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A UNICITY THEOREM FOR MEROMORPHIC MAPPINGS

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ABSTRACT. We prove a unicity theorem of Nevanlinna for meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^m$.

1. INTRODUCTION

As an application of Nevanlinna's second main theorem and Borel's lemma, R. Nevanlinna proved that for any two meromorphic functions in the complex plane $\mathbb{C}$ on which they share four distinct values, then, these two meromorphic functions are the same up to a Möbius transformation. Since then, there have been a number of papers (e.g. [4], [2], and [8]) working towards this kind of problems. Recently, motivated by the accomplishment of the second main theorem for moving targets (cf. [6]), M. Shirosaki [9] has proved a unicity theorem of meromorphic functions for moving targets, i.e. replacing four values in the original problem by four 'small' functions. However, his result is only dealing with one complex variable. In this paper, we extend this kind of theorem to the case of meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^m$ for moving targets. Broadly speaking, for any two meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^m$ sharing $2(m+1)$ 'small' mappings in a certain sense, then, there is a non-zero bilinear function vanishing on these two meromorphic mappings. Particularly, when $m = 1$, these two meromorphic mappings in $\mathbb{C}^n$ are the same up to a Möbius transformation. Thus, Shirosaki's result is a special case of ours when $n = m = 1$.

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2. Preliminaries and our Results

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we define, for any $r \in \mathbb{R}^+$,

$$\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}, \quad \text{and} \quad B_n(r) = \{z \in \mathbb{C}^n;\ \|z\| < r\},$$
$$S_n(r) = \{z \in \mathbb{C}^n;\ \|z\| = r\}, \quad \text{and} \quad B_n[r] = \{z \in \mathbb{C}^n;\ \|z\| \le r\}.$$

Let $d = \partial + \bar\partial$ and $d^c = (\partial - \bar\partial)/4\pi i$, we write,

$$\omega_n(z) = dd^c \log \|z\|^2, \quad \text{and} \quad \sigma_n(z) = d^c \log \|z\|^2 \wedge \omega_n^{n-1}(z),$$
$$\nu(z) = dd^c \|z\|^2 \quad \text{for} \quad z \in \mathbb{C}^n \setminus \{0\}.$$

Thus $\sigma_n(z)$ defines a positive measure on $S_n(r)$ with total measure one, and $\nu_n^n(z)$ defines a positive measure on $B_n(r)$ with total measure one.

Let $F:\ \mathbb{C}^n \to \mathbb{P}^m$ be a meromorphic mapping, then F can be represented by a holomorphic mapping $f:\ \mathbb{C}^n \to \mathbb{C}^{m+1}$ such that $f = (f_0, f_1, \cdots, f_m)$, and

$$I_f := \{z \in \mathbb{C}^n;\ f_0(z) = f_1(z) = \cdots = f_m(z) = 0\}$$

is an analytic subvariety of $\mathbb{C}^n$ of codimension at least 2 and $F = \pi \circ f$ on I_f on $\mathbb{C}^n \setminus I_f$, where $\pi:\ \mathbb{C}^{m+1} \setminus \{0\} \to \mathbb{P}^m$ is $\pi(w) \equiv [w]$ = complex line through 0 and w. We call I_f the set of indeterminacy. We call f as a reduced representative of F (the only factors common to $f_0, \cdots, f_m$ are units). F will often be identified with its reduced representative f.

Let $I = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index with $\alpha_j \in \mathbb{Z}^+ \cup \{0\}$ with $1 \le j \le n$. We denote the length of the I by $|I| = \sum_{j=1}^n \alpha_j$ and define

$$\partial^I f = \left(\frac{\partial^{|I|} f_1}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}, \ \cdots, \ \frac{\partial^{|I|} f_m}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \right)$$

and $f_{z_j^k} = \partial^k f / \partial z_j^k = (\partial^k f_1 / \partial z_j^k, \ \cdots, \ \partial^k f_m / \partial z_j^k)$, for any holomorphic map

$$f = (f_1, \ \cdots, \ f_m):\ \mathbb{C}^n \to \mathbb{C}^m.$$

For all $0 < s < r$, the growth of a meromorphic mapping $f:\ \mathbb{C}^n \to \mathbb{P}^m$ is measured by its characteristic function

$$\begin{aligned} T_f(r,s) &= \int_s^r \frac{1}{t^{2n-1}} \int_{B_n[t]} f^*(\omega_0) \wedge \nu_n^{n-1}\, dt \\ &= \int_s^r \frac{1}{t^{2n-1}} \int_{B_n[t]} dd^c \log \|f\|^2 \wedge \nu_n^{n-1}\, dt, \end{aligned}$$

where ω_0 is the Fubini-Study metric on $\mathbb{P}^m$. Sometimes, for simplicity, we write $T_f(r)$ instead of $T_f(r,s)$ if no confusion occurs.

A meromorphic mapping $a : \mathbb{C}^n \to \mathbb{P}^m$ is 'small' with respect to mapping f of $\mathbb{C}^n$ into $\mathbb{P}^m$ if $T_a(r) = o(T_f(r))$. Let $a = (a_0,\ a_1,\ \cdots,\ a_m)$ be a reduced representation of a, we denote by

$$N_{f,a}(r) = \int_{S_n(r)} \log |(f,\ a)|\, \sigma_n + O(1), \quad \text{and} \quad m_{f,a}(r) = \int_{S_n(r)} \log \frac{\|f\|}{|(f,\ a)|}\, \sigma_n,$$

where $(f,\ a) = \sum_0^m f_j a_j$. Moreover, the first main theorem states

$$T_f(r) = N_{f,a}(r) + m_{f,a}(r) + O(1) = \int_{S_n(r)} \log \|f\|\, \sigma_n + O(1).$$

If f is a meromorphic function in $\mathbb{C}^n$ and $a \in \mathbb{C} \cup \{\infty\}$, then we adopt the standard notations for $N_f(a,r)$, $m_f(a,r)$, and etc. Thus we have

$$N_{f,a}(r) = N_{(f,a)}(0,r)$$

if a is a non-zero meromorphic mapping.

For any $q \geq m+1$, let $a_1, \cdots, a_q$ be q 'small' meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^m$ with reduced representations $a_j = (a_{j0},\ \cdots,\ a_{jm})$ $(0 \leq j \leq q)$. We say that a_j is in general position if for any $0 \leq j_0,\ j_1, \cdots,\ j_m \leq q$, $\det(a_{j_k l}) \not\equiv 0$.

Let $\mathcal{R}(\{a_j\}_1^q)$ be the smallest subfield containing $\{a_{jk}\} \cup \mathbb{C}$ of the meromorphic mappings field on $\mathbb{C}$. Then, for any $h \in \mathcal{R}(\{a_j\}_1^q)$, h is a 'small' mapping with respect to f. Furthermore, we call that f is non-degenerate over $\mathcal{R}(\{a_j\}_1^q)$ if $f_0,\ f_1,\ \cdots,\ f_m$ are linearly independent over $\mathcal{R}(\{a_j\}_1^q)$.

Suppose $f(r)$ and $g(r)$ are two positive functions in $\mathbb{R}^+$. "$f(r) \leq g(r)\|$" means that $f(r) \leq g(r)$ for all large r outside a set of finite Lebesgue measure.

Assume f and $\{a_t\}_1^q$ $(q \geq m+1)$ are meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^m$, and $\{a_t\}_1^q$ are 'small' with respect to f. If f is non-degenerate over $\mathcal{R}(\{a_j\}_1^q)$, then, Nevanlinna's second main theorem for moving targets can be described as, for any $\epsilon > 0$,

$$(1) \qquad (q - m - 1 - \epsilon)T_f(r) \leq \sum_{j=1}^{q} N_{f,a_j}(r) + o(T_f(r))\|.$$

It follows that a corresponding defect relation for moving targets holds. This was proved by M. Ru and W. Stoll [6] in a more general setting. Recently, M. Shirosaki [7] has given another proof of (1) when f is a holomorphic mapping of $\mathbb{C}$ into $\mathbb{P}^m$. In fact, Shirosaki's proof of (1) can be carried over to any meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^m$. We now state our unicity theorem.

Theorem 2.1. *Let $f, g : \mathbb{C}^n \to \mathbb{P}^m$ be non-constant meromorphic mappings, and let $\{a_t\}_{t=1}^{2(m+1)}$ be 'small' (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^m$*

with in general position, and f non-degenerate over $\mathcal{R} = \mathcal{R}(\{a_t\}_{t=1}^{2(m+1)})$. Assume (i) there are nowhere vanishing holomorphic functions $\psi_j : \mathbb{C}^n \to \mathbb{C}$ such that

$$F_j := (f,\ a_j) = \psi_j(g,\ a_j) =: \psi_j G_j, \quad j = 1, 2, \cdots, 2(m+1); \tag{2}$$

(ii) if $F_l(z) = G_l(z) = 0$ for some $z \in \mathbb{C}^n$ and some $l \in \{1, \cdots, 2(m+1)\}$, then there is a $c(z) \in \mathbb{C} \setminus \{0\}$ such that $f(z) = c(z)g(z)$.
Then there exists $(m+1) \times (m+1)$ non-zero matrix Y with elements in $\mathcal{R}$ such that

$$(f_0(z),\ f_1(z),\ \cdots,\ f_m(z))Y(z) \begin{pmatrix} g_0(z) \\ g_1(z) \\ \vdots \\ g_m(z) \end{pmatrix} \equiv 0,$$

where $(f_0, \cdots, f_m)$ and $(g_0, \cdots, g_m)$ are reduced representatives of f and g, respectively.

Corollary 2.2. *Let $f, g : \mathbb{C}^n \to \mathbb{P}^1$ be meromorphic mappings, and suppose $\{a_t\}_{t=1}^4$ are 'small' (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^1$ with in general position, and f is non-degenerate over $\mathcal{R}(\{a_t\}_{t=1}^4)$. Assume there are nowhere vanishing holomorphic functions $\psi_j : \mathbb{C}^n \to \mathbb{C}$ such that*

$$(f,\ a_j) = (g,\ a_j)\psi_j, \quad j = 1, 2, 3, 4;$$

then, there are A, B, C, and D in $\mathcal{R}(\{a_t\}_{t=1}^4)$ with $AD - BC \not\equiv 0$ such that

$$f = \frac{Ag + B}{Cg + D}.$$

Remark. The unicity theorem proved by Shirosaki in [9] and [8] is a special case of our Corollary when $n = 1$. Moreover, we can see from the proof of Theorem that the determinant of the matrix Y may be identically equal to zero in some cases.

3. Lemmas

In order to prove our theorem, we need the following lemmas.

Lemma 3.1. *Let $f_0,\ f_1,\ \cdots,\ f_m$ be entire functions in $\mathbb{C}^n$ and linearly independent over $\mathbb{C}$. Then there are multi-indices $\beta_1,\ \cdots,\ \beta_m$ such that $1 \le |\beta_j| \le j$ for any $j = 1, \cdots, m$, and $f,\ \partial^{\beta_1} f,\ \cdots,\ \partial^{\beta_m} f$ are linearly independent over $\mathbb{C}$, where $f = (f_0,\ f_1,\ \cdots,\ f_m)$.*

The proof of the lemma can be found in [11] and [3].

For any multi-indices $\beta_1, \cdots, \beta_m$, let

$$W_{\beta_1, \cdots, \beta_m}(f_0, \cdots, f_m) = \begin{vmatrix} f_0 & f_1 & \cdots & f_m \\ \partial^{\beta_1} f_0 & \partial^{\beta_1} f_1 & \cdots & \partial^{\beta_1} f_m \\ \cdots & \cdots & \cdots & \cdots \\ \partial^{\beta_m} f_0 & \partial^{\beta_m} f_1 & \cdots & \partial^{\beta_m} f_m \end{vmatrix}.$$

Thus, $W_{\beta_1, \cdots, \beta_m}(f_0, \cdots, f_m) \equiv 0$ if and only if $f, \partial^{\beta_1} f, \cdots, \partial^{\beta_m} f$ are linearly dependent, where $f = (f_0, \cdots, f_n)$. Moreover, we have

Lemma 3.2. *Holomorphic functions $f_0, f_1, \cdots, f_m$ are linearly dependent over $\mathbb{C}$ if and only if*

$$W_{\beta_1, \cdots, \beta_m}(f_0, \cdots, f_m) \equiv 0,$$

for any multi-indices $\beta_1, \cdots, \beta_m$ with $|\beta_j| \leq j$ for $j = 1, \cdots, m$.

PROOF. If $f_0, f_1, \cdots, f_m$ are linearly dependent over $\mathbb{C}$, then there are $m+1$ complex numbers c_i $(i = 0, \cdots, m)$ in $\mathbb{C}$ such that $|c_0| + |c_1| + \cdots + |c_m| \neq 0$, and for any $z \in \mathbb{C}^n$,

$$c_0 f_0(z) + c_1 f_1(z) \cdots c_m f_m(z) = 0.$$

Consequently, for any multi-indices β_j $(j = 1, \cdots, m)$ with $|\beta_j| \geq 1$,

$$c_0 \partial^{\beta_j} f_0(z) + c_1 \partial^{\beta_j} f_1(z) \cdots c_m \partial^{\beta_j} f_m(z) = 0.$$

It follows

$$W_{\beta_1, \cdots, \beta_m}(f_0, \cdots, f_m) \equiv 0,$$

for any multi-indices $\beta_1, \cdots, \beta_m$ with $|\beta_j| \leq j$ for $j = 1, \cdots, m$.

Conversely, if $f_0, f_1, \cdots, f_m$ are linearly independent over $\mathbb{C}$, then we have from Lemma 1 that there are multi-indices $\beta_1, \cdots, \beta_m$ such that $1 \leq |\beta_j| \leq j$ for any $j = 1, \cdots, m$, and $f, \partial^{\beta_1} f, \cdots, \partial^{\beta_m} f$ are linearly independent over $\mathbb{C}$, where $f = (f_0, f_1, \cdots, f_m)$. Thus,

$$W_{\beta_1, \cdots, \beta_m}(f_0, \cdots, f_m) \not\equiv 0,$$

for some multi-indices $\beta_1, \cdots, \beta_m$ with $1 \leq |\beta_j| \leq j$ for $j = 1, \cdots, m$. This is a contradiction. Therefore Lemma 2 is proved completely. □

Lemma 3.3. *Suppose $m \geq 1$ is an integer, and $h_1, \cdots, h_m$ are nowhere vanishing entire functions in $\mathbb{C}^n$, and $a_1, \cdots, a_m$ are non-zero meromorphic functions in $\mathbb{C}^n$ with*

(3) $$T_{a_j}(r) = o(T(r)) + O(1),$$

where $T(r) = \sum_1^m T_{h_j}(r)$ *(Note: if one of* T_{h_j} *'s is unbounded, then (3) is* $T_{a_j}(r) = o(T(r))$*; and otherwise, all* h_j *'s and* a_j *'s are constants). Assume that*

$$a_1h_1 + a_2h_2 + \cdots + a_mh_m \equiv 1. \tag{4}$$

Then $a_1h_1,\ a_2h_2,\ \cdots,\ a_mh_m$ *are linearly dependent over* $\mathbb{C}$*. Moreover, if* $m = 1$*, then* h_1 *and* a_1 *are constants; and if* $m = 2$*, then* $h_1,\ h_2,\ a_1,$ *and* a_2 *are constants.*

PROOF. First, we consider the case of $m = 1$. Suppose h_1 is not a constant, then T_{h_1} is unbounded. Therefore, we have from (4), the first main theorem, and (3) that

$$T_{h_1}(r) = T_{1/a_1}(r) + O(1) = T_{a_1}(r) + O(1) = o(T_{h_1}(r)).$$

This is impossible. Thus, h_1 is a constant, so is a_1.

Second, we deal with the case of $m \geq 2$. Put $H_j = a_jh_j$. Without loss of generality, we assume H_j $(1 \leq j \leq m)$ is not identically equal to zero and $m \geq 2$. For any multi-indices $\beta_1,\ \cdots,\ \beta_{m-1}$ with $|\beta_j| \geq 1$ for $j = 1, \cdots, m-1$, differentiating both sides of (4) gives

$$\sum_{j=1}^m \frac{\partial^{\beta_i} H_j}{H_j} H_j = 0, \qquad (1 \leq i \leq m-1),$$

which, with (4), form a system of m equations $G_{\beta_1,\ \cdots,\ \beta_{m-1}} H = E$, where

$$E = (1, 0, \cdots, 0)^t, \qquad H = (H_1, H_2, \cdots, H_m)^t, \qquad \text{and}$$

$$G_{\beta_1,\ \cdots,\ \beta_{m-1}} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{\partial^{\beta_1} H_1}{H_1} & \frac{\partial^{\beta_1} H_2}{H_2} & \cdots & \frac{\partial^{\beta_1} H_m}{H_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{\beta_{m-1}} H_1}{H_1} & \frac{\partial^{\beta_{m-1}} H_2}{H_2} & \cdots & \frac{\partial^{\beta_{m-1}} H_m}{H_m} \end{pmatrix}.$$

We claim $\det G_{\beta_1,\ \cdots,\ \beta_{m-1}} \equiv 0$. In fact, if it is not so, then, $G_{\beta_1,\ \cdots,\ \beta_{m-1}} H = E$ has unique solutions. Moreover, each solution is composed of logarithmic derivatives $\partial^{\beta_j} H_i / H_i$. Thus we have from a logarithmic derivatives lemma (e.g. see [10] or [11]) and the definition of H_j that

$$T_{h_j}(r) \leq O(\sum_{i=1}^m T_{a_i}(r)) + o(\sum_{i=1}^m T_{H_i}(r)) + O(1)\|.$$

Consequently, if one of T_{h_j}'s is unbounded, then we get from (3) and $H_i = a_ih_i$ that

$$\sum_{j=1}^m T_{h_j}(r) \leq O(\sum_{i=1}^m T_{a_i}(r)) + o(\sum_{i=1}^m T_{h_i}(r)) + O(1) \leq o(\sum_{i=1}^m T_{h_i}(r))\|.$$

This is a contradiction. So, if one of T_{h_j}'s is unbounded, then

$$\det G_{\beta_1, \cdots, \beta_{m-1}} \equiv 0,$$

for any β_j's. However, if all T_{h_j}'s are bounded, then all h_j's are constants, and so all a_j's are constants, too. Hence, in either case, we always have, for $m \geq 2$,

$$\det G_{\beta_1, \cdots, \beta_{m-1}} \equiv 0, \quad \text{for any} \quad \beta_1, \cdots, \beta_m. \tag{5}$$

Hence, $W_{\beta_1, \cdots, \beta_{m-1}}(H_1, \cdots, H_m) \equiv 0$ for any $\beta_1, \cdots, \beta_m$. It follows from Lemma 3.2 that the first part of the lemma is proved.

If $m = 2$, then, for any $\beta_j = (0, \cdots, 1, \cdots, 0)$ (the j-th component is 1), we have from (5) that $\det G_{\beta_j} \equiv 0$. Accordingly, we have two equations:

$$\begin{aligned} H_1 + H_2 &= 1, \\ \frac{\partial^{\beta_j} H_1}{H_1} - \frac{\partial^{\beta_j} H_2}{H_2} &= 0, \end{aligned}$$

from (5) and the definition of G_{β_j}. Solving the system presents $\partial H_i / \partial z_j \equiv 0$ for $i = 1, 2$ and $j = 1, \cdots, n$. It turns out each H_i $(i = 1, 2)$ is a constant. Therefore, T_{a_i} and T_{h_i} have the same order of magnitude. Now suppose that if one of h_1 and h_2 is not constant, then $T(r) = T_{h_1}(r) + T_{h_2}(r)$ is unbounded. Thus, (3) implies

$$T_{h_1}(r) + T_{h_2}(r) = T_{a_1}(r) + T_{a_2}(r) + O(1) = o(T(r)),$$

which is a contradiction. Hence, each h_i is a constant, so is each a_i. It follows that Lemma 3.3 is proved completely. □

Remark. Clearly, Lemma 3.3 extends the classical Borel Lemma (e.g. see [5]) which is the case of $n = 1$ in Lemma 3.3; and the frame of Lemma 3.3 is influenced by [5]. In addition, ones can find other versions of extension of the classical Borel lemma (e.g. see [1]).

4. Proof of the Theorem

For simplicity, we write,

$$f = (f_0, \cdots, f_m), \quad \text{and} \quad g = (g_0, \cdots, g_m);$$

and let

$$a_t = (a_{t0}, \cdots, a_{tm}), \quad t = 1, \cdots, 2(m+1).$$

We first show that our Theorem is a consequence of the following claim.

Claim: There exist j and k with $j \neq k$ such that ψ_j / ψ_k is a non-zero constant.

Suppose the claim is true, then there is a non-zero constant b such that

$$\psi_j = b\psi_k.$$

It turns out from (2) that

$$b(f,\ a_k)(g,\ a_j) = (f,\ a_j)(g,\ a_k),$$

which is

(6)
$$\begin{pmatrix} f_0 \\ \vdots \\ f_m \end{pmatrix}^t \left(b \begin{pmatrix} a_{k0} \\ \vdots \\ a_{km} \end{pmatrix} \begin{pmatrix} a_{j0} \\ \vdots \\ a_{jm} \end{pmatrix}^t - \begin{pmatrix} a_{j0} \\ \vdots \\ a_{jm} \end{pmatrix} \begin{pmatrix} a_{k0} \\ \vdots \\ a_{km} \end{pmatrix}^t \right) \begin{pmatrix} g_0 \\ \vdots \\ g_m \end{pmatrix} \equiv 0.$$

This implies our Theorem.

We now start to prove Claim. Let j and k be any positive integers with $j \neq k$ and $1 \leq j, k \leq 2(m+1)$, then (2) gives

$$\text{(7)} \qquad \frac{\psi_j}{\psi_k} - 1 = \frac{F_j G_k - F_k G_j}{F_k G_j} = \frac{1}{F_k G_j} \begin{vmatrix} (f,\ a_j) & (g,\ a_j) \\ (f,\ a_k) & (g,\ a_k) \end{vmatrix}$$

$$\text{(8)} \qquad = \frac{1}{F_k G_j} \sum_{q=0}^{m} \sum_{p=0}^{m} \begin{vmatrix} f_p a_{jp} & g_q a_{jq} \\ f_p a_{kp} & g_q a_{kq} \end{vmatrix}$$

$$=: \frac{1}{F_k G_j}(f_0,\ f_1,\ \cdots,\ f_m) S_{jk} \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_m \end{pmatrix},$$

where the matrix S_{jk} is anti-symmetric, i.e. $S_{jk}^t = -S_{jk}$. Note that if S_{jk} is identically equal to zero, i.e. ψ_j/ψ_k is identically equal to one, for some j and k with $j \neq k$, then Claim is proved.

Let j and k be fixed and $j \neq k$, and suppose that ψ_j/ψ_k is not identically equal to a constant. Assume l is neither equal to j nor equal to k, and with $1 \leq l \leq 2m+2$. If $F_l(z) = 0$ for some $z \in \mathbb{C}^n$, then the equation(2) and the nowhere vanishness of ψ_j's give that $G_l(z) = 0$. Hence we have from the condition (ii) of Theorem that $f(z) = c(z)g(z)$ for some $c(z) \in \mathbb{C} \setminus \{0\}$. It follows from (7) that

$$\frac{\psi_j}{\psi_k}(z) - 1 = \frac{c(z)}{F_k G_j} g(z) S_{jk}(z) g^t(z) = 0.$$

It turns out that, (noting $N_{f,a_l}(r) = N_{F_l}(0,\ r)$)

$$\sum_{l \neq j,k; 1 \leq l \leq 2m+2} N_{f,a_l}(r) = \sum_{l \neq j,k; 1 \leq l \leq 2m+2} N_{F_l}(0,\ r) \leq N_{\psi_j/\psi_k}(1,\ r).$$

Therefore, for $j \neq k$, $u \neq v$ and $\#\{j,k,u,v\} \geq 3$ (here we denote the number of distinct integers $n_1, n_2, \cdots, n_p$ by $\#\{n_1, n_2, \cdots, n_p\}$), we obtain from (1) that

$$\begin{aligned} T_{\psi_j/\psi_k}(r) + T_{\psi_u/\psi_v}(r) &\geq N_{\psi_j/\psi_k}(1,\ r) + N_{\psi_u/\psi_v}(1,r) \\ &\geq \sum_{l\neq j,k;1\leq l\leq 2m+2} N_{f,a_l}(r) + \sum_{l\neq u,v;1\leq l\leq 2m+2} N_{f,a_l}(r) \\ &\geq \sum_{at\ least\ 2m+1\ different\ terms} N_{f,a_k}(r) \\ &\geq (2m+1-m-1-\epsilon)T_f(r) - o(T_f(r)) \geq \frac{m}{2}T_f(r)\|. \end{aligned} \tag{9}$$

It follows from (9) that, for $j \neq k$, $u \neq v$, $(\#\{j,k,u,v\} \geq 3)$,

$$T_{a_t}(r) = o(T_f(r)) = o(T_{\psi_j/\psi_k}(r) + T_{\psi_u/\psi_v}(r)), \quad \text{for any} \quad t = 1, \cdots, 2m+2.$$

Consequently, for any $h \in \mathcal{R}(\{a_t\}_{t=1}^{2(m+1)})$, we have

$$T_{h_t}(r) = o(T_{\psi_j/\psi_k}(r) + T_{\psi_u/\psi_v}(r)). \tag{10}$$

Since $F_j - \psi_j G_j = 0$ for $j = 1, \cdots, 2(m+1)$ and f is non-degenerate, then, by Cramer's rule for solving systems of equations, we obtain

$$\begin{aligned} \Delta &:= \\ &\begin{vmatrix} a_{10} & \cdots & a_{1m} & -\psi_1 a_{10} & \cdots & -\psi_1 a_{1m} \\ a_{20} & \cdots & a_{2m} & -\psi_2 a_{20} & \cdots & -\psi_2 a_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(m+1)0} & \cdots & a_{(m+1)m} & -\psi_{m+1} a_{(m+1)0} & \cdots & -\psi_{m+1} a_{(m+1)m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(2m+2)0} & \cdots & a_{(2m+2)m} & -\psi_{2m+2} a_{(2m+2)0} & \cdots & -\psi_{2m+2} a_{(2m+2)m} \end{vmatrix} \\ &= \sum_{1\leq i_0, i_1, \cdots, i_m \leq 2m+2} A_{i_0 i_1 \cdots i_m} \psi_{i_0}\psi_{i_1}\cdots\psi_{i_m} \equiv 0, \end{aligned}$$

where

$$A_{i_0 i_1 \cdots i_m} = (-1)^{sign(i_0, i_1, \cdots, i_m)} \begin{vmatrix} a_{i_0 0} & \cdots & a_{i_0 m} \\ a_{i_1 0} & \cdots & a_{i_1 m} \\ \vdots & \vdots & \vdots \\ a_{i_m 0} & \cdots & a_{i_m m} \end{vmatrix} \begin{vmatrix} \psi_{i_0} a_{i_0 0} & \cdots & a_{i_0 m} \\ \psi_{i_1} a_{i_1 0} & \cdots & a_{i_1 m} \\ \vdots & \vdots & \vdots \\ \psi_{i_m} a_{i_m 0} & \cdots & a_{i_m m} \end{vmatrix}.$$

Note first that the determinant is the cofactor of the second determinant in Δ and

$$A_{i_0 i_1 \cdots i_m} \in \mathcal{R}(\{a_t\}_{t=1}^{2(m+1)}).$$

Furthermore, $A_{i_0 i_1 \cdots i_m}$ is not identically equal to zero since $a_i's$ are in general position. Therefore, we have

$$\Delta = \sum_{1 \le i_0, i_1, \cdots, i_m \le 2m+2} A_{i_0 i_1 \cdots i_m} \psi_{i_0} \psi_{i_1} \cdots \psi_{i_m} \equiv 0. \tag{11}$$

In order to avoid an unnecessary complexity of computation, we begin to finish the proof of our Claim by using induction with respect to m. First let $m = 1$, thus (11) can be written as

$$A_{12}\psi_1\psi_2 + A_{13}\psi_1\psi_3 + A_{14}\psi_1\psi_4 + A_{23}\psi_2\psi_3 + A_{24}\psi_2\psi_4 + A_{34}\psi_3\psi_4 = 0. \tag{12}$$

We first show that we can eliminate any term in (12) as long as the identity (12) contains more than two terms. Without loss of generality, let us eliminate the term $A_{34}\psi_3\psi_4$. Indeed, (12) can be written as

$$\frac{A_{12}}{A_{34}}\frac{\psi_1\psi_2}{\psi_3\psi_4} + \frac{A_{13}}{A_{34}}\frac{\psi_1\psi_3}{\psi_3\psi_4} + \frac{A_{14}}{A_{34}}\frac{\psi_1\psi_4}{\psi_3\psi_4} + \frac{A_{23}}{A_{34}}\frac{\psi_2\psi_3}{\psi_3\psi_4} + \frac{A_{24}}{A_{34}}\frac{\psi_2\psi_4}{\psi_3\psi_4} = -1,$$

since ψ_k is nowhere vanishing and A_{34} is not identically equal to zero. Because the number of the terms in the left hand side of the equation is more than one, (10) ensures the condition (3) is satisfied. Hence from Lemma 3.3, there are $c_{ij} \in \mathbb{C}$, not all zero, such that

$$c_{12}A_{12}\psi_1\psi_2 + c_{13}A_{13}\psi_1\psi_3 + c_{14}A_{14}\psi_1\psi_4 + c_{23}A_{23}\psi_2\psi_3 + c_{24}A_{24}\psi_2\psi_4 = 0.$$

Clearly, this identity is one term shorter than the identity (11); i.e. the term $A_{34}\psi_3\psi_4$, which is in (11), has disappeared. Repeating the above procedure again and again, we get that there are some indices i_0, i_1; j_0, j_1 and k_0, k_1; and constants $c_{i_0 i_1}$, $c_{j_0 j_1}$ and $c_{k_0 k_1}$, not all zero, such that

$$c_{i_0 i_1} A_{i_0 i_1} \psi_{i_0} \psi_{i_1} + c_{j_0 j_1} A_{j_0 j_1} \psi_{j_0} \psi_{j_1} + c_{k_0 k_1} A_{k_0 k_1} \psi_{k_0} \psi_{k_1} = 0,$$

and noting $i_0 \ne i_1$; $j_0 \ne j_1$ and $k_0 \ne k_1$. Without loss of generality, we assume that $c_{k_0 k_1}$ is not equal to zero. Furthermore, either $c_{i_0 i_1}$ or $c_{j_0 j_1}$ is not equal to zero since we know that ψ_k is nowhere vanishing and A_K is not identically equal to zero. Thus,

$$\frac{c_{i_0 i_1}}{c_{k_0 k_1}} \frac{\psi_{i_0}\psi_{i_1}}{\psi_{k_0}\psi_{k_1}} + \frac{c_{j_0 j_1}}{c_{k_0 k_1}} \frac{\psi_{j_0}\psi_{j_1}}{\psi_{k_0}\psi_{k_1}} = -1.$$

It follows from Lemma 3.3 that both

$$\frac{\psi_{i_0}\psi_{i_1}}{\psi_{k_0}\psi_{k_1}} \quad \text{and} \quad \frac{\psi_{j_0}\psi_{j_1}}{\psi_{k_0}\psi_{k_1}} \tag{13}$$

are constants. Now we have to consider two cases.

Case one: Either $\#\{i_0, i_1; k_0, k_1\} = 2$ or $\#\{j_0, j_1; k_0, k_1\} = 2$, recall

$$\#\{n_0,\ n_1,\ \cdots,\ n_t\}$$

means the number of distinct integers in the set $\{n_0, n_1, \cdots, n_t\}$.

It is straightforward to see that the Claim follows this case by noting $i_0 \neq i_1$; $j_0 \neq j_1$; $k_0 \neq k_1$; and $\{q_0, q_1\} \neq \{p_0, p_1\}$ for $q \neq p$, and $p, q = i, j, k$.

Case two: Both $\#\{i_0, i_1; k_0, k_1\}$ and $\#\{j_0, j_1; k_0, k_1\}$ are greater than 2, i.e. equal to 4.

We now show that this is an impossible case (it is possible when $m > 1$). In fact, since

$$\#\{i_0, i_1; k_0, k_1\} = 4,$$

so, $\{i_0, i_1; k_0, k_1\} = \{1,\ 2,\ 3,\ 4\}$. It turns out from the fact $\{j_0, j_1\} \neq \{k_0, k_1\}$ that $\#\{j_0, j_1; k_0, k_1\}$ must be 2, which contradicts to the fact that $\#\{j_0, j_1; k_0, k_1\}$ is greater than 2.

It follows that Theorem is proved for $m = 1$. For simplicity, we consider the Claim $m = 2$ by using the fact that the Claim holds for $m = 1$. For $m = 2$, we write (11) as

$$\sum_{1 \leq i_0 < i_1 < i_2 \leq 6} A_{i_0 i_1 i_2} \psi_{i_0} \psi_{i_1} \psi_{i_2} = 0. \tag{14}$$

From the discussion we have done for $m = 1$, we know that we can eliminate any terms which do not have the factor ψ_6 in (14). In this procedure, we still have two cases as we have had for $m = 1$; either $\min(\#\{p_0, p_1; k_0, k_1\}) = 2$ for $p = i, j$, then the claim follows; or $\min_{p=i,j}(\#\{p_0, p_1; k_0, k_1\}) > 2$. If we have both that $\#\{i_0, i_1, i_2; k_0, k_1, j_2\}$ and $\#\{j_0, j_1; k_0, k_1, k_2\}$ are greater than 2, then, similar to (13) there are two non-zero constants a and b in $\mathbb{C}$ such that

$$\psi_{i_0}\psi_{i_1}\psi_{i_2} = a\psi_{k_0}\psi_{k_1}\psi_{k_2} \quad \text{and} \quad \psi_{j_0}\psi_{j_1}\psi_{j_2} = b\psi_{k_0}\psi_{k_1}\psi_{k_2}.$$

Substituting these two equations into the identity (14), we get a new identity. Clearly the new identity is at least two terms shorter than the identity (14) (Remark: it would be three terms shorter when $A_{i_0 i_1 i_2} a + A_{j_0 j_1 j_2} b + A_{k_0 k_1 k_2} = 0$). Thus we can eventually eliminate all terms which do not have factor ψ_6, i.e. we get an identity in which every term has the factor ψ_6. Thus dividing both sides of the identity by ψ_6, we obtain an identity having terms $\psi_{i_0}\psi_{i_1}$ (where $1 \leq i_0 < i_1 \leq 5$) rather than $\psi_{i_0}\psi_{i_1}\psi_{i_2}$ (where $1 \leq< i_0 < i_1 < i_2 \leq 6$). Similarly, we can get rid of the terms with the factor ψ_5 from the new identity only having terms $\psi_{i_0}\psi_{i_1}$ where $1 \leq i_0 < i_1 \leq 5$. Thus we have an identity which only contains terms $\psi_{i_0}\psi_{i_1}$ where $1 \leq i_0 < i_1 \leq 4$. This is the situation we have had in the proof for

$m = 1$. Thus the Claim is proved for $m = 2$. In the same manner, we can prove the Claim for $m = 3, 4, \cdots$. It follows that the Theorem is proved.

5. Proof of Corollary

Let

$$f = (f_0,\ f_1), \quad g = (g_0,\ g_1), \quad \text{and} \quad a_t = (a_{t0},\ a_{t1}), \quad t = 1, 2, 3, 4.$$

If $F_l(z) = G_l(z) = 0$ for some $z \in \mathbb{C}^n$, then the vectors $f(z)$ and $g(z)$ in $\mathbb{C}^2$ are perpendicular to the vector $a_l(z)$ in $\mathbb{C}^2$. So, there exists a $c(z) \in \mathbb{C} \setminus \{0\}$ such that $f(z) = c(z)g(z)$. It follows that the condition (ii) of the Theorem is redundant when $m = 1$. Thus, we know from (6) there are j and k with $j \neq k$ and $1 \leq j, k \leq 4$ and a non-zero constant b such that

$$\begin{aligned}
&(f_0,\ f_1)\begin{pmatrix} (b-1)a_{k0}a_{j0} & ba_{k0}a_{j1} - a_{k1}a_{j0} \\ ba_{k1}a_{j0} - a_{k0}a_{j1} & (b-1)a_{k1}a_{j1} \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} \\
&=: (f_0,\ f_1)E\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} \equiv 0.
\end{aligned}$$

Consequently,

$$\det E = \det\left(\begin{pmatrix} a_{j0} & ba_{k0} \\ a_{j1} & ba_{k1} \end{pmatrix}\begin{pmatrix} -a_{k0} & -a_{k1} \\ a_{j0} & a_{j1} \end{pmatrix}\right) = b\begin{vmatrix} a_{j0} & a_{j1} \\ a_{k0} & a_{k1} \end{vmatrix}^2.$$

It turns out from the general position of a_t's that $\det E$ is not identically equal to zero. Therefore, let

$$\begin{aligned}
A = -ba_{k0}a_{j1} + a_{k1}a_{j0}, \quad & B = -(b-1)a_{k0}a_{j0}, \\
C = (b-1)a_{k1}a_{j1}, \quad & D = ba_{k1}a_{j0} - a_{k0}a_{j1};
\end{aligned}$$

now our corollary can be verified after a little computation.

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