$. We choose $f \in (X/X^h)^*$, with $\|f\| = 1$, such that $|\langle \sigma(h), f\rangle| = \|\sigma(h)\|$, and we obtain

$$\|\sigma(h)\| = |\langle \sigma(h), f\rangle| \leq \beta \|f\| = \beta,$$

i.e. $\|\sigma\| = \sup\{\|\sigma(h)\| : h \in \Delta\} \leq \beta.$ □

Thus, the question of when an A-module X is BSE can now be studied by using only elements of $\Gamma^b(\pi)$, the bounded sections of the bundle $\pi : \mathcal{E} \to \Delta$. There is a relationship between $\Gamma_{BSE}(\pi)$ and $\Gamma_{BSE}(\pi')$.

Proposition 3.2. *Let $\psi : \Gamma^b(\pi) \to \Gamma^b(\pi')$, $\phi : \Gamma^b(\pi') \to \Gamma^b(\pi)$ be the topological linear isomorphisms described at the end of the previous section. If $\sigma \in \Gamma_{BSE}(\pi')$ (arising from X_e), then $\phi(\sigma) \in \Gamma_{BSE}(\pi)$ (arising from X). Conversely, if $A = C_0(\Delta)$ is a C^*-algebra, and if $\sigma \in \Gamma_{BSE}(\pi)$, then $\psi(\sigma) \in \Gamma_{BSE}(\pi')$.*

PROOF. First, let $h \in \Delta$, let $f \in (X_h)^*$, and let $ax \in X_e$. Then

$$\begin{aligned} \langle ax, f \circ \rho_h\rangle &= \left\langle ax + X^h, f\right\rangle \\ &= \left\langle \phi_h(ax + (X_e)^h), f\right\rangle \\ &= \left\langle ax + (X_e)^h, \phi_h^*(f)\right\rangle \\ &= \left\langle ax, \phi_h^*(f) \circ \rho_h'\right\rangle. \end{aligned}$$

That is, $\phi_h^*(f) \circ \rho_h' \in (X_e)^*$ is the restriction to $X_e \subset X$ of $f \circ \rho_h \in X^*$, where $\rho_h : X \to X_h$ and $\rho_h' : X_e \to (X_e)_h$ are the quotient maps.

With this in mind, we now let $h_i \in \Delta$, $f_i \in (X_{h_i})^*, i = 1, ..., n$, and suppose that $\sigma \in \Gamma_{BSE}(\pi')$. Then

$$\begin{aligned} |\sum \langle [\phi(\sigma)](h_i), f_i\rangle| &= |\sum \langle \phi_{h_i}(\sigma(h_i)), f_i\rangle| \\ &= \left|\sum \left\langle \sigma(h_i), \phi_{h_i}^*(f_i)\right\rangle\right| \\ &\leq \beta_\sigma \left\|\sum \phi_{h_i}^*(f_i) \circ \rho_{h_i}'\right\|_{(X_e)^*} \\ &\leq \beta_\sigma \|\sum f_i \circ \rho_{h_i}\|_{X^*}. \end{aligned}$$

Thus, $\phi(\sigma) \in \Gamma_{BSE}(\pi)$.

Now, let A be a C^*-algebra. We first note that, given $h_i \in \Delta$ $(i = 1,..,n)$ we can choose our e_{h_i} to have disjoint support. We also note that, if $ax \in X_e$ and $f \in (X_e)^*_h$, we have

$$\begin{aligned} \langle ax, \psi^*_h(f) \circ \rho_h \rangle &= \left\langle ax + X^h, \psi^*_h(f) \right\rangle \\ &= \left\langle \psi_h(ax + X^h), f \right\rangle \\ &= \left\langle ax + (X_e)^h, f \right\rangle \\ &= \left\langle ax, f \circ \rho'_h \right\rangle . \end{aligned}$$

That is, $f \circ \rho'_h \in (X_e)^*$ is the restriction to X_e of $\psi^*_h(f) \circ \rho_h \in X^*$.

Let $h_i \in \Delta, f_i \in (X_e)^*_{h_i}$, $(i = 1,...,n)$. Let $\sigma \in \Gamma_{BSE}(\pi)$; we claim that $\psi(\sigma) \in \Gamma_{BSE}(\pi')$. We have

$$\begin{aligned} \left|\sum \langle [\psi(\sigma)](h_i), f_i \rangle\right| &= \left|\sum \langle \psi_{h_i}(\sigma(h_i)), f_i \rangle\right| \\ &= \left|\sum \left\langle \sigma(h_i), \psi^*_{h_i}(f_i) \right\rangle\right| \\ &\leq \beta_\sigma \left\|\sum \psi^*_{h_i}(f_i) \circ \rho_{h_i}\right\|_{X^*} . \end{aligned}$$

If $\varepsilon > 0$ is given, we can choose $x \in X, \|x\| = 1$, such that

$$\beta_\sigma \left\| \sum_i \psi^*_{h_i}(f_i) \circ \rho_{h_i} \right\|_{X^*} < \beta_\sigma \left| \sum_i \left\langle x, \psi^*_{h_i}(f_i) \circ \rho_{h_i} \right\rangle \right| + \varepsilon.$$

From our choice of e_{h_i} $(i = 1,...,n)$ to have disjoint support, we see that $\left\|\sum_j e_{h_j}\right\| = 1$ and that $[\psi^*_{h_i}(f_i) \circ \rho_{h_i}] \cdot (e_{h_j}) = \delta_{ij}[\psi^*_{h_i}(f_i) \circ \rho_{h_i}]$, where δ_{ij} is the Kronecker δ. It follows that

$$\begin{aligned} \beta_\sigma \left|\sum_i \left\langle x, \psi^*_{h_i}(f_i) \circ \rho_{h_i} \right\rangle\right| + \varepsilon &= \beta_\sigma \left|\sum_i \left\langle x, [\psi^*_{h_i}(f_i) \circ \rho_{h_i}] \cdot (\textstyle\sum_j e_{h_j}) \right\rangle\right| + \varepsilon \\ &= \beta_\sigma \left|\sum_i \left\langle (\textstyle\sum_j e_{h_j})x, \psi^*_{h_i}(f_i) \circ \rho_{h_i} \right\rangle\right| + \varepsilon \\ &= \beta_\sigma \left|\sum_i \left\langle (\textstyle\sum_j e_{h_j})x, f_i \circ \rho'_{h_i} \right\rangle\right| + \varepsilon \\ &\leq \beta_\sigma \left\|\sum_i f_i \circ \rho'_{h_i}\right\|_{(X_e)^*} + \varepsilon, \end{aligned}$$

because $(\sum_j e_{h_j})x \in X_e$, by the restriction argument above and $\left\|\left(\sum_j e_{h_j}\right)x\right\| \leq 1$. Thus, $\psi(\sigma) \in \Gamma_{BSE}(\pi')$. $\square$

Corollary 3.3. *If A is a C^*-algebra and if X_e is a BSE A-module, then so is X.*

PROOF. Let X_e be BSE, and let $\sigma \in \Gamma_{BSE}(\pi)$. Then $\psi(\sigma) \in \Gamma_{BSE}(\pi')$, and so there exists $T' \in M(X_e)$ such that $\widetilde{T'} = \psi(\sigma)$. If $i : X_e \to X$ is the inclusion, then $T = i \circ T' : A \to X$, and $T \in M(X)$. We have

$$\begin{aligned} \widetilde{(i \circ T')}(h) &= (i \circ T')(e_h) + X^h \\ &= T'(e_h) + X^h \\ &= \phi_h(T'(e_h) + (X_e)^h) \\ &= (\phi_h \circ \psi_h)(\sigma(h)) \\ &= \sigma(h), \end{aligned}$$

that is,$\widetilde{(i \circ T')} = \sigma$. $\square$

We now turn to some special cases involving commutative C^*-algebras. If $A = C_0(\Delta)$ is a commutative C^*-algebra, an A-module X is said to be $C_0(\Delta)$-locally convex if (among other equivalent formulations) we have $\|ay_1 + by_2\| = \max\{\|ay_1\|, \|by_2\|\}$ for all $a, b \in A$ with disjoint support, and for all $y_1, y_2 \in X$. If X is $C_0(\Delta)$-locally convex, and if X is essential, then there is an isometric $C_0(\Delta)$-isomorphism of X and $\Gamma_0(\pi)$, the space of sections of the multiplier bundle for X which disappear at infinity. (See [2] or [4] for details.)

Proposition 3.4. *Suppose that $A = C_0(\Delta)$ is a commutative C^*-algebra, and suppose that X is an A-module such that X_e, the essential part of X, is $C_0(\Delta)$-locally convex. Then 1) each element of $\Gamma^b(\pi)$ is BSE; and 2) $\widetilde{M(X)} = \Gamma^b(\pi)$.*

PROOF. For 1), let $\sigma \in \Gamma^b(\pi)$, and choose arbitrary $h_i \in \Delta, f_i \in (X_{h_i})^* = (X/X^{h_i})^*$ $(i = 1, ..., n)$. Choose $e_{h_i} \in A$ with disjoint support and such that $\|e_{h_i}\| = e_{h_i}(h_i) = 1$, and choose $x_i \in X$ such that $\sigma(h_i) = x_i + X^{h_i} = \widetilde{x_i}(h_i)$. Given $\varepsilon > 0$, for each $i = 1, ..., n$ choose $z_i \in K_{h_i}X + (1 - e_{h_i})X \subset X^{h_i}$ such that

$$\|\sigma(h_i)\| \leq \|x_i + z_i\| < \|\sigma\| + \varepsilon.$$

Set $w = \sum e_{h_i}(x_i + z_i)$. Then $w \in X_e$, and so

$$\begin{aligned} \|w\| &= \|\textstyle\sum e_{h_i}(x_i + z_i)\| \\ &= \max\{\|e_{h_i}\| \, \|x_i + z_i\|\} \\ &\quad \text{(because } X_e \text{ is } C_0(\Delta)\text{-locally convex and the } e_{h_i} \\ &\quad\quad \text{have disjoint support)} \\ &< \|\sigma\| + \varepsilon. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} \left|\textstyle\sum_i \langle \sigma(h_i), f_i\rangle\right| &= \left|\textstyle\sum_i \langle x_i + z_i, f_i \circ \rho_{h_i}\rangle\right| \\ &= \left|\textstyle\sum_i \langle e_{h_i}(x_i + z_i), f_i \circ \rho_{h_i}\rangle\right| \\ &= \left|\textstyle\sum_i \left\langle \sum_j e_{h_j}(x_j + z_j), f_i \circ \rho_{h_i} \right\rangle\right| \\ &\qquad \text{(since } e_{h_i}(h_j) = \delta_{ij} = \text{ Kronecker } \delta) \\ &= \left|\textstyle\sum_i \langle w, f_i \circ \rho_{h_i}\rangle\right| \\ &\le \|w\| \, \|\textstyle\sum_i f_i \circ \rho_{h_i}\|_{X^*} \\ &< (\|\sigma\| + \varepsilon) \, \|\textstyle\sum_i f_i \circ \rho_{h_i}\|_{X^*} \, . \end{aligned}$$

(We note that for $x \in X$ and $a \in A$, we have $(f \circ \rho_h)(ax) = f(\widetilde{ax}(h)) = f(a(h)\widetilde{x}(h)) = a(h)(f \circ \rho_h)(x)$).

For part 2), suppose that $\phi : \Gamma_0(\pi) \to X_e$ is the isometric $C_0(\Delta)$-isomorphism of the assumption. Among other properties of ϕ, we have $[\phi(\sigma)]^\sim(h) = \sigma(h)$ for each $\sigma \in \Gamma_0(\pi)$. Now, let $\sigma \in \Gamma^b(\pi)$. We define $T_\sigma : A \to X$ by $T_\sigma(a) = \phi(a\sigma)$. Then, for $b \in C_0(\Delta)$, we have

$$bT_\sigma(a) = b\phi(a \cdot \sigma) = \phi(ba \cdot \sigma) = T_\sigma(ba).$$

Clearly, T_σ is bounded, and so $T \in M(X)$. Moreover, for $h \in \Delta$, we have

$$\widetilde{T_\sigma}(h) = [T_\sigma(e_h)]^\sim(h) = [\phi(e_h \cdot \sigma)]^\sim(h) = (e_h \cdot \sigma)(h) = \sigma(h).$$

□

Two examples, worked out at some length in [7], then follow as corollaries:

Corollary 3.5. *Let A be a commutative C^*-algebra, and let $I \subset A$ be a closed ideal. Then I is BSE as an A-module, and A is BSE as an I-module.*

PROOF. As an A-module, $I = I_e$ is essential, and since $I \subset C_0(\Delta_A)$ is $C_0(\Delta_A)$-locally convex, the result follows. On the other hand, as an I-module, $A_e = I = C_0(\Delta_I)$, which is $C_0(\Delta_I)$-locally convex. □

Corollary 3.6. *Let A be a quasi-central C^*-algebra, with center Z. Then A is BSE as a Z-module.*

PROOF. From the proof in ([7], Theorem 3.2), A is essential as a Z-module. Note that $Z \simeq C_0(\Delta_Z)$. A variant of a result by Varela ([9], Theorem 3.5) shows that A is isometrically isomorphic to the space $\Gamma_0(\pi)$ of sections of $\pi : \mathcal{E} \to \Delta_Z$ which vanish at infinity, and hence that A is $C_0(\Delta_Z)$-locally convex. □

We now address questions asked by Takahasi in [7], as to whether $\Gamma_{BSE}(\pi) \subset \Gamma^b(\pi)$ is a Banach A-module.

Proposition 3.7. *Let A and X be as generally given. Then $\Gamma_{BSE}(\pi)$ is an A-module.*

PROOF. Let $\sigma, \tau \in \Gamma_{BSE}(\pi)$. Choose β_σ and β_τ as in the definition of BSE, and let $h_i \in \Delta, f_i \in (X_{h_i})^*$ $(i = 1, ..., n)$. Then

$$\begin{aligned} \left|\sum \langle (\sigma+\tau)(h_i), f_i\rangle\right| &\le \left|\sum \langle \sigma(h_i), f_i\rangle\right| + \left|\sum \langle \tau(h_i), f_i\rangle\right| \\ &\le (\beta_\sigma + \beta_\tau) \left\|\sum f_i \circ \rho_{h_i}\right\|_{X^*}, \end{aligned}$$

so that $\sigma + \tau \in \Gamma_{BSE}(\pi)$. Similarly, let $a \in A$. Then

$$\begin{aligned} \left|\sum \langle (a\cdot\sigma)(h_i), f_i\rangle\right| &= \left|\sum \langle \widehat{a}(h_i)\sigma(h_i), f_i\rangle\right| \\ &= \left|\sum \langle \sigma(h_i), \widehat{a}(h_i) f_i\rangle\right| \\ &\le \beta_\sigma \left\|\sum \widehat{a}(h_i)(f_i \circ \rho_{h_i})\right\|_{X^*} \\ &= \beta_\sigma \left\|\sum (f_i \circ \rho_{h_i}) \cdot a\right\|_{X^*} \\ &\le \beta_\sigma \|a\| \left\|\sum f_i \circ \rho_{h_i}\right\|_{X^*}, \end{aligned}$$

since $\sum (f_i \circ \rho_{h_i}) \cdot a = \sum (f_i \circ \rho_{h_i})\widehat{a}(h_i)$. Thus, $a \cdot \sigma \in \Gamma_{BSE}(\pi)$. □

However, as the following example shows, $\Gamma_{BSE}(\pi)$ may not be a Banach space, even when A is about as nice as it can be.

Example 3.1: Let $A = C([0,1])$, and let $X = A^*$, the set of bounded Borel measures on $[0,1]$. Since A has an identity, we have $M(X) = X$, and it can be shown (see [10] or [4]) that, for $h \in \Delta = [0,1]$, we have $X_h \simeq \mathbb{C}$. Under this identification, for $\mu \in X$ and $h \in [0,1]$ we have $\widetilde{\mu}(h) = \mu(\{h\})$, so that $\ker(\tilde{\ })$ is the space of continuous measures on $[0,1]$. Evidently, for any $\mu \in X = M(X)$, $\widetilde{\mu}$ has only countable support in $[0,1]$, and we can identify $\Gamma(\pi) = \Gamma^b(\pi)$ with $c_0([0,1])$, the closure under the sup-norm of the space of functions on $[0,1]$ which vanish off finite sets.

Now, A is a regular algebra, and so each element of $\widetilde{X} = \widetilde{M(X)}$ is BSE. We will describe an element $\sigma \in \Gamma(\pi)$ such that $\sigma \neq \widetilde{\mu}$ for any $\mu \in X$ but such that there is a sequence $\{\mu_n\} \subset X$ such that $\sigma = \lim \widetilde{\mu_n}$ in $\Gamma(\pi)$; thus $\Gamma_{BSE}(\pi)$ is not complete.

For each $h \in [0,1]$, we have $X_h \simeq \mathbb{C}$, so that for $f \in (X_h)^*$ the action of f on X_h can be identified with multiplication by some $\alpha = \alpha_f \in \mathbb{C}$. We show that, given $h_i \in [0,1]$ and $f_i \in (X_{h_i})^*$ $(i = 1, ..., n)$, we have $\|\sum f_i \circ \rho_{h_i}\|_{X^*} = \max\{|\alpha_{f_i}| : i = 1, ..., n\}$. First, let $\varepsilon > 0$ be given. We can choose $\mu \in X$, $\|\mu\| = 1$ such that

$$\begin{aligned}
\|\textstyle\sum f_i \circ \rho_{h_i}\|_{X^*} &< |\textstyle\sum \langle \mu, f_i \circ \rho_{h_i}\rangle| + \varepsilon \\
&= |\textstyle\sum \alpha_{f_i}\mu(\{h_i\})| + \varepsilon \\
&< \textstyle\sum |\alpha_{f_i}|\,|\mu(\{h_i\})| + \varepsilon \\
&\leq \max\{|\alpha_{f_i}|\} \textstyle\sum |\mu(\{h_i\})| + \varepsilon \\
&\leq \max\{|\alpha_{f_i}|\} + \varepsilon,
\end{aligned}$$

since $\sum |\mu(\{h_i\})| \leq \|\mu\| = 1$. Hence, $\|\sum f_i \circ \rho_{h_i}\|_{X^*} \leq \max\{|\alpha_{f_i}|\}$. On the other hand, for each $j = 1, ..., n$, we let $\mu_j \in X$ be the unit point mass at h_j. Then $\|\mu_j\| = 1$, and

$$\left|\sum_i \langle \mu_j, f_i \circ \rho_{h_i}\rangle\right| = |\alpha_{f_j}\mu_j(\{h_j\})| = |\alpha_{f_j}| \leq \left\|\sum_i f_i \circ \rho_{h_i}\right\|_{X^*},$$

so that $\max\{|\alpha_{f_i}|\} \leq \|\sum f_i \circ \rho_{h_i}\|_{X^*}$.

Now, consider $\sigma \in \Gamma(\pi)$ given by $\sigma(h) = h$, if $h = 1/k$ for some $k = 1, 2, ...$, $\sigma(h) = 0$ otherwise. Let $h_j = 1/j$ for $j = 1, ..., n$, and let $f_j \in (X_{h_j})^*$ be determined by $\alpha_{f_j} = 1$. Then

$$\left|\sum_{j=1}^{n} \langle \sigma(h_j), f_j\rangle\right| = \sum_{j=1}^{n} 1/j,$$

but $\left\|\sum f_j \circ \rho_{h_j}\right\|_{X^*} = 1$, so that $\sigma \notin \Gamma_{BSE}(\pi)$. However, let $\mu_n \in X$ be the discrete measure on $[0,1]$ such that $\mu_n(\{1/j\}) = 1/j$ for each $j = 1, ..., n$. Then $\widetilde{\mu_n} \in \Gamma_{BSE}(\pi)$ and $\sigma = \lim \widetilde{\mu_n}$.

Example 3.2: It is also shown in [7] that when G is a compact abelian group, each of the convolution $L^1(G)$-modules $C(G)$, $L^p(G)$ $(1 \leq p \leq \infty)$ and $M(G)$ is BSE, and the question is asked whether the same is true for the case of non-compact G. This is true, at least for $L^p(G)$ when $1 < p < \infty$, but the reason turns out not to be especially interesting, as the following shows:

We have noted that for algebras A of the sort we are using, and A-modules X, we have $a(x + X^h) = \widehat{a}(h)(x + X^h)$. Thus, if $f \in (X_h)^* = (X/X^h)^* \simeq (X^h)^\perp$, we may write $(f \cdot a)(x + X^h) = \widehat{a}(h)f(x + X^h)$, that is, $f \cdot a = \widehat{a}(h)f$ in $(X_h)^*$. Hence f (actually, its isomorphic image in $(X^h)^\perp$) generates a one-dimensional submodule in X^*. Conversely, each element of X^* which generates a one-dimensional submodule in X^* clearly annihilates X^h, and therefore has an isomorphic image in $(X_h)^*$.

It is shown in [3] that for any locally compact abelian group G, the one dimensional submodules in $L^p(G)$ $(1 \leq p \leq \infty)$ are scalar multiples of characters of G. But when G is non-compact, these characters are not in $L^p(G)$ for $1 \leq p < \infty$, and so $L^p(G)$ has no one-dimensional submodules. It follows that if $1 < p < \infty$, $X = L^p(G)$, and G is non-compact, then $X_h = 0$ for each character $h \in \Delta_{L^1(G)} = \widehat{G}$. In this case, the only section of the multiplier bundle for $L^p(G)$ as a module over $L^1(G)$ is the zero section, which is trivially BSE.

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