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QUOTIENT HARDY MODULES

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ABSTRACT. Suppose $H^2(D^n)$ is the Hardy space over the unit polydisk D^n, and $[h]$ is the closed submodule generated by a function $h \in H^\infty(D^n)$. The quotient $H^2(D^n) \ominus [h]$ is an $A(D^n)$ module and the coordinate functions $z_1, z_2, ..., z_n$ act on $H^2(D^n) \ominus [h]$ as bounded linear operators. In this paper, we first make a study of the spectral properties of these operators and reveal how these properties are related to the function h. Then we will have a look at the analytic continuation problem. At last, we will show a rigidity phenomenon of quotient Hardy modules.

0. INTRODUCTION

We let $\mathbf{C}^n$ denote the cartesian product of n copies of the complex field. The points of $\mathbf{C}^n$ are thus ordered n-tuples $z = (z_1, z_2, ..., z_n)$. D^n will be the unit polydisk in $\mathbf{C}^n$ with distinguished boundary T^n, where T is the unit circle. The closure of polynomials over D^n under the supremum norm will be denoted by $A(D^n)$ and called the polydisk algebra. The Hardy space $H^2(D^n)$ is the collection of holomorphic functions over D^n which satisfy the inequality

$$sup_{0\le r<1} \int_{T^n} |f(rz)|^2 dm < \infty,$$

with norm

$$\|f\|_2 = \left(sup_{0\le r<1} \int_{T^n} |f(rz)|^2 dm \right)^{1/2},$$

where dm is the normalized Lebesgue measure on T^n. $H^\infty(D^n)$ is the space of all bounded holomorphic functions in D^n with

$$\|f\|_\infty = sup|f(z)|, \quad z \in D^n,$$

and it is easily seen that $H^\infty(D^n)$ is a Banach algebra with pointwise multiplication and addition. The collection of invertible elements in algebra $H^\infty(D^n)$

is denoted by $[H^\infty(D^n)]^{-1}$. The H^∞ spaces over other domains are similarly defined. Suppose Ω is any open set that contains $\overline{D^n}$. For any natural number j less than or equal to n and any $\mu \in \overline{D}$, we set

$$S^j_\mu := \{z \in \Omega | z_j = \mu\},$$

which is called the *slice* of Ω at $z_j = \mu$. In many places of this paper, we will assume some functions to be holomorphic in a neighborhood of $\overline{D^n}$. The slice sets will be useful in the discussion there. For any h holomorphic in a domain, $Z(h)$ will denotes the set of zeros of h in that domain.

It is well known that the space $H^2(D^n)$ is an $A(D^n)$ -module with action defined by the pointwise multiplication by $A(D^n)$ functions. For any $h \in H^2(D^n)$, we let

$$[h] := \overline{A(D^n)h}^{H^2}$$

be the submodule generated by the function h. A function $h \in H^2(D^n)$ is called *Helson outer*(denoted outer(H)) if $[h]$ is equal to $H^2(D^n)$ and is called *inner* if $|h(z)|$ is equal to 1 almost everywhere on T^n. It is easy to see that when h is inner,

$$[h] = hH^2(D^n).$$

Normally $[h]$ is a proper subspace of $H^2(D^n)$. If we let

$$p : H^2(D^n) \longrightarrow [h], \quad q : H^2(D^n) \longrightarrow H^2(D^n) \ominus [h]$$

be projections, then one checks that the map $S : A(D^n) \longrightarrow B(H^2(D^n) \ominus [h])$ defined by

$$S_f g := qfg, \quad f \in A(D^n), \ g \in H^2(D^n) \ominus [h],$$

is a homomorphism which turns $H^2(D^n) \ominus [h]$ into a quotient $A(D^n)$-module. One sees that the operators S_{z_1}, S_{z_2}, ..., S_{z_n} are compressions of the Toeplitz operators T_{z_1}, T_{z_2}, ..., T_{z_n} onto $H^2(D^n) \ominus [h]$. For convenience we denote S_{z_j} simply by S_j, $j = 1, 2, ..., n$.

In section 1, we make a study of the the spectra of the operators S_1, S_2, ..., S_n as well as the joint spectrum of the n-tuple $(S_1, S_2, ..., S_n)$. Section 2 is devoted to the study of some functional properties of certain functions in $H^2(D^n)$. In section 3, we establish a rigidity phenomenon of quotient modules.

We thank the referee for bringing [CR] to our attention and making other comments on this paper.

1. Spectrum

We recall from section 0 that S_1, S_2, ..., S_n are the compressions of the Toeplitz operators T_{z_1}, T_{z_2}, ..., T_{z_n} to the quotient module $H^2(D^n) \ominus [h]$. Cowen and Rubel made a study of the spectral properties of the operators S_1, S_2, ..., S_n and showed a close relation between the *joint* spectrum and the zero set of h([CR]). In this section we will show that under certain conditions the spectrum of S_j is exactly the projection of the zero set to the *jth* coordinate.

We proceed by proving the following

Lemma 1.1. *If h is holomorphic in a neighborhood of $\overline{D^n}$ and $h(\lambda, z') \in [H^\infty(S^1_\lambda)]^{-1}$, then $\lambda \in \rho(S_1)$, the resolvent set of S_1.*

Proof. Consider the function

$$F(z_1, z') = \frac{1 - h(z_1, z')h^{-1}(\lambda, z')}{z_1 - \lambda}.$$

By the Weierstrass Preparation Theorem[Kr, Thm. 6.4.5], the numerator of F has $z_1 - \lambda$ as a factor, and hence F is a bounded holomorphic function over D^n. So S_F is a bounded operator on $H^2(D^n) \ominus [h]$.

For every $f \in H^2(D^n) \ominus [h]$,

$$\begin{aligned}(S_1 - \lambda)S_F f &= q(z_1 - \lambda)Ff \\ &= q(1 - h(\cdot,\cdot)h^{-1}(\lambda,\cdot))f \\ &= qf - qh(\cdot,\cdot)h^{-1}(\lambda,\cdot)f \\ &= qf = f.\end{aligned}$$

This shows that

$$(S_1 - \lambda)S_F = I.$$

Since S_1 commutes with S_F, we also have that

$$S_F(S_1 - \lambda) = I$$

i.e. $\lambda \in \rho(S_1)$. □

Similar statements are true for the operators S_2, ..., S_n with the corresponding assumptions on h.

In essence, if λ is not the j-th coordinate of any of the zeros of h in $\overline{D^n}$, then λ is in the resolvent set of S_j. It is actually possible to give a complete description of the spectra of these compression operators when $Z(h)$ satisfies certain conditions, but it is convenient to have a look at their *joint spectrum* first.

Let us first give the definition of the joint spectrum. A good reference for this subject is Chapter III in [Hö].

Suppose $\mathcal{B}$ is a commutative Banach algebra with unit e, and

$$a = (a_1,\ a_2,\ \ldots,\ a_n)$$

is a tuple of elements in $\mathcal{B}$. We say that a is *non-singular* if there are elements $b_1,\ b_2,\ \ldots,\ b_n \in \mathcal{B}$ with

$$\sum_{i=1}^{n} a_i b_i = e.$$

The tuple a is called *singular* if it is not non-singular. The *joint spectrum* of the tuple a is defined as

$$\sigma(a) := \{z \in \mathbf{C}^n :\ a - ze \textit{ is singular.}\}$$

Here $a - ze$ denotes the tuple $(a_1 - z_1,\ a_2 - z_2,\ \ldots,\ a_n - z_n)$.

Now we state two more lemmas which are special cases of results in [CR] with only slightly different proofs.

Lemma 1.2. *Suppose h is holomorphic in a neighborhood of $\overline{D^n}$ and*

$$\overline{Z(h)} \cap \overline{D^n}$$

is not empty, then the joint spectrum

$$\sigma(S_1, S_2, \ldots, S_n) \subset \overline{Z(h)} \cap \overline{D^n}.$$

PROOF. Without loss of generality we may assume that h is holomorphic in a pseudoconvex neighborhood U of $\overline{D^n}$. Then, for any $\lambda = (\lambda_1,\ \lambda_2, \ldots,\ \lambda_n) \in \overline{D^n} \setminus \overline{Z(h)} \cap \overline{D^n}$, we can write h as

$$h(z_1, z_2, \ldots, z_n) = h(\lambda_1, \lambda_2, \ldots, \lambda_n) + \sum_{j=1}^{n} (z_j - \lambda_j) g_j$$

for some functions g_j that are also holomorphic in U[Kr, Thm.7.2.9]. Since

$$h(z)h^{-1}(\lambda) = 1 + h^{-1}(\lambda) \sum_{j=1}^{n} (z_j - \lambda_j) g_j,$$

it follows for any $f \in H^2(D^n) \ominus [h]$ that

$$\begin{aligned} -h^{-1}(\lambda)\sum_{j=1}^{n}(S_j - \lambda_j)S_{g_j} f &= -q(h^{-1}(\lambda)\sum_{j=1}^{n}(z_j - \lambda_j)g_j)f \\ &= q(1 - h(\cdot)h^{-1}(\lambda))f \\ &= qf = f. \end{aligned}$$

This implies that

$$-h^{-1}(\lambda)\sum_{j=1}^{n}(S_j - \lambda_j)S_{g_j} = I,$$

and hence $\lambda \in \rho(S_1, S_2, ..., S_n)$ for any $\lambda \in \overline{D^n} \setminus \overline{Z(h)} \cap \overline{D^n}$, or equivalently

$$\sigma(S_1, S_2, ..., S_n) \subset \overline{Z(h)} \cap \overline{D^n}.$$

□

Here we note that $\overline{Z(h)} \cap \overline{D^n}$ not empty doesn't imply that $[h]$ is proper. For example, $[z + w + 2]$ is equal to $H^2(D^2)$ ([Ge]). In Lemma 1.2 we excluded the trivial case $[h] = H^2(D^2)$. In case $Z(h) \cap D^n$ is not empty, we have an inclusion in the other direction.

Lemma 1.3. *If h is holomorphic in a neighborhood of $\overline{D^n}$ and $Z(h) \cap D^n$ is not empty, then*

$$Z(h) \cap D^n \subset \sigma(S_1, S_2, ..., S_n).$$

PROOF. Suppose $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in Z(h) \cap D^n$. It is easy to see that

$$\overline{\sum_{j=1}^{n}(S_j - \lambda_j)(H^2(D^n) \ominus [h]) + [h]} \subset \overline{\sum_{j=1}^{n}[z_j - \lambda_j] + [h]},$$

but λ is a common zero of the functions $z_1 - \lambda_1$, $z_2 - \lambda_2, ..., z_n - \lambda_n$ and h, so $\overline{\sum_{j=1}^{n}[z_j - \lambda_j] + [h]}$ is a proper subset of $H^2(D^n)$. This implies that

$$\sum_{j=1}^{n}(S_j - \lambda_j)(H^2(D^n) \ominus [h]) \neq H^2(D^n) \ominus [h]$$

i.e. $\lambda \in \sigma(S_1, S_2, ..., S_n)$. □

Using module resolution and tensor product one can prove the inclusion in this lemma for other submodules. But the statement here is good enough for our purpose.

Combining lemmas 1.2 and 1.3 , we have the following

Theorem 1.4. *If h is holomorphic in a neighborhood of $\overline{D^n}$ such that*

$$(*) \qquad \overline{Z(h)} \cap \overline{D^n} = \overline{Z(h) \cap D^n},$$

then

$$\sigma(S_1,\ S_2, ...,\ S_n) = \overline{Z(h) \cap D^n}.$$

For $j = 1,\ 2,\ 3,\ ...,\ n$ and any $z \in \overline{D^n}$, we let

$$\pi_j z \stackrel{\text{def}}{=} z_j$$

be the projection to the j-th coordinate of z. It is well known that,

$$\pi_j \sigma(S_1,\ S_2, ...,\ S_n) \subset \sigma(S_j).$$

Combining lemma 1.1 and the above theorem, we have

Theorem 1.5. *If h is holomorphic in a neighborhood of $\overline{D^n}$ that satisfies condition (*), then for $j = 1,\ 2,\ ...,\ n$,*

$$\sigma(S_j) = \pi_j(\overline{Z(h) \cap D^n}).$$

PROOF. It suffices to show that

$$\sigma(S_j) \subset \pi_j(\overline{Z(h) \cap D^n}).$$

In fact, if μ is inside the complement of $\pi_j(\overline{Z(h) \cap D^n})$, then fixing $z_j = \mu$, h doesn't vanish on the closure of S^j_μ. Lemma 1.1 then concludes that $\mu \in \rho(S_j)$. □

Theorem 1.5 will be used in section 2 to make a study of the analytic continuation problem.

2. ANALYTIC CONTINUATION

In [AC], Ahern and Clark made a study on the analytic continuation of functions in certain quotient Hardy modules. In this section, we are going to use a result from their work and the results obtained in section 1 to study the analytic continuation problem. Again we find that the zero set plays an important role.

Corollary 2.1. *If h is holomorphic in a neighborhood of $\overline{D^n}$ and h is in $[H^\infty(S^j_\lambda)]^{-1}$ setting $z_j = \lambda$ with $|\lambda| = 1$, then every function in $H^2(D^n) \ominus [h]$ has an analytic continuation to a neighborhood of $D^{j-1} \times \{\lambda\} \times D^{n-j}$.*

PROOF. We prove the assertion for $j = 1$.

Every function of $H^2(D^n) \ominus [h]$ has the property that

$$f(\lambda_1, \lambda_2, ..., \lambda_n) = < f, (I - \overline{\lambda_1} S_1)^{-1}(I - \overline{\lambda_2} S_2)^{-1} \cdots (I - \overline{\lambda_n} S_n)^{-1} q1 >,$$

where q is the projection from $H^2(D^n)$ onto $H^2(D^n) \ominus [h]$([AC]). Therefore f extends analytically in the first variable to a neighborhood of λ_1 if $1/\overline{\lambda_1}$ is in the resolvent set of S_1. If $\lambda_1 = \lambda$, then $1/\overline{\lambda} = \lambda$ is in the resolvent set of S_1 by Lemma 1.1 and hence the corollary follows. □

In essence, this corollary means that if a $\lambda \in T$ is not the j-th coordinate of any of the zeros of h in $\overline{D^n}$, then every function of $H^2(D^n) \ominus [h]$ has an analytic continuation to a fixed neighborhood of $D^{j-1} \times \{\lambda\} \times D^{n-j}$.

Example : If $h(z, w) = z - \mu w$ for some nonzero $\mu \in D$, then $h(\lambda, w)$ is holomorphic in a neighborhood of $\overline{D}$ for every $\lambda \in T$ and $h(\lambda, \cdot)$ is invertible in $H^\infty(D)$. Then by the corollary, all the functions of $H^2(D^2) \ominus [z - \mu w]$ are analytic in a neighborhood of the unit disk in the first variable.

When $n = 1$, it is well known that every function of $H^2(D)$, the Hardy space over the unit disk, has an inner-outer(H) factorization. But that is far from the case even when $n = 2$([Ru, pp 63]). (Here we alert the reader that the notion of outer function considered in [Ru] is not the same as that used here even though they are the same when n=1. We refer the reader to [Ru] for a detailed discussion.) Equipped with corollary 2.1 and a theorem from [AC], we find a simple way to determine that certain functions have no inner-outer(H) factorization.

Theorem 2.2. *Suppose h is holomorphic in a neighborhood of $\overline{D^n}$ satisfying the condition (*) in theorem 1.4 such that*

1. $Z(h) \cap D^n$ is not a subset of a countable union of slices of D^n, and

2. there is an integer $j \leq n$ such that $\pi_j(\overline{Z(h) \cap D^n})$ doesn't contain the unit circle T.

Then h has no inner-outer(H) factorization.

We note that condition 1 demands in particular that $Z(h) \cap D^n$ is not empty. The proof uses the following theorem of P.Ahern and D.Clark[AC].

Theorem 2.3. *Suppose $M = gH^2(D^n)$ where g is inner, $\lambda = (\lambda_1, \lambda_2, ...\lambda_n) \in \partial D^n$ with $|\lambda_j| = 1$, and there is a neighborhood B of λ such that every function in $H^2(D^n) \ominus M$ has an analytic continuation into B. Then g is a function of z_j*

alone. In particular , if more than one of the λ_i has modulus 1, g is a constant and $M^{\perp} = 0$.

We now come to the proof of theorem 2.2.

PROOF. Suppose h is a function with the properties mentioned in the theorem. If h has the inner-outer(H) factorization GF, then

$$Z(h) \cap D^n = Z(G) \cap D^n$$

and $[h] = GH^2(D^n)$.

If $\mu \in T \setminus \pi_j\overline{(Z(h) \cap D^n)}$, then $\mu \in \rho(S_j)$ from theorem 1.5. Let $B(\mu) \subset \rho(S_j)$ be a neighborhood of μ. Corollary 2.1 shows that each f of $H^2(D^n) \ominus GH^2(D^n)$ has an analytic continuation into $D^{j-1} \times B(\mu) \times D^{n-j}$. Then the theorem of Ahern and Clark implies that G depends on z_j only. Therefore $Z(h) \cap D^n$ must be a subset of a countable union of slices of D^n which contradicts the assumption. □

In view of the results in [ACD], theorem 2.2 enables one to construct many examples of submodules that are not equivalent to $H^2(D^n)$.

3. Quotient module

Submodules with *thin* zero sets exhibit the so called *rigidity phenomenon* [DY][Pa]. Things are much different when the zero sets are hyper-surfaces. For example, it is well known that M is equivalent to gM for any submodule if g is inner. But the zero sets of M and gM can be quite different. In this section, we prove a theorem which shows that for quotient modules this is by far not the case.

Let us first recall some definitions. If H_1 and H_2 are two $A(D^n)$ modules, then H_1 is said to be *unitarily equivalent(similar)* to H_2 if there is a unitary(invertible) module map from H_1 to H_2. A bounded module map T from H_1 to H_2 is called *quasi-affine* if it is one to one and has dense range. H_1 and H_2 are said to be *quasi-similar* if there are quasi-affine module maps from H_1 to H_2 and from H_2 to H_1. One sees that similarity implies quasi-similarity .

In [DF], the first author and C. Foias have shown that if H_1 and H_2 are two submodules of $H^2(D^n)$ then $H^2(D^n) \ominus H_1$ is unitarily equivalent to $H^2(D^n) \ominus H_2$ if and only if $H_1 = H_2$. In [DC], the first author and Xiaoman Chen were able to prove that if either J_1 or J_2 is principal in $C[z, w]$, then $H^2(\Omega) \ominus [J_1]$ is quasi-similar to $H^2(\Omega) \ominus [J_2]$ if and only if $J_1 = J_2$, where Ω can be any bounded domain. They proved the result through a detailed analysis of the zero varieties of the two ideals. We refer the reader to [FS] and [Wo] for definitions of Hardy spaces over general domains.

Recently we discovered a direct approach which generalize the results in [DC]. This approach was also suggested by Keren Yan in a less general context some years ago. We state our result in the polydisk case.

Proposition 3.1. *If M_1 and M_2 are submodules of $H^2(D^n)$ such that there is a quasi-affine module map from $H^2(D^n)\ominus M_1$ to $H^2(D^n)\ominus M_2$, then every bounded function in M_1 is also contained in M_2.*

PROOF. Suppose

$$p_1 : H^2(D^n) \longrightarrow M_1, \quad q_1 : H^2(D^n) \longrightarrow H^2(D^n)\ominus M_1,$$
$$p_2 : H^2(D^n) \longrightarrow M_2, \quad q_2 : H^2(D^n) \longrightarrow H^2(D^n)\ominus M_2,$$

are projections and let the operator $T : H^2(D^n)\ominus M_1 \longrightarrow H^2(D^n)\ominus M_2$ be the quasi-affine module map. Then

$$q_2 f T = T q_1 f, \qquad \textit{for any } f \in A(D^n).$$

As multiplication operators acting on $H^2(D^n)$, $H^\infty(D^n)$ is the *weak* operator closure of $A(D^n)$, so the equality

$$q_2 f T = T q_1 f$$

holds for every $f \in H^\infty(D^n)$. In particular, for any bounded function $g \in M_1$, it follows

$$q_2 g T = T q_1 g = 0,$$

and hence the operator $q_2 g = 0$ since T has dense range. If we choose $q_2 1 \in H^2(D^n)\ominus M_2$, then

$$0 = q_2 g(q_2 1) = q_2(g).$$

This shows that $g \in M_2$. □

This proposition leads to the following theorem which shows the rigidity phenomenon of quotient Hardy modules.

Theorem 3.2. *If M_1 and M_2 are submodules both generated by bounded holomorphic functions , then $H^2(D^n)\ominus M_1$ and $H^2(D^n)\ominus M_2$ are quasi-similar $A(D^n)$ modules if and only if $M_1 = M_2$.*

PROOF. Sufficiency is obvious.

From the above theorem every bounded function of M_1 also belongs in M_2. Since M_1 is generated by bounded functions and M_2 is a closed submodule, we have that

$$M_1 \subset M_2.$$

Similarly we also have the inclusion

$$M_2 \subset M_1,$$

and hence

$$M_1 = M_2.$$

□

Here we note that the above theorem in the case of the Hardy space over the unit disk is also implied by the Livsic-Moeller theorem[Ni]. We also point out that the proofs work for other $A(D^n)$ modules, such as the weighted Hardy modules and even the Bergman modules.

We end this paper by a conjecture suggested by theorem 3.2.

Conjecture. *If M_1 and M_2 are submodules of $H^2(D^n)$, then $H^2(D^n) \ominus M_1$ is similar to $H^2(D^n) \ominus M_2$ if and only if $M_1 = M_2$.*

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