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WAVELET ANALYSIS ON THE CANTOR DYADIC GROUP

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ABSTRACT. Compactly supported orthogonal wavelets are built on the Cantor dyadic group (the dyadic or 2-series local field). Necessary and sufficient conditions are given on a trigonometric polynomial scaling filter for a multiresolution analysis to result. A Lipschitz regularity condition is defined and an unconditional L^p-convergence result is given for regular wavelet expansions ($p > 1$). Wavelets are given whose scaling filter is a trigonometric polynomial with 2^n many terms; regular wavelets with filters with 8 terms are detailed. These wavelets are identified with certain Walsh series on the real line. A Mallat tree algorithm is given for the wavelets.

1. INTRODUCTION

To better understand the construction of orthogonal wavelets and to better understand analysis on groups, we study compactly supported orthogonal wavelets on the locally compact Cantor dyadic group; this group, also known as the 2-series local field, is rather different in structure than other groups for which wavelet constructions have been carried out.

Our purpose here is to extend and strengthen the results in Lang [10]. In that paper, compactly supported orthogonal wavelets were constructed on the locally compact Cantor dyadic group. These wavelets were identified with certain Walsh series on the real line, and form bases for L^2 on the line. These bases included the familiar Haar basis, and a basis arising from a scaling filter of length 4 (4 nonzero terms, each a Walsh function).

Here, necessary and sufficient conditions analogous to those of A. Cohen are given on a scaling filter for a multiresolution analysis, and wavelets forming a basis of L^2, to arise. We also give a Lipschitz regularity condition and a Calderón-Zygmund theory which leads to an unconditional L^p-convergence result ($p > 1$) for our wavelet expansions.

We describe length 2^n wavelets (compactly supported orthogonal wavelets arising from scaling filters with 2^n many terms). We detail length 8 wavelets, some of which satisfy our Lipschitz-regularity condition. We also describe completely the regularity of the length 4 wavelets of Lang [10].

We give a fast tree algorithm for expansions in these wavelets; these wavelets may be identified with Walsh series on the real line, where analysis using the tree algorithm is possible.

We note that our work complements the work of S. Dahlke [1], who constructs wavelets on locally compact abelian groups (under certain natural assumptions), resembling spline wavelets. Spline wavelets may be built by repeated convolutions of a simple Haar-like scaling function (a "box" function). That construction on the Cantor dyadic group would actually be trivial: the natural choice for the Haar-like scaling function has itself as its convolution powers. Other constructions of wavelets on locally compact groups include Gröchenig and Madych [6], and Lemarie [12]. Also see [8], Chapter 5, sections 8, 9 and 10.

1.1. **Notation.** For the convenience of the reader, we reiterate the description in Lang [10] of the locally compact Cantor dyadic group.

Let

$$\begin{aligned} G &= \prod_{n=-\infty}^{\infty}{}^{*} Z/(2) \\ &= \{(x_j)_{j\in Z} : x_j \in \{0,1\} \text{ for all } j \text{ and } x_j = 0 \text{ for all } j > n, \text{ for some } n \in Z\}. \end{aligned}$$

Then G is an abelian group with coordinate-wise addition; G is locally compact under the cartesian product topology.

We may think of $x \in G$ as a binary expansion $x = x_n x_{n-1} \cdots x_1 x_0.x_{-1}x_{-2}\cdots$ and we identify G as a measure space with the half real line $[0,\infty)$ under the mapping $x \to |x| = \sum_{j\in Z} x_j 2^j$. This induces Haar measure on G.

Let $\Lambda \subseteq G$ be the 'lattice' subgroup $\Lambda = \{x \in G : x_j = 0 \text{ for } j < 0\}$ and let $D = \{x \in G : x_j = 0 \text{ for } j \geq 0\}$. Thus Λ is countable and closed, D is compact, and $G/\Lambda = D$. Under the map $x \to |x|$, Λ is identified with the nonnegative integers in the half real line, and D is identified with the interval $[0,\infty)$.

We now define dilation on G by $\rho : G \to G$ where $\rho(x)_j = x_{j-1}$ for $x \in G$. (This corresponds to $x \to 2x$ on the real line, under the map $x \to |x|$.) We let $\sigma : G \to G$ be the inverse mapping $\sigma(x)_j = x_{j+1}$ for $x \in G$. We let ρ^j be $\rho \circ \cdots \circ \rho$ (j times) if $j > 0$ and $\sigma \circ \cdots \circ \sigma$ (j times) if $j < 0$; ρ^0 is the identity mapping.

The Pontryagin dual group $\hat{G}$ of G is topologically isomorphic to G. We may write

$$\hat{G} = \{\omega = (\omega_j)_{j\in Z} : \omega_j \in \{0,1\} \text{ for all } j \text{ and } \omega_j = 0 \text{ for all } j > n \text{ for some } n\}$$

and define $\omega(x) = \prod_{j\in Z}(-1)^{\omega_{-1-j}x_j}$ for $x = (x_j)_j$ in G. This gives $\omega \in G$ as a group character on G. Note the minus sign in the exponent, in the subscript of ω, so that the product actually has only finitely many terms different than 1.

We define dilation on $\hat{G}$ just as on G: $\rho(\omega)_j = \omega_{j-1}$ for $\omega = (\omega_j)_j$ in $\hat{G}$. We find that the character $\rho(\omega)$ evaluated at $x = (x_j)_j$ is equal to ω evaluated at $\rho(x)$, i.e., $\omega \circ \rho = \rho(\omega)$. This is analogous to the situation for characters on the real line: $e^{i(2x)y} = e^{ix(2y)}$. We define $\Gamma \in \hat{G}$ to be the set of Λ-periodic characters on G: $\omega(x+n) = \omega(x)$ for $x \in G$, $n \in \Lambda$. The lattice subgroup Λ in $\hat{G}$ turns out to be exactly Γ.

We will use the notation $W_\omega(x) = \omega(x)$ for $\omega \in \hat{G}$ and $x \in G$. If $\omega \in \Gamma$ then $|\omega|$ is an (ordinary) integer, and W_ω is identified with an ordinary Walsh function on the real line. Here we will use the Paley enumeration: if we let $r_k(x) = \text{sign}(\sin(2\pi 2^k x))$ be the k-th Rademacher function, then the Walsh function W_n is the product of the functions r_k such that the k-th bit in the binary expansion of n is a 1. Thus W_1 is the function constantly equal to 1 on the interval $0 \le x < 1/2$ and constantly equal to -1 on $1/2 < x < 1$.

We will also use the notation $W_x(\omega) = \omega(x)$ for $x \in G$ and $\omega \in \hat{G}$ to write characters on $\hat{G}$; by Pontryagin duality, all characters on $\hat{G}$ arise in this way.

The Fourier transform is as usual: for ϕ a function on G, let

$$\hat{\phi}(\omega) = \int_G \phi(x) W_\omega(x)\, dx.$$

See Hewitt and Ross [7]; F. Schipp, W. R. Wade, P. Simon [15]; M. Taibleson [17]; B. Golubov, A. Efimov, Skvortsov [5]; or R. E. Edwards, G. I. Gaudry [4] for development of the harmonic analysis of these groups. In [17], the locally compact group G is an example of a local field, the 2 series field. Our notation follows that of Edwards and Gaudry, and Hewitt and Ross.

1.2. **Multiresolution analyses.** We define a *multiresolution analysis* (or *MRA*) to be a sequence of closed subspaces $V_j \subset L^2(G)$ $(j \in Z)$ such that

1. $V_j \subset V_{j+1}$ for all $j \in Z$
2. $f \in V_j$ if and only if $\rho f = f \circ \rho \in V_{j+1}$
3. $f \in V_0$ if and only if $\tau_n f \in V_0$ for all $n \in \Lambda$ (where $\tau_y f(x) = f(x+y)$ is our notation for translation); so $f \in V_0$ if and only if $\rho^j \tau_n f \in V_j$.

4. $\cup V_j$ is dense in $L^2(K)$ and $\cap V_j = \{0\}$
5. There is $\phi \in V_0$ such that $\{\tau_n \phi : n \in \Lambda\}$ is a Riesz basis of V_0.

2. Scaling filters

Suppose for $\phi \in L^2(G)$ we define V_j to be the L^2-closure of the linear span of $\{\phi(\rho^j(x) + n) : n \in \Lambda\}$, for $j \in Z$. To have an MRA we need $V_j \subseteq V_{j+1}$ for all $j \in Z$. In particular, we require $V_{-1} \subseteq V_0$. This entails $\phi(\sigma x) = 2\sum_{n\in\Lambda} a_n \phi(x + n)$ for some $\{a_n\}$. Under the Fourier transform, we have $\hat{\phi}(\rho\omega) = m_0(\omega)\hat{\phi}(\omega)$ where $m_0(\omega) = \sum_{n\in\Lambda} a_n W_n(\omega)$. We call m_0 a *scaling filter*. We seek conditions on m_0 such that $\hat{\phi}(\omega) = m_0(\sigma\omega)m_0(\sigma^2\omega)m_0(\sigma^3\omega)\cdots$ converges, and such that $\{\tau_n\phi : n \in \Lambda\}$ is an orthonormal set whose norm-closed linear span is a space V_0 which generates an MRA. Here ϕ is the inverse Fourier transform of $\hat{\phi}$; we call ϕ a *scaling function* for the resulting MRA.

We may show, similar to S. Mallat [13], that the product $m_0(\sigma\omega)m_0(\sigma^2\omega)$ $m_0(\sigma^3\omega)\cdots$ converges to $\hat{\phi} \in L^2(G)$ if $m_0(0) = 1$ and $|m_0(\omega+0.1)|^2 + |m_0(\omega)|^2 \equiv 1$. If the scaling filter m_0 is a trigonometric polynomial, we may give necessary and sufficient conditions similar to those of A. Cohen on m_0 for the translates of ϕ by Λ to be orthogonal. For $E \subseteq G$ we say *E is congruent to D modulo Λ* if $|E| = 1$ (Haar measure) and if for all $y \in E$ there exists $n \in \Lambda$ such that $y + n \in D$.

We then have our analogue of A. Cohen's theorem (see Daubechies [2], 6.3):

Theorem 2.1. *Suppose m_0 is a trigonometric polynomial with $m_0(\omega) = 1$ and $|m_0(\omega+0.1)|^2 + |m_0(\omega)|^2 \equiv 1$. Define $\hat{\phi}(\omega) = \prod_{j=1}^{\infty} m_0(\sigma^j\omega)$. Then the following are equivalent:*

1. *There exists a compact set E congruent to D modulo Λ and containing a neighborhood of 0 such that $\inf_{j>0}\inf_{\omega\in E}|m_0(\sigma^j\omega)| > 0$*
2. *The translates of ϕ by Λ are orthogonal.*

The proof of this theorem is essentially the same as the proof of the original version, as detailed in I. Daubechies [2], 6.3.

For m_0 and ϕ as in Theorem 2.1, we have the result:

Theorem 2.2. *The translates of ϕ by Λ form an orthonormal basis for a space V_0, the dilates of which form an MRA.*

Proof. The proof uses the following version of a lemma of Mallat [13]: Let $A(x,y) = \sum_{k\in\Lambda} g(x-k)\bar{g}(y-k)$ where g is a compactly supported bounded integrable function on G, and let T_j be the integral operator with kernel $2^j A(\rho^j(x), \rho^j(y))$. If $\int_G A(x,y)\,dy = 1$ for almost all x, then T_j converges to the identity operator.

Note that ϕ as in Theorem 2.1 is compactly supported. This follows from $\hat{\phi}(\omega) = m_0(\sigma\omega)m_0(\sigma^2\omega)\cdots$ being constant on cosets of $\rho^j D$ for some j (m_0 is a trigonometric polynomial).

We also make note of the following version of Poisson summation: if $\phi \in L^2(G)$ is compactly supported then $\sum_{k\in\Lambda} \phi(x-k) = \sum_{n\in\Lambda} \hat{\phi}(n)W_n(x)$ (i.e., the right-hand side converges in L^2 to the Λ-periodic function on the left-hand side). (See Kluvanek [9] or Theorem 8.3.2 of Chapter 5 of [8].)

This with Theorem 2.1 may be used to show that $A(x,y) = \sum \phi(x-k)\bar{\phi}(x-k)$ satisfies $\int A(x,y)\,dy = 1$ for almost all x, so the Mallat lemma applies. □

3. The wavelet ψ

Suppose we have an MRA $(V_j)_j$ of $L^2(G)$. We will construct wavelets from this MRA. We will follow the approach of S. Mallat [13], or Y. Meyer [14] chapter III section 2. (This is similar to Lang [10].)

We know $V_j \subseteq V_{j+1}$. Let W_j be the orthogonal complement of V_j in V_{j+1}, so $V_j \oplus W_j = V_{j+1}$. Let ϕ be the scaling function whose translates span V_0. We seek ψ whose translates span W_0, whose translates are orthogonal, and such that the translates of ψ are orthogonal to the translates of ϕ. We should have $\phi(\sigma x) = 2\sum_{n\in\Lambda} a_n\phi(x+n)$ and $\psi(\sigma x) = 2\sum_{n\in\Lambda} b_n\phi(x+n)$. Upon taking Fourier transforms, we have $\hat{\phi}(\rho\omega) = m_0(\omega)\hat{\phi}(\omega)$ and $\hat{\psi}(\rho\omega) = m_1(\omega)\hat{\phi}(\omega)$, where $m_0(\omega) = \sum_{n\in\Lambda} a_n W_n(\omega)$ and $m_1(\omega) = \sum_{n\in\Lambda} b_n W_n(\omega)$. The orthogonality conditions are met when

$$\begin{bmatrix} m_0(\omega) & m_0(\omega) \\ m_1(\omega + 0.1) & m_1(\omega+0.1) \end{bmatrix}$$

is a unitary matrix. This can be arranged by letting $m_1(\omega) = W_{0.1}(\omega)\bar{m}_0(\omega + 0.1)$ (note $W_{0.1}(\omega+0.1) = -W_{0.1}(\omega)$). So

$$\psi(x) = \sum_{n\in\Lambda} (-1)^{|n|}\bar{a}_n\phi(\rho(x)+n+1.0).$$

This is (3.2) of Lang [10].

4. Calderón-Zygmund theory

We wish to show that expansions in the wavelets on the Cantor dyadic group converge unconditionally in $L^p(G)$ for $p > 1$. We will do so for wavelets arising from scaling filters as in Theorem 2.1, which satisfy the following Lipschitz-type regularity condition:

Definition 1. We say ψ is *regular* if $|\psi(x) - \psi(y)| \le C|x-y|$ for some constant C, for all $x, y \in G$.

This condition differs from the regularity condition of Lang [10].

Our proof of the L^p-convergence result (Theorem 4.4 below) uses the technique of Y. Meyer (see Daubechies [2] chapter 9). This entails showing that the integral operator corresponding to wavelet expansion is a Calderón-Zygmund singular integral operator, and hence of type weak(1,1); it then follows by the Marcinkowicz interpolation theorem that the operator is bounded on $L^p(G)$ for $p > 1$. See M. Taibleson [17] for more information and references concerning singular integral operator theory on local fields.

Our definition of a Calderón-Zygmund singular integral operator is as follows:

Definition 2. Let T be a bounded integral operator on $L^2(G)$, whose kernel $M(x,y)$ satisfies for $y \in y_0 + \rho^j(D)$,

$$\int_{G\setminus(y_0+\rho^{j+1}(D))} |M(x,y) - M(x,y_0)|\, dx \le C < \infty.$$

Then T is a *Calderón-Zygmund singular integral operator.*

This definition is similar to that given in Dyn'kin [3] (3.24), p. 237. Also see E. Stein [16], the discussion following equation (2′), p. 34. The proof that such operators are of type weak(1,1) proceeds as usual via the Calderón-Zygmund decomposition theorem (e.g., Edwards and Gaudry [4] Theorem 2.3.2), and the following lemma:

Lemma 4.1. *Let T be a Calderón-Zygmund singular integral operator. Let $b \in L^1(G)$ be supported on $y_0+\rho^j(D)$ and let $\int_G b = 0$. Then $\int_{G\setminus(y_0+\rho^{j+1}(D))} |Tb(x)|\, dx \le C \int_G |b(y)|\, dy$.*

Lemma 4.1 is similar to equation (3.25) of section 5.1 of [3], p. 237. We have:

Theorem 4.2. *Any Calderón-Zygmund singular integral operator T is of type weak(1,1): There is a constant $C > 0$ such that meas$\{x \in G : |Tf(x)| > \lambda\} \le \frac{C\|f\|_1}{\lambda}$ for any $\lambda > 0$.*

We have the following theorem:

Theorem 4.3. *Suppose ψ has support in $\rho^m(D)$ for some $m \in Z$, and suppose ψ satisfies the regularity condition. Let $M(x,y) = \sum_{j\in Z, k\in\Lambda} \theta_{jk}\psi_{jk}(x)\bar{\psi}_{jk}(y)$*

where $\psi_{jk}(x) = 2^{j/2}\psi(\rho^j(x)+k)$ *and* $\theta_{jk} \in \{1,-1\}$. *Let* $y \in y_0 + \rho^\ell(D)$. *Let* $I = \int_{G\setminus(y_0+\rho^{\ell+1}(D))} |M(x,y) - M(x,y_0)|\,dx$. *Then* I *is bounded by a constant.*

Proof. For each j let Q_j be the set of $k \in \Lambda$ such that $\psi(\rho^j x - k)[\bar\psi(\rho^j y - k) - \bar\psi(\rho^j y_0 - k)] \neq 0$. We find that Q_j has cardinality at most 2^{m+1}. So

$$I = \int_{G\setminus(y_0+\rho^{\ell+1}(D))} \left| \sum_j \sum_{k\in Q_j} 2^j \theta_{jk}\psi(\rho^j x - k)[\bar\psi(\rho^j y - k) - \bar\psi(\rho^j y_0 - k)] \right| dx.$$

Now $x \notin y_0 + \rho^{\ell+1}(D)$ and $y \in y_0 + \rho^\ell(D)$. Consequently, by the triangle inequality, if $j > m - \ell$ we know $\rho^j x - k$ and $\rho^j y - k$ cannot both belong to $\rho^\ell(D)$ for any k, and $\rho^j x - k$ and $\rho^j y_0 - k$ cannot both belong to $\rho^\ell(D)$ for any k. So if $j > m - \ell$ it follows that $\psi(\rho^j x - k)[\bar\psi(\rho^j y - k) - \bar\psi(\rho^j y_0 - k)] = 0$.

So

$$I = \int_{G\setminus(y_0+\rho^{\ell+1}(D))} \left| \sum_{j\leq m-\ell} \sum_{k\in Q_j} 2^j \theta_{jk}\psi(\rho^j x - k)[\bar\psi(\rho^j y - k) - \bar\psi(\rho^j y_0 - k)] \right| dx.$$

We may estimate this as follows:

$$\begin{aligned}
I &\leq \int_G \left| \sum_{j\leq m-\ell} \sum_{k\in Q_j} 2^j \theta_{jk}\psi(\rho^j x - k)[\bar\psi(\rho^j y - k) - \bar\psi(\rho^j y_0 - k)] \right| dx \\
&\leq \sum_{j\leq m-\ell} \sum_{k\in Q_j} \left(2^j \int_G |\psi(\rho^j x - k)|\,dx \right) |\bar\psi(\rho^j y - k) - \bar\psi(\rho^j y_0 - k)| \\
&\leq \sum_{j\leq m-\ell} \sum_{k\in Q_j} C 2^j (2^{-j}||\psi||_1) |(\rho^j y - k) - (\rho^j y_0 - k)| \\
&\leq \sum_{j\leq m-\ell} \sum_{k\in Q_j} C 2^j ||\psi||_1 |y - y_0| \quad \leq \quad \sum_{j\leq m-\ell} C 2^{m+1} 2^j ||\psi||_1 |y - y_0| \\
&\leq C 2^{m+1} 2^{m-\ell+1} ||\psi||_1 |y - y_0| \quad \leq \quad C 2^{2m+2} ||\psi||_1.
\end{aligned}$$

□

Theorem 4.4. *Let* ψ *be a wavelet associated as in section 3 with a scaling filter as in theorem 2.1. Suppose* ψ *obeys the regularity condition of definition 1. Then convergence of wavelet expansions is unconditional in* L^p *for* $p > 1$.

The proof of this follows from the previous theorem, together with theorem 2.2 (which implies that the operator with kernel $M(x,y)$ is bounded on $L^2(G)$ and hence is type weak(1,1)), and the Marcinkowicz interpolation theorem (e.g., Edwards and Gaudry [4] theorem A.2.1).

5. Examples of wavelets on the Cantor dyadic group

Here we describe compactly supported orthogonal wavelets on the Cantor dyadic group resulting from scaling filters which are trigonometric polynomials. We first discuss the general situation.

5.1. **Length 2^n wavelets.** Consider $m_0(\omega) = \sum_{k\in\Lambda,|k|<2^n} a_k W_k(\omega)$ where $n > 0$. This has 2^n many terms. We wish to select $\{a_k\}$ in such a way that the conditions of theorem 2.1 are met. This can be done as follows.

Consider $\sigma^n(D)$ in $\hat{G}$. This set corresponds to the real interval $[0, 2^{-n})$ under the map $\omega \to |\omega|$. This set is also a subgroup of G, with 2^n many cosets in the group D. These cosets may be written as $I_{k,n} = \sigma^n(k) + \sigma^n(D)$ for $k \in \Lambda$, $|k| < 2^n$. Then $I_{k,n}$ is identified with the dyadic interval $[k2^{-n}, (k+1)2^{-n})$; $0 \le k < 2^n$. Now W_k for $|k| < 2^n$ is constant on the sets $I_{k,n}$ so the same is true for m_0. Let s_k be the value of m_0 on $I_{k,n}$. Note that $\{W_k : k \in \Lambda, |k| < 2^n\}$ is a complete set of 2^n many characters on the order 2^n abelian group formed by the cosets $I_{k,n}$ of $\sigma^n(D)$. This implies that given any choice of values $\{s_k\}$, we may solve the system of 2^n many linear equations $m_0(\sigma^n(k)) = s_k$, $|k| < 2^k$, for the 2^n many unknowns $\{a_k\}$.

To meet the condition $|m_0(\omega)|^2 + |m_0(\omega + 0.1)|^2 \equiv 1$, all we need to do is select $\{s_k\}$ in such a way that $|s_k|^2 + |s_{k'}|^2 = 1$, where $k' = k + 2^{n-1}$. (Here by 2^{n-1} we mean that member of $\hat{G}$ whose $(n-1)$-th coordinate is 1, and whose other coordinates are 0.) Addition here is in Λ, so the mapping $k \to k + 2^{n-1}$ is injective on the set of $k \in \Lambda$ with $|k| < 2^n$. To meet the remaining conditions of theorem 2.1, we set $s_0 = 1$ and we require that $|s_k| > 0$ for some selection of at least half of the sets $I_{k,n}$.

We are then assured that m_0 leads to ϕ whose translates by Λ span orthogonally a space V_0 which generates an MRA, by theorem 2.2.

We now detail some examples, in which we will describe the regularity (definition 1) of the resulting wavelets.

5.2. **Length 2 wavelets.** Here $n = 1$ and $m_0(\omega) = a_0 W_0(\omega) + a_1 W_1(\omega)$. On $I_{0,1}$ we let m_0 have value 1. Then m_0 must have value 0 on $I_{1,1}$. It follows that $m_0(\omega) = \frac{1}{2} W_0(\omega) + \frac{1}{2} W_1(\omega)$. As in Lang [10], this leads to the Haar wavelets.

5.3. **Length 4 wavelets.** We now have $m_0 = a_0 W_{00} + a_1 W_{01} + a_2 W_{10} + a_3 W_{11}$. (Here we use binary notation to indicate the elements of Λ in the subscripts of the Walsh functions W.) We meet the conditions of theorem 2.1 by setting m_0 to have value 1 on $I_{00,2}$, value a on $I_{01,2}$, value 0 on $I_{10,2}$, and value b on $I_{11,2}$, where

$|a|^2+|b|^2=1$. As in Lang [10], this leads to $a_0=(1+a+b)/4$, $a_1=(1+a-b)/4$, $a_2=(1-a-b)/4$, and $a_3=(1-a+b)/4$. This leads to

$$\begin{aligned}\hat{\phi}(\omega) &= m_0(\sigma(\omega))m_0(\sigma^2(\omega))m_0(\sigma^3(\omega))\cdots \\ &= f(\omega)+af(\omega+0.1)+abf(\omega+1.1) \\ &\quad + ab^2f(\omega+11.1)+ab^3f(\omega+111.1)+\cdots .\end{aligned}$$

As in [10], the inverse Fourier transform of this corresponds to the lacunary Walsh function series (on the half real line)

$$\phi(x) = \frac{1}{2}1_{[0,2)}(x)\left[1+aW_1(x/2)+abW_3(x/2)+ab^2W_7(x/2)+ab^3W_{15}(x/2)+\cdots\right].$$

For information about ψ and for graphs of these functions, see [10].

We now consider regularity of these wavelets. Let $x,y\in\rho(D)$. Then $|\phi(x)-\phi(y)|\le\frac{1}{2}\left(\sum_{j=0}^{\infty}|ab^j|2^j\right)|x-y|$ so ϕ is regular if $|b|<1/2$. The same is true for ψ (being a combination of four translates of dilates of ϕ). This condition is also necessary, as is seen by considering $x=0$ and $y=2^{-m}$ (i.e., the element in G corresponding to that power of 2): $|\phi(0)-\phi(y)|=\left|\frac{1}{2}\sum_{j=0}^{\infty}ab^j\right|\approx C|b|^m$.

5.4. **Length 8 wavelets.** Here $m_0=a_0W_{000}+a_1W_{001}+a_2W_{010}+a_3W_{011}+a_4W_{100}+a_5W_{101}+a_6W_{110}+a_7W_{111}$. We meet the conditions of theorem 2.1 by setting m_0 to have value 1 on $I_{000,3}$, value a on $I_{001,3}$, value b on $I_{010,3}$, value c on $I_{011,3}$, value 0 on $I_{100,3}$, value d on $I_{101,3}$, value e on $I_{110,3}$, and value f on $I_{111,3}$; where $|a|^2+|d|^2=1$, $|b|^2+|e|^2=1$, and $|c|^2+|f|^2=1$. Also, at most two of a,b,c may be zero. It requires solving a linear system of eight equations in eight unknowns to obtain the coefficients a_k; we omit this here.

It is possible to develop an expansion for $\phi(x)$ in terms of Walsh functions, as for the length 4 wavelets. An expansion of the first 255 terms (87 of which are nonzero) has been computed by the present author using C language code; no simple patterns appear in the coefficients.

If we choose certain a,b,c (and hence d,e,f), we do obtain some simple expansions.

If $a=b=c=1$, we recover the Haar wavelets. (The resulting m_0 is 1 on $\sigma(D)$ and 0 on the rest of D.) If $a=1$ and $b=c$ (so $d=0$ and $e=f$), we recover the length 4 wavelets.

If $a = 1$, $b = 0$ and $|c| < 1$, then $d = 0$, $e = 1$, and $|f| < 1$. In this case, we obtain $\phi(x) = \frac{1}{4}F(x/4)1_{[0,4)}(x)$, where

$$F = 1 + W_1 + cW_3 + \sum_{j=3}^{\infty} c\left(f^{j-3}W_{2^j-2} + f^{j-2}W_{2^j-1}\right).$$

This is regular if $|f| < 1/2$.

If $|a| < 1$, $b = 1$ and $c = 0$, then $|d| < 1$, $c = 0$ and $f = 1$. In this case we find that $\phi(x) = \frac{1}{4}F(x/4)1_{[0,4)}(x)$, where

$$F = 1 + aW_1 + aW_3 + \sum_{j=3}^{\infty} ad^j\left(W_{m_j} + W_{2m_j}\right),$$

where $m_j = 1 + 4 + 16 + \cdots + 4^j$. This is regular if $|d| < 1/4$.

In figure 5.4.1, we give graphs of ϕ for certain a, b, c. These illustrate the variety of waveforms possible. These graphs were produced using the machine-generated expansion of ϕ described above.

6. Algorithms on the line

Here we describe a Mallat tree-type algorithm which may be used to analyze functions using the wavelets of the previous section.

For f a function on G, let $c_k^j = \int_G f(x)\phi(\rho^j(x) - k)2^{j/2}\,dx$ and $d_k^j = \int_G f(x)\psi(\rho^j(x) - k)2^{j/2}\,dx$ for $j \in Z$ and $k \in \Lambda$. Then the reconstruction algorithm is

$$c_k^{j+1} = 2^{1/2}\left(\sum_{n\in\Lambda} c_n^j a_{k-\rho(n)} + \sum_{n\in\Lambda} d_n^j b_{k-\rho(n)}\right)$$

and the decomposition algorithm is

$$c_k^j = 2^{1/2}\sum_{n\in\Lambda} a_{n-\rho(k)} c_n^{j+1} \text{ and } d_k^j = 2^{1/2}\sum_{n\in\Lambda} b_{n-\rho(k)} c_n^{j+1}.$$

Here the coefficients $\{a_k\}$, $\{b_k\}$ are those associated with the scaling filters m_0 and m_1 respectively (see section 3). These algorithms are easy to develop in the usual manner.

Since the Cantor dyadic group may be identified as a measure space with the real half-line, this algorithm may be applied to analyze functions on the real line. See figure 6.1 for typical "level-j resolutions" of the function $f(x) = \sqrt{1-x^2}$ over $[0, 1]$ using the length 4 wavelets, for certain values of the parameter a of section 5.3. The resolution with $a = 1.0$ is of course just the Haar approximation; on each dyadic interval of length $1/8$, the function is replaced by its average on that interval.

See [11] for further discussion of the nature and structure of this algorithm.

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