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INVOLUTIVE STONE ALGEBRAS AND REGULAR α-DE MORGAN ALGEBRAS

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ABSTRACT. A piggyback duality and a translation process between this one and a Priestley duality for each subvariety of involutive Stone algebras and regular α-De Morgan algebras is presented. As a consequence we describe free algebras and the prime spectrum of each subvariety.

1. INTRODUCTION

The two varieties in the title were introduced in a paper of R. Cignoli and M.S. de Gallego [1] concerning De Morgan algebras with operators. These algebras are connected with the theory of n-valued Lukasiewicz algebras, and in those we study in this paper the operators are lattice endomorphisms or dual lattice endomorphisms.

An algebra $(A; \vee, \wedge, 0, 1, \sim, \nabla)$ $[(A; \vee, \wedge, 0, 1, \sim, \alpha)]$ of type $(2, 2, 0, 0, 1, 1)$ is an involutive Stone algebra [regular α-De Morgan algebra] if the reduct $(A; \vee, \wedge, 0, 1, \sim)$ is a De Morgan algebra and the operator $\nabla : A \to A$ $[\alpha : A \to A]$ satisfies the following equations:

$$\begin{array}{ll} (S_1)\ \nabla 0 = 0 & \Big[(R_1)\ \sim \alpha x \vee \alpha x = 1 \\ (S_2)\ x \wedge \nabla x = x & (R_2)\ (\sim x \vee \alpha x) \wedge x = x \\ (S_3)\ \nabla(x \wedge y) = \nabla x \wedge \nabla y & (R_3)\ \alpha x \wedge \alpha \sim x = 0 \\ (S_4)\ \sim \nabla x \wedge \nabla x = 0 & (R_4)\ \alpha(x \vee y) = \alpha x \vee \alpha y \\ & (R_5)\ \alpha(x \wedge y) = \alpha x \wedge \alpha y \Big]. \end{array}$$

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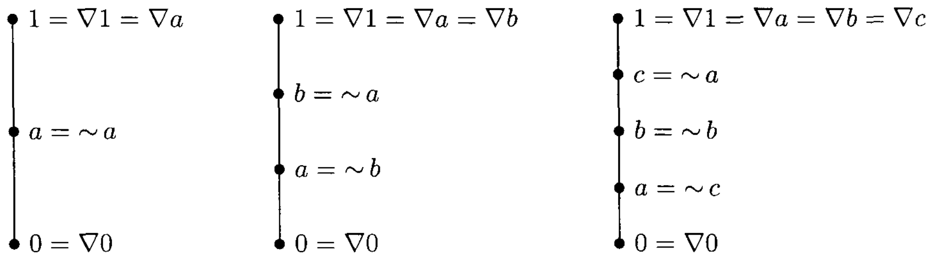
Let $\mathbf{S}$ be the variety of involutive Stone algebras and $\mathbf{R}$ be the variety of regular α-De Morgan algebras. It is shown in [1] that $\mathbf{S}$ is a subclass of the class of double Stone lattices, so if $A \in \mathbf{S}$ then A is a pseudocomplemented and a dual pseudocomplemented lattice. In fact, $\sim \nabla x = x^*$ is the pseudocomplement of x and $\nabla \sim x = x^+$ is the dual pseudocomplement of x. Developing a dual category equivalence the authors determined the subdirectly irreducible algebras in $\mathbf{S}$. The lattice of subvarieties of $\mathbf{S}$ is the chain $\mathbf{S}_1 \subset \mathbf{S}_2 \subset \ldots \subset \mathbf{S}_6 = \mathbf{S}$, where $\mathbf{S}_1$ is the class of one element algebras and $\mathbf{S}_j$, for $2 \leq j \leq 6$, is the equational subclass of $\mathbf{S}$ generated by $\underline{L}_j$ (see [1], Theorem 2.8). The diagrams of these $\underline{L}_j$ for $j \geq 3$ are those of figure 1. Analogously, the subdirectly irreducible algebras of $\mathbf{R}$ were determined in Theorem 3.10 of [1], and we denote them also by $\underline{L}_j$, with $1 \leq j \leq 5$, because it will be clear in each case if we are referring to an algebra of $\mathbf{S}$ or to an algebra of $\mathbf{R}$. Their diagrams, with the exception of those of $\underline{L}_2$ and $\underline{L}_1$, are presented in figure 2.



$\underline{L}_3$ $\underline{L}_4$ $\underline{L}_5$

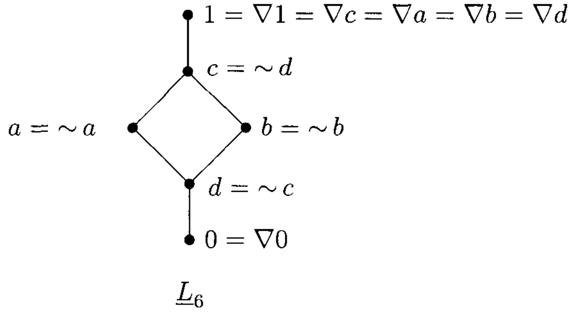


Figure 1

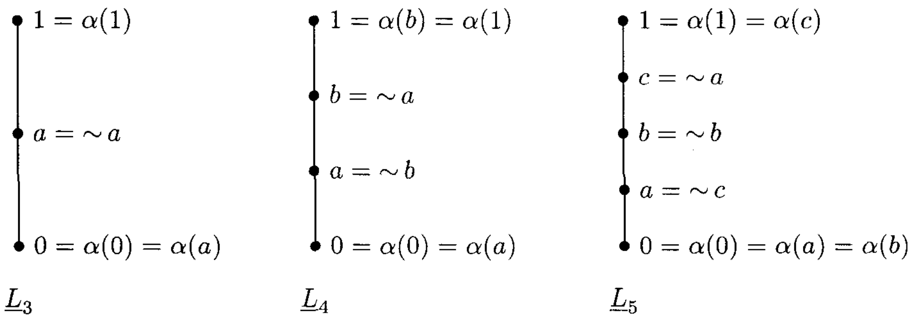


Figure 2

The lattice of the subvarieties of $\mathbf{R}$ is described in [1, page 390] as the finite distributive lattice having as join-irreducible elements the classes $\mathbf{R}_i$ ordered as follows: $\mathbf{R}_2 \leq \mathbf{R}_3$, $\mathbf{R}_2 \leq \mathbf{R}_4 \leq \mathbf{R}_5 = \mathbf{R}$ and $\mathbf{R}_3$ not comparable with $\mathbf{R}_4$. We observe that $\mathbf{R}_2$ as well as $\mathbf{S}_2$ are doubled Boolean algebras since ∇ and α must be the identity. For this reason we will not be concerned with them.

When $j \in \{3, 5, 6\}$ the subdirectly irreducible algebras in $\mathbf{S}_j$ are isomorphic to subalgebras of $\underline{L}_j$ so $\mathcal{HSP}(\underline{L}_j) = \mathcal{ISP}(\underline{L}_j)$. When $j = 4$ it results, applying Jónsson's Lemma and using the fact that $\underline{L}_3 \in \mathcal{H}(\underline{L}_4)$, that $\mathbf{S}_4 = \mathcal{ISP}(\underline{L}_3, \underline{L}_4)$. In what follows we will not be concerned with $\mathbf{S}_3$ since this class of algebras coincides with the class of regular α-De Morgan algebras denoted by $\mathbf{R}_3$ (see [1], pages 381 and 390).

For the subvarieties of $\mathbf{R}$ it is clear that $\mathbf{R}_j = \mathcal{ISP}(\underline{L}_j)$ for $j \in \{3,4,5\}$, and that $\mathbf{R}_3 \vee \mathbf{R}_4 = \mathcal{ISP}(\underline{L}_3, \underline{L}_4)$ follows once more by Jónsson's Lemma.

2. Piggyback dualities for involutive Stone algebras and for regular α-De Morgan algebras

About the theory of natural dualities it must be emphasized that a very useful survey is presented in [6]. The book [3] is recommended.

We suppose the reader familiar with the fundamental facts of how to obtain restricted Priestley dualities and natural dualities, how to translate between natural and restricted Priestley dualities, and how to apply them in the description of free algebras.

In this section we present a piggyback duality (see [7] and [12]) for each variety under consideration and we obtain a nice translation process between this duality and a restricted Priestley duality. We apply the Generalized Piggyback Duality Theorem (see [7], Theorem 2.5) in its brute force "multiple algebra, single carrier" formulation. This theory as it applies to distributive-lattice-ordered algebras was presented in [7]. We use the refinement of Theorem 2.5 of [7] as presented in [13]. We will not repeat the definitions here, and we will use the habitual symbology.

Theorem 2.1. (The Generalized Piggyback Duality Theorem, for distributive-lattice-ordered algebras) *Suppose that $\mathcal{A} = \mathcal{ISP}(\underline{\Pi})$, where $\underline{\Pi}$ is a finite set of finite algebras of a given fixed type each having a $\mathbf{D}$-reduct. For each $\underline{P}$ in $\underline{\Pi}$ let $\Omega_{\underline{P}}$ be a (possibly empty) subset of $\mathbf{D}(\underline{P}, \underline{2})$.*

Let $\underset{\sim}{\Pi} = (\Pi; \tau, R_{\mathcal{A}})$ be the topological relational structure on $\dot{\cup}\, \{P \mid \underline{P} \in \underline{\Pi}\}$ in which

i) *τ is the discrete topology,*
ii) *$R_{\mathcal{A}} = S_{\mathcal{A}} \cup T_{\mathcal{A}}$ where*
  a) *$S_{\mathcal{A}}$ is the collection of maximal $\mathcal{A}$-subalgebras of sublattices of the form*

$$(\alpha, \beta)^{-1}(\leq) = \Big\{(a,b) \in P \times Q \mid \alpha(a) \leq \beta(b)\Big\},$$

  where $\alpha \in \Omega_{\underline{P}}$, $\beta \in \Omega_{\underline{Q}}$ ($\underline{P}, \underline{Q} \in \underline{\Pi}$), and
  b) *$T_{\mathcal{A}}$ is the set of graphs of a set $\mathcal{E}_{\mathcal{A}} \subseteq \bigcup\{\mathcal{A}(\underline{P}, \underline{Q}) \mid \underline{P}, \underline{Q} \in \underline{\Pi}\}$ of endomorphisms satisfying the following separation condition* (S)*:*
  for all $\underline{P} \in \underline{\Pi}$, given $a, b \in \underline{P}$ with $a \neq b$, there exist $\underline{Q} \in \underline{\Pi}$, $u \in \mathcal{A}(\underline{P}, \underline{Q}) \cap \mathcal{E}_{\mathcal{A}}$ and $\alpha \in \Omega_{\underline{Q}}$ such that $\alpha(u(a)) \neq \alpha(u(b))$.

Then $R_{\mathcal{A}}$ yields a duality on $\mathcal{A}$.

When we say that $R_{\mathcal{A}}$ yields a duality on $\mathcal{A}$ we mean that the structure $\underset{\sim}{\Pi}$ generates a class $\mathcal{X} = \mathcal{IS}_c\mathcal{P}(\underset{\sim}{\Pi})$ of multi-sorted objects such that the natural hom-functors D and E into $\underline{\Pi}$ and $\underset{\sim}{\Pi}$ set up a dual equivalence, as described in [7].

We will now prepare the way to apply Theorem 2.1 to our concrete cases.

First we must choose $\underline{\Pi}$ and $\Omega_{\underline{P}}$ for each $\underline{P} \in \underline{\Pi}$. We have several choices. We could minimize the size of $\underline{\Pi}$; but then we would have $|\Omega_{\underline{P}}| > 1$ in general. As in [14], we find that the description of the translation process is most transparent when we allow multiple copies of an algebra $\underline{P}$ in $\underline{\Pi}$ and then choose $|\Omega_{\underline{P}}| = 1$ for all $\underline{P} \in \underline{\Pi}$.

Let's consider $\mathbf{S}_4 = \mathcal{ISP}(\underline{P}_{04}, \underline{P}_{14}, \underline{P}_{24})$ where $\underline{P}_{04} = \underline{P}_{14} = \underline{L}_3$ and $\underline{P}_{24} = \underline{L}_4$. Let's consider $\mathbf{S}_5 = \mathcal{ISP}(\underline{P}_{05}, \underline{P}_{15})$, where $\underline{P}_{05} = \underline{P}_{15} = \underline{L}_5$, and $\mathbf{S}_6 = \mathcal{ISP}(\underline{P}_{06})$ where $\underline{P}_{06} = \underline{L}_6$. We will abbreviate this by saying that $\mathbf{S}_j = \mathcal{ISP}(\underline{\Pi}_i \mid i \in I_j)$ where $I_j = \{0,1,2\}$ if $j = 4$, $I_j = \{0,1\}$ if $j = 5$ and $I_j = \{0\}$ if $j = 6$. Where convenient we will refer to $\mathbf{S}_j = \mathcal{ISP}(\underline{\Pi})$ to be more concise.

We are now going to specify suitable carriers for the natural dualities we are setting up. For each case $\Omega_{\underline{P}_{ij}} = \{\alpha_{ij}\}$ where α_{04}, α_{14} are respectively the bottom and the top of $H(\underline{L}_3)$, α_{24} is the midpoint of $H(\underline{L}_4)$, α_{05}, α_{15} are respectively the element of $H(\underline{L}_5)$ that covers the bottom and the one that is covered by the top, and α_{06} is the element of $H(\underline{L}_6)$ such that $\alpha_{06}^{-1}(1) = [a)$.

In the case of regular α-De Morgan algebras it will be convenient for our purposes to consider $\mathbf{R}_j = \mathcal{ISP}(\underline{P}_{0j}, \underline{P}_{1j})$ with $\underline{P}_{0j} = \underline{P}_{1j} = \underline{L}_j$ for $3 \le j \le 5$ and $\mathbf{R}_3 \vee \mathbf{R}_4 = \mathcal{ISP}(\underline{P}_{03}, \underline{P}_{13}, \underline{P}_{04}, \underline{P}_{14})$. We will abbreviate this writing $\mathbf{R}_j = \mathcal{ISP}(\underline{\Pi}_i \mid i \in I_j)$ where $I_j = \{0,1\}$ if $3 \le j \le 5$ and

$$\mathbf{R}_3 \vee \mathbf{R}_4 = \mathcal{ISP}\Big(\underline{\Pi}_i \cup \underline{\Pi}_{i'} \mid i \in I_3,\ i' \in I_4\Big)\ .$$

Where convenient we will refer to these varieties as $\mathcal{ISP}(\underline{\Pi})$.

For $3 \le j \le 5$, and $i \in \{0,1\}$ we take $\Omega_{\underline{P}_{ij}} = \{\alpha_{ij}\}$ where α_{0j}, α_{1j} are respectively the top and the bottom of $H(\underline{L}_j)$.

Since we have so many different cases to handle we will pick out the most interesting one and we will work with it. The more relevant results of the other cases will be tabulated in table 1. We observe that, in this table, for the subvariety

$\mathbf{R}_3 \vee \mathbf{R}_4$ each column is obtained doing the union of the corresponding columns of $\mathbf{R}_3$ and $\mathbf{R}_4$.

Often in what follows $\mathcal{A} = \mathbf{S}_4 = \mathcal{ISP}(\underline{P}_{04}, \underline{P}_{14}, \underline{P}_{24}) = \mathcal{ISP}(\underline{\Pi}_i \mid i \in I_j = \{0,1,2\})$. However we continue specifying $\mathcal{A} = \mathbf{S}_4$ to make it clear when we are dealing with $\mathbf{S}_4$ and when we are dealing with a general subvariety $\mathcal{A}$.

For each $\underline{A} \in \mathcal{A}$, let $X^i = \mathcal{A}(\underline{A}, \underline{P}_{ij})$ and

$$Y^i = \left\{y \in H(\underline{A}) \mid \exists\, x \in X^i,\ \alpha_{ij} \circ x = y\right\} \text{ for each } i \in I_j\ .$$

Lemma 2.2. *For each $\underline{A} \in \mathcal{A} = \mathbf{S}_4$ consider*

$$\Phi_{\alpha_{i4}} \colon X^i \to Y^i$$

such that $\Phi_{\alpha_{i4}}(x) = \alpha_{i4} \circ x$. Then $\Phi_{\alpha_{i4}}$ are bijections.

PROOF. Consider

$$\Phi_{\alpha_{24}} \colon X^2 = \mathcal{A}(\underline{A}, \underline{P}_{24}) \to Y^2 = \left\{y \in H(\underline{A}) \mid \exists\, x \in \mathcal{A}(\underline{A}, \underline{P}_{24}),\ \alpha_{24} \circ x = y\right\}$$
$$x \to \alpha_{24} \circ x$$

We want to show that if $\alpha_{24} \circ x_1 = \alpha_{24} \circ x_2$ (i.e., $(\forall\, s \in A)\ \alpha_{24}(x_1(s)) = \alpha_{24}(x_2(s))$) then $x_1 = x_2$. We must consider the following four cases:

Case 1) $x_1(s) = 0$.

We have $\alpha_{24}(x_1(\nabla s)) = \alpha_{24}(\nabla x_1(s)) = \alpha_{24}(\nabla 0) = \alpha_{24}(0) = 0 = \alpha_{24}(x_2(\nabla s))$. Then $\nabla x_2(s) \in \{a, 0\}$ and $x_2(s) = 0$.

Case 2) $x_1(s) = a$.

Since $\alpha_{24}(x_1(s)) = \alpha_{24}(a) = 0 = \alpha_{24}(x_2(s))$, then $x_2(s) \in \{a, 0\}$. But $\alpha_{24}(x_1(\nabla s)) = \alpha_{24}(\nabla x_1(s)) = \alpha_{24}(\nabla a) = \alpha_{24}(1) = 1 = \alpha_{24}(x_2(\nabla s)) = \alpha_{24}(\nabla x_2(s))$. Then $\nabla x_2(s) \in \{1, b\}$, $x_2(s) \in \{1, a, b\}$ and consequently $x_2(s) = a$.

Case 3) $x_1(s) = b$ and *Case* 4) $x_1(s) = 1$ can be proved by entirely analogous reasoning.

If we consider $\Phi_{\alpha_{04}}$ or $\Phi_{\alpha_{14}}$ it can be easily seen that they are also bijections. $\square$

To apply the Generalized Piggyback Duality Theorem for $\mathbf{S}_4$ we need to determine the monoid of endomorphisms of $\underline{L}_4$ and $\underline{L}_3$ which in fact reduce to the identity map. It will also be convenient to consider the homomorphism $u \colon \underline{P}_{24} \to \underline{P}_{14}$, the graph of which is $\{(1,1), (b,a), (a,a), (0,0)\}$, the identity maps g_{04} from $\underline{P}_{04}$

to $\underline{P}_{14}$ and g_{14} from $\underline{P}_{14}$ to $\underline{P}_{04}$ and id_{i4}, the identity map on $\underline{P}_{i4}$ for $i = 0, 1$ and 2.

For any of the varieties that we will consider, g_{ij} as well as id_{ij} will have a similar meaning to that we presented above for the case of $\mathbf{S}_4$.

Using the restricted Priestley duality as it appears in [1] or using a merely algebraic method we can easily determine $\mathbf{S}(\underline{L}_j, \underline{L}_j)$ for $j \in \{5,6\}$ [or $\mathbf{R}(\underline{L}_j, \underline{L}_j)$] the monoid of endomorphisms of $\underline{L}_j$. In fact for $\underline{L}_j \in \mathbf{S}$ it is $\langle u^{j1}, u^{j2}\rangle$ with $j \in \{5,6\}$ where, using the same symbol for an endomorphism and its graph, we have

$$u^{51} = \Big\{(1,1),\ (c,b),\ (b,b),\ (a,b),\ (0,0)\Big\}\ ,\ u^{52} = \mathrm{id}_{\underline{L}_5}\ ,$$
$$u^{61} = \Big\{(0,0),\ (d,d),\ (b,a),\ (a,b),\ (c,c),\ (1,1)\Big\}\ ,$$
$$u^{62} = \Big\{(0,0),\ (d,a),\ (b,a),\ (a,a),\ (c,a),\ (1,1)\Big\}\ .$$

For $\underline{L}_j \in \mathbf{R}$, when $j = 3$ it reduces to the identity map and when $j \in \{4,5\}$ it is $\langle v^{j1}, v^{j2}\rangle$, where we have $v^{41} = \mathrm{id}_{\underline{L}_4}$, $v^{42} = \{(0,0),(a,0),(b,1),(1,1)\}$, $v^{51} = \mathrm{id}_{\underline{L}_5}$ and $v^{52} = \{(0,0),(a,0),(b,b),(c,1),(1,1)\}$.

Consider $\mathcal{E}_{\mathcal{A}} = \mathcal{E}_{\mathbf{S}_4} = \{g_{04}, g_{14}, \mathrm{id}_{24}, \mathrm{id}_{04}, \mathrm{id}_{14}, u\}$. We can state the following lemma:

Lemma 2.3. *Let $\mathcal{A} = \mathbf{S}_4 = \mathcal{ISP}(\underline{\Pi})$ and $T_{\mathcal{A}}$ be the graphs of the elements of $\mathcal{E}_{\mathcal{A}}$, then the separation condition (S) of the Theorem 2.1 is verified.*

PROOF. We will see what happens with pairs of elements of $\underline{P}_{24}$. Consider $(a,0)$ and $(b,1)$. We have $\alpha_{14}(u(a)) = 1 \neq \alpha_{14}(u(0)) = 0$ and also $\alpha_{04}(u(b)) = 0 \neq \alpha_{04}(u(1)) = 1$. For the other pairs of $\underline{P}_{24}$ it is enough to consider $\alpha_{24} \circ \mathrm{id}_{24}$ to get them separated. We can act analogously to verify the condition (S) on the pairs of elements of $\underline{P}_{04}$ and of $\underline{P}_{14}$. □

Since in the algebras we are dealing with the unary operations are homomorphisms or dual homomorphisms, we obtain the following proposition:

Proposition 2.4. *Let $\underline{A} \in \mathbf{S}$ [$\underline{A} \in \mathbf{R}$]. Let B be a sublattice of $\underline{A}$, then the maximal $\mathbf{S}$-subalgebra [$\mathbf{R}$-subalgebra] of B is*

$$B^0 = \Big\{x \in B \mid\ \sim x,\ \nabla x,\ \sim\nabla x,\ \nabla\sim x,\ \sim\nabla\sim x,\ \nabla\sim\nabla x\ \in\ B\Big\}$$
$$\Big[B^0 = \Big\{x \in B \mid\ \sim x,\ \alpha x,\ \sim\alpha x,\ \alpha\sim x,\ \sim\alpha\sim x,\ \alpha\sim\alpha x\ \in\ B\Big\}\Big]\ .$$

PROOF. It results easily as a consequence of Lemma 3.5 of [7] using among others the equations $\alpha^2 x = \alpha x$ and $\alpha\sim\alpha x = \ \sim\alpha x$ [$\nabla^2 x = \nabla x$ and $\nabla\sim\nabla x = \ \sim\nabla x$]. □

In what follows instead of saying that B^0 is a maximal $\mathcal{A}$-subalgebra of B we will simply say that it is a "maximal subalgebra of".

We will denote in $\mathcal{A} = \mathbf{S}_j$ with $j \in \{4,5,6\}$ and $\mathcal{A} = \mathbf{R}_j$ with $j \in \{3,4,5\}$, $(\alpha_{ij}, \alpha_{kj})^{-1}(\leq)^0$ by r_{ik}. We adopt this notation because the context makes clear the variety we are referring to. For each $\underline{A} \in \mathcal{A}$ the pointwise extension of r_{ik} to $\mathcal{A}(\underline{A}, \underline{P}_{ij}) \times \mathcal{A}(\underline{A}, \underline{P}_{kj})$ will be denoted by r_{ik}^A.

For $\mathcal{A} = \mathbf{S}_4$ we let $S_{\mathcal{A}}$ consist of the following relations:

$$
\begin{aligned}
r_{01} &= \Big\{(0,0),\, (a,a),\, (1,1)\Big\}\,; & r_{10} &= \Big\{(1,1),\, (0,0)\Big\}\,;\\
r_{00} &= \Big\{(0,0),\, (a,a),\, (1,1)\Big\}\,; & r_{11} &= \{(0,0),\, (a,a),\, (1,1)\}\,;\\
r_{02} &= \Big\{(0,0),\, (a,a),\, (a,b),\, (1,1)\Big\}\,; & r_{20} &= \Big\{(1,1),\, (0,0)\Big\}\,;\\
r_{21} &= \Big\{(0,0),\, (a,a),\, (b,a),\, (1,1)\Big\}\,; & r_{12} &= \Big\{(0,0),\, (1,1)\Big\}\,;\\
r_{22} &= \Big\{(0,0),\, (a,a),\, (b,b),\, (1,1)\Big\}\,.
\end{aligned}
$$

We can now state the following theorem:

Theorem 2.5. *Let $\mathcal{A} = \mathbf{S}_4 = \mathcal{ISP}(\underline{\Pi})$, $T_{\mathcal{A}}$ and $S_{\mathcal{A}}$ as indicated above. Let $\underset{\sim}{\Pi} = (\dot\cup\, P_{ij}; R_{\mathcal{A}}, \tau)$ in which*

a) *τ is the discrete topology.*

b) *$R_{\mathcal{A}} = T_{\mathcal{A}} \cup S_{\mathcal{A}}$.*

Then $R_{\mathcal{A}}$ yields a duality on $\mathcal{A}$.

Although we have $|T_{\mathcal{A}}| = 6$ and $|S_{\mathcal{A}}| = 9$, we shall see that the resulting duality is easily applied, for example, to give the translation process between the natural duality and the restricted Priestly duality and to find free algebras.

The question of whether this natural duality is optimal remains open. At this stage, no general theory which explains the relationship between piggyback dualities and optimal dualities has been developed.Nevertheless, the techniques developed in Davey and Priestly [9, 10] could be applied to settle this question.

We will see in what follows that the natural duality we have determined for $\mathbf{S}_4$ has the advantage of providing us with a nice translation process between this one and the known restricted Priestley duality (see [1] and section 6.3 of [5]).

Here, as in [1], the symbol g denotes the continuous involutive order reversing map from $H(\underline{A})$ to $H(\underline{A})$ associated with $\sim$, the De Morgan negation on $\underline{A}$.

We observe that f_∇ denotes the continuous order-preserving map from $H(\underline{A})$ to $H(\underline{A})$ associated with the $\{0,1\}$-lattice homomorphism $\nabla\colon \underline{A} \to \underline{A}$. The symbol f_α has analogous meaning in the case of regular α-De Morgan algebras.

Proposition 2.6. *Let $\underline{A} \in \mathcal{A} = \mathbf{S}_4 = \mathcal{ISP}(\underline{\Pi})$. Let $i, k \in I_4 = \{0, 1, 2\}$, $x \in X^i$ and $y \in X^k$ then $x\, r^A_{ik}\, y$ if and only if*

$$\Phi_{\alpha_{i4}}(x) \le \Phi_{\alpha_{k4}}(y) \ \text{in } H(\underline{A}) \ .$$

PROOF. Let consider $x \in X^0$, $y \in X^1$ and $x\, r^A_{01}\, y$. Then

$$\begin{aligned}(\forall\, s \in \underline{A})\ \Big(x(s), y(s)\Big) \in (\alpha_{04}, \alpha_{14})^{-1}(\le)^0 \iff& \\ \iff& (\forall\, s \in \underline{A})\ (\alpha_{04} \circ x)(s) \le (\alpha_{14} \circ y)(s) \\ & \text{(by Proposition 2.4)} \\ \iff& \Phi_{\alpha_{04}}(x) \le \Phi_{\alpha_{14}}(y)\ .\end{aligned}$$

We can argue analogously with the other relations of the form r^A_{ik}. □

Let $\mathcal{A} = \mathbf{S}_j$, $j \in \{4, 5, 6\}$ or $\mathcal{A} = \mathbf{R}_j$ with $j \in \{3, 4, 5\}$, we will denote by B^A_{ik} and M^A_{ik} certain sublattices of $\underline{P}_{ij} \times \underline{P}_{kj}$, that in the case of $\mathcal{A} = \mathbf{S}_4$ we will define in the following proof. Whenever we use these sublattices it will be clear which variety we are working in, so we will write B_{ik} and M_{ik}, respectively. The utility of these sublattices will be clear in the proof of the next theorem and in table 1.

For the second part of the next theorem, which gives the translation process from the natural dual to the Priestley dual of an algebra $A \in \mathbf{S}_4$, we need to introduce some notation. Thus for each pair (r_{ik}, r_{ki}) we define

$$i^{ik} = \Big\{(a_i, b_k) \in P_{ij} \times P_{kj} \mid (a_i, b_k) \in r_{ik},\ (b_k, a_i) \in r_{ki}\Big\}\ .$$

For each $\underline{A} \in \mathcal{A}$ we denote by I^{ik} the pointwise extension to $D(\underline{A})$ of i^{ik} where $i, k \in I_j$ and by

$$I = \mathrm{id} \cup \Big(\bigcup_{i,k \in I_j} \Big(I^{ik} \cup (I^{ik})^{-1}\Big)\Big)\ .$$

Theorem 2.7. *Let $\mathcal{A} = \mathbf{S}_4 = \mathcal{ISP}(\underline{\Pi})$ and $\underline{A} \in \mathcal{A}$. From the Priestley space $Y = H(\underline{A}) = (Y, \tau, \le, g, f_\nabla)$ form the $\underline{\Pi}$-indexed structure $\underset{\sim}{Y} = (\dot\bigcup\, Y^i; G', R', \tau')$ as follows:*

i) *τ' is the union of the topologies induced by τ on the sets Y^i;*
ii) *$G' = \{g|_{Y^i} \mid i \in I_j\} \cup \{f_\nabla|_{Y^i} \mid i \in I_j\}$;*
iii) *$R' = \{\le \cap Y^i \times Y^k \mid i, k \in I_j\}$.*

Then $\underset{\sim}{X} = D(\underline{A}) \le \underline{\Pi}^A$ is isomorphic to $\underset{\sim}{Y}$.

Conversely, from $D(\underline{A})$ define $U = (Z, \tau_Z, \le_Z, g_Z, f_{\nabla_Z})$ as follows:

1. *$Z = X/\rho$ where ρ is the equivalence relation on X given by $x \rho y$ iff $(x,y) \in I$; τ_Z is the quotient topology;*
2. *Denoting by $\pi \colon X \to X/\rho$ the canonical projection, $\pi(x) \leq_Z \pi(y)$ in Z iff $(x,y) \in r_{ik}^A$ for some $(i,k) \in I_j^2$;*
3. *g_Z is the function whose restriction to X^0, X^1, X^2 are respectively g_{04}, g_{14}, id_{24} and f_{∇_Z} is the function whose restriction to X^0, X^1, X^2 are respectively g_{04}, id_{14}, u.*

Then U is a well defined involutive Stone space isomorphic to $H(\underline{A})$.

PROOF. We will prove 3) since the other statements of the theorem can be proved following very closely the proof of the corresponding parts in Theorem 3.8 of [7].

Consider the sublattices B_{ik} of $P_{i4} \times P_{k4}$ such that $B_{ik} = \{(a,b) \in P_{i4} \times P_{k4} \mid g(\alpha_{i4})(a) = \alpha_{k4}(b)\}$, where $(i,k) \in \{(0,1),(1,0),(2,2)\}$. We can see that $B_{01}^0 = g_{04}$, $B_{10}^0 = g_{14}$ and $B_{22}^0 = \mathrm{id}_{24}$. It is now easy to verify 3) in what respects g_Z. We will see that g_Z is the function whose restriction to X^0 is g_{04}.

For any $x \in \mathcal{A}(\underline{A}, \underline{P}_{04})$, $y \in \mathcal{A}(\underline{A}, \underline{P}_{14})$ we have

$$\begin{aligned}
(x,y) \in g_{04} &\iff (\forall\, s \in \underline{A})\ (x(s), y(s)) \in g_{04} \\
&\iff (\forall\, s \in \underline{A})\ g(\alpha_{04})(x(s)) = \alpha_{14}(y(s)) \ (\text{since } g_{04} = B_{01}^0) \\
&\iff (\forall\, s \in \underline{A})\ \Big(\alpha_{04}(\sim x(s)) = 0 \ \Leftrightarrow\ \alpha_{14}(y(s)) = 1\Big) \\
&\qquad \text{(by Priestley duality for De Morgan algebras)} \\
&\iff (\forall\, s \in \underline{A})\ \Big(g(\alpha_{04} \circ x) = 1 \ \Leftrightarrow\ (\alpha_{14} \circ y)(s) = 1\Big) \\
&\iff g(\alpha_{04} \circ x) = \alpha_{14} \circ y \\
&\iff g(\Phi_{\alpha_{04}}(x)) = \Phi_{\alpha_{14}}(y)\ .
\end{aligned}$$

Using B_{10}^0 and B_{22}^0 we can see that g_Z is the function whose restrictions to X^1 and X^2 are g_{14} and id_{24}, respectively.

To see what f_{∇_Z} is, we must consider the following sublattices M_{ik} of the lattices $\underline{P}_{i4} \times \underline{P}_{k4}$

$$M_{ik} = \Big\{(a,b) \in \underline{P}_{i4} \times \underline{P}_{k4} \mid \alpha_{i4} \circ \nabla(a) = \alpha_{k4}(b)\Big\}\ ,$$

where $(i,k) \in \{(0,1),(1,1),(2,1)\}$. Then it is easy to verify that $M_{01}^0 = g_{04}$, $M_{11}^0 = \mathrm{id}_{14}$ and $M_{21}^0 = u$.

We will see that f_{∇_Z} is the function whose restriction to X^0 is g_{04}. The other two cases can be proved analogously.

For any $x \in \mathcal{A}(\underline{A}, \underline{P}_{04})$, $y \in \mathcal{A}(\underline{A}, \underline{P}_{14})$ we have

$$\begin{aligned}
(x, y) \in g_{04} &\iff (\forall\, s \in \underline{A})\ x(s)\, g_{04}\, y(s) \\
&\iff (\forall\, s \in \underline{A})\ \alpha_{04} \circ \nabla(x(s)) = \alpha_{14}(y(s))\ (\text{since } g_{04} = M^0_{01}) \\
&\iff (\forall\, s \in \underline{A})\ \alpha_{04}(x(\nabla s)) = \alpha_{14}(y(s)) \\
&\iff (\forall\, s \in \underline{A})\ f_\nabla(\alpha_{04} \circ x)(s) = \alpha_{14}(y(s)) \\
&\iff f_\nabla(\Phi_{\alpha_{04}}(x)) = \Phi_{\alpha_{14}}(y)\ .
\end{aligned}$$

□

Table 1

$\mathcal{A}$	sublattices associated to g	
$\mathbf{S}_4$	$B_{ik} = \Big\{(a,b) \in P_{i4} \times P_{k4} \mid g(\alpha_{i4})(a) = \alpha_{k4}(b)\Big\}$ $(i,k) \in \{(0,1),(1,0),(2,2)\}$	$gz\vert_{X^0} = g_{04} = B^0_{01}$ $gz\vert_{X^1} = g_{14} = B^0_{10}$ $gz\vert_{X^2} = \mathrm{id}_{24} = B^0_{22}$
$\mathbf{S}_5$	$B_{ik} = \Big\{(a,b) \in P_{i5} \times P_{k5} \mid g(\alpha_{i5})(a) = \alpha_{k5}(b)\Big\}$ $(i,k) \in \{(0,1),(1,0)\}$	$gz\vert_{X^0} = g_{05} = B^0_{01}$ $gz\vert_{X^1} = g_{15} = B^0_{10}$
$\mathbf{S}_6$	$B_{ii} = \Big\{(a,b) \in P_{i6} \times P_{i6} \mid g(\alpha_{i6})(a) = \alpha_{i6}(b)\Big\}$ $(i,i) \in \{(0,0)\}$	$gz\vert_{X^0} = u^{61} = B^0_{00}$
$\mathbf{R}_3$	$B_{ik} = \Big\{(a,b) \in P_{i3} \times P_{k3} \mid g(\alpha_{i3})(a) = \alpha_{k3}(b)\Big\}$ $(i,k) \in \{(0,1),(1,0)\}$	$gz\vert_{X^0} = g_{03} = B^0_{01}$ $gz\vert_{X^1} = g_{13} = B^0_{10}$
$\mathbf{R}_4$	$B_{ik} = \Big\{(a,b) \in P_{i4} \times P_{k4} \mid g(\alpha_{i4})(a) = \alpha_{k4}(b)\Big\}$ $(i,k) \in \{(0,1),(1,0)\}$	$gz\vert_{X^0} = g_{04} = B^0_{01}$ $gz\vert_{X^1} = g_{14} = B^0_{10}$
$\mathbf{R}_5$	$B_{ik} = \Big\{(a,b) \in P_{i5} \times P_{k5} \mid g(\alpha_{05})(a) = \alpha_{15}(b)\Big\}$ $(i,k) \in \{(0,1),(1,0)\}$	$gz\vert_{X^0} = g_{05} = B^0_{01}$ $gz\vert_{X^1} = g_{15} = B^0_{10}$

Table 1 (cont.)

$\mathcal{A}$	sublattices associated to $\nabla[\alpha]$	
$\mathbf{S}_4$	$M_{ik} = \left\{(a,b) \in P_{i4} \times P_{k4} \mid \alpha_{i4} \circ \nabla(a) = \alpha_{k4}(b)\right\}$ $(i,k) \in \{(0,1),(1,1),(2,1)\}$	$f_{\nabla_Z}\|_{X^0} = g_{04} = M_{01}^0$ $f_{\nabla_Z}\|_{X^1} = \mathrm{id}_{14} = M_{11}^0$ $f_{\nabla_Z}\|_{X^2} = u = M_{21}^0$
$\mathbf{S}_5$	$M_{ik} = \left\{(a,b) \in P_{i5} \times P_{k5} \mid \alpha_{i5} \circ \nabla(a) = \alpha_{k5}(b)\right\}$ $(i,k) \in \{(0,1),(1,1)\}$	$f_{\nabla_Z}\|_{X^0} = g_{05} \circ u^{51} = M_{01}^0$ $f_{\nabla_Z}\|_{X^1} = u^{51} = M_{11}^0$
$\mathbf{S}_6$	$M_{ii} = \left\{(a,b) \in P_{i6} \times P_{i6} \mid g(\alpha_{i6}) \circ \nabla(a) = \alpha_{i6}(b)\right\}$ $(i,i) \in \{(0,0)\}$	$f_{\nabla_Z}\|_{X^0} = u^{62} = M_{00}^0$
$\mathbf{R}_3$	$M_{ik} = \left\{(a,b) \in P_{i3} \times P_{k3} \mid \alpha_{i3} \circ \alpha(a) = \alpha_{k3}(b)\right\}$ $(i,k) \in \{(0,1),(1,1)\}$	$f_{\alpha_Z}\|_{X^0} = g_{03} = M_{01}^0$ $f_{\alpha_Z}\|_{X^1} = \mathrm{id}_{13} = M_{11}^0$
$\mathbf{R}_4$	$M_{ik} = \left\{(a,b) \in P_{i4} \times P_{k4} \mid \alpha_{i4} \circ \alpha(a) = \alpha_{k4}(b)\right\}$ $(i,k) \in \{(0,1),(1,1)\}$	$f_{\alpha_Z}\|_{X^0} = g_{04} \circ v^{42} = M_{01}^0$ $f_{\alpha_Z}\|_{X^1} = v^{42} = M_{11}^0$
$\mathbf{R}_5$	$M_{ik} = \left\{(a,b) \in P_{i5} \times P_{k5} \mid \alpha_{i5} \circ \alpha(a) = \alpha_{k5}(b)\right\}$ $(i,k) \in \{(0,1),(1,1)\}$	$f_{\alpha_Z}\|_{X^0} = g_{05} \circ v^{52} = M_{01}^0$ $f_{\alpha_Z}\|_{X^1} = v^{51} = M_{11}^0$

3. Free algebras

Since we have obtained a translation process between the natural duality and the Priestley duality for the subvarieties of both $\mathbf{S}$ and $\mathbf{R}$, we can now very easily describe the free algebras on n generators with $n < \omega$ (see [7], Lemma 1.2).

However, before doing that, we will show that the maps f_∇ and f_α can be nicely described. In fact using the Priestley duality obtained by R. Cignoli and M.S. de Gallego in [1], for the two classes of algebras we are dealing with, we can show the following two propositions:

Proposition 3.1. *Let $\underline{A} \in \mathbf{S}$. Given the $\{0,1\}$-lattice homomorphism*

$$\nabla\colon \underline{A} \to \underline{A}$$

the associated continuous order preserving map

$$f_\nabla\colon H(\underline{A}) \to H(\underline{A})$$

is such that, for each $y \in H(\underline{A})$, $f_\nabla(y) = n_y$ where n_y is the unique element of $\operatorname{Max} H(\underline{A})$ *such that $y \leq n_y$.*

PROOF. It is known that $f_\nabla : H(\underline{A}) \to H(\underline{A})$ is such that $f_\nabla(y) = \mathrm{Max}\{x \in H(\underline{A}) \,|\, y \in \nabla[x)\}$, for each $y \in H(\underline{A})$. But if $x \in H(\underline{A})$ and $y \in \nabla[x)$ then $y \in (n_x]$, $n_x = n_y$ and $x \leq n_y$ (see [1], page 382).. Since $y \in \nabla[n_y)$ we have that $f_\nabla(y) = n_y$. $\square$

Proposition 3.2. *Let $\underline{A} \in \mathbf{R}$. Given the $\{0,1\}$-lattice homomorphism*

$$\alpha\colon \underline{A} \to \underline{A}$$

the associated continuous order preserving map

$$f_\alpha\colon H(\underline{A}) \to H(\underline{A})$$

is defined, for each $y \in H(\underline{A})$, by $f_\alpha(y) = m_y$, where m_y is the unique element of $\mathrm{Max}\{x \in H(\underline{A}) : x \leq g(x)\}$ *which is comparable with y.*

PROOF. We have to show that $\mathrm{Max}\{x \in H(\underline{A}) \mid y \in \alpha[x)\} = m_y$ for each $y \in H(\underline{A})$. Applying Lemma 3.6 of [1] we have $y \leq m_y$ or $m_y \leq y$ and also that $y \in \alpha[x)$ iff $m_y \in [x)$. So we can conclude that for each $x \in H(\underline{A})$ such that $y \in \alpha[x)$ we have $x \leq m_y$. Since $y \in \alpha[m_y)$ the proposition is proved. $\square$

We can now describe the distributive lattice duals and the distributive lattice reducts of the free algebras on n generators for the varieties we are working with. As a corollary we also obtain the free spectra of these varieties. In order to do this we need to establish some notation.

Let P and Q be disjoint finite ordered sets such that $Q = P^\delta$. Let ϕ be a bijection from $\mathrm{Min}\, Q$ onto $\mathrm{Max}\, P$. The new partially ordered set obtained superimposing Q over P and then collapsing each minimal point a of Q with the element $\phi(a)$ of P is called the *collapsed sum* of P and Q and it will be denoted by $P \oplus_{\mathrm{c}} Q$.

Denote by $\mathcal{O}(Q)$ the lattice of all up-sets of Q. Consider $P \oplus_{\mathrm{c}} Q$ and let $\emptyset \neq Y \subseteq \mathrm{Min}\, Q$. In what follows Y_Q denotes the element $[Y)$ of $\mathcal{O}(Q)$ and Y_P denotes the element $[\phi(Y))$ of $\mathcal{O}(P)$. Letting $\mathcal{O}(Q)' = \{Y_Q \mid \emptyset \neq Y \subseteq \mathrm{Min}\, Q\}$ and $\mathcal{O}(P)' = \{Y_P \mid \emptyset \neq Y \subseteq \mathrm{Min}\, Q\}$ we have that the map φ from $\mathcal{O}(Q)'$ onto $\mathcal{O}(P)'$ such that $\varphi(Y_Q) = Y_P$ is a bijection.

Remark. The partially ordered set $\mathcal{O}(P \oplus_{\mathrm{c}} Q)$ may be obtained by superimposing $\mathcal{O}(P)$ over $\mathcal{O}(Q)$, deleting the element 0 of $\mathcal{O}(P)$ and collapsing each $Y_Q \in \mathcal{O}(Q)'$ with $\varphi(Y_Q) \in \mathcal{O}(P)'$. We will call $\mathcal{O}(P \oplus_{\mathrm{c}} Q)$ the *special collapsed sum* of $\mathcal{O}(Q)$ and $\mathcal{O}(P)$ and it will be denoted by $\mathcal{O}(Q) \oplus_{\mathrm{sc}} \mathcal{O}(P)$.

Starting from the results of Theorem 2.7 and those we tabulated in table 1, we will represent in the figures numbered from 3 to 9, the diagrams of $DF\mathcal{A}(1)$.

These are exactly the multi-sorted schizophrenic objects on which the duality is based. In the figures we will only evidence that part of the relational structure of these objects that pertains to the partial orders underlying $HF\mathcal{A}(1)$ and nothing more.

We note that in the figures which follow, with the exception of figure 8, the subscript on the elements is the i in P_{ij}, and that we will refer to the elements with subscript i as the P_{ij} component of the figure. In fact the bijection from $\underline{P}_{ij}$ to $\mathcal{A}(F\mathcal{A}(1), \underline{P}_{ij})$ explains this notation.

In the case of figure 8 we attribute the subscript 2 to the elements of the component P_{14} of figure 7, and the subscript 3 to the elements of the component P_{04} of figure 7. The subscripts 0 and 1 were attributed according to those of figure 6.

Furthermore to a better understanding of these figures we point out that, for example, figure 3 is constructed from the relations $r_{ij}^{F\mathbf{S}_4(1)}$ with $0 \leq i, j \leq 2$, where r_{ij} are the elements of $S_{\mathcal{A}}$ for $\mathcal{A} = \mathbf{S}_4$, that we presented before.

It must be said that the referred figures are sort of "generalized Hasse diagrams" (where the loops are to be understood as the quasi-order relation going both ways) for the quasi-orders whose partial order quotients are exactly the underlying orders of the Priestley duals, $HF\mathcal{A}(1)$.

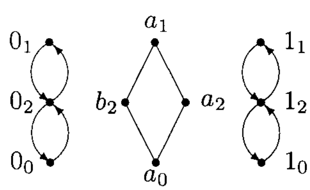


Figure 3 – $D F\mathbf{S}_4(1)$.

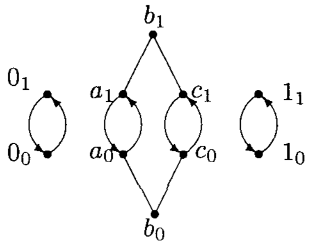


Figure 4 – $DF\mathbf{S}_5(1)$.

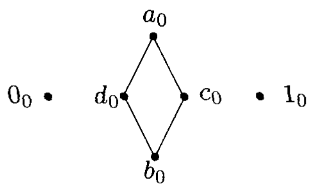


Figure 5 – $DF\mathbf{S}_6(1)$.

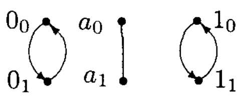


Figure 6 – $DF\mathbf{R}_3(1)$.

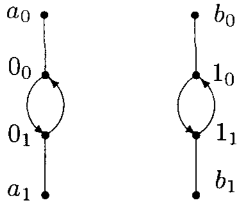


Figure 7 – $DF\mathbf{R}_4(1)$.

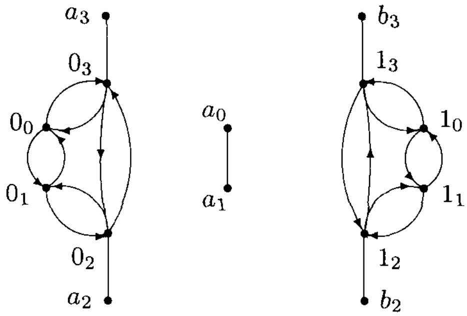


Figure 8 – $DF(\mathbf{R}_3 \vee \mathbf{R}_4)(1)$.

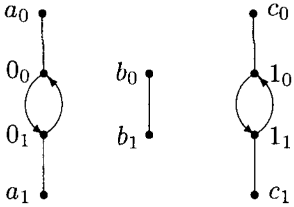


Figure 9 – $DF\mathbf{R}_5(1)$.

On the figures mentioned above we did not indicate the maps g in those that respect the subvarieties of $\mathbf{S}$ nor in those that respect the subvarieties of $\mathbf{R}$. In fact since in these varieties the free algebras have a Kleene negation dual to the map g this one is uniquely determined. The maps f_∇ and f_α have been explicitly described in Propositions 3.1 and 3.2, respectively. Thus it is unnecessary to describe these maps on the figures.

We observe that $HF\mathbf{S}_j(n)$ is derived from $DF\mathbf{S}_j(n)$ which, as is well known, is obtained by taking the n-fold power of $DF\mathbf{S}_j(1)$ for $j = 4, 5$ and 6. In our cases the partial order is obtained via the pointwise extension of the relations "maximal subalgebras of". The maps g and f_∇ in $HF\mathbf{S}_j(n)$ are obtained in the obvious way.

Analogous comments can be made about the process of obtaining $H(\mathbf{R}_3 \vee \mathbf{R}_4)(n)$ and $HF\mathbf{R}_j(n)$ with $j \in \{3, 4, 5\}$ starting with the natural dual of the corresponding free algebra on one generator.

In what follows $\underline{\mathbf{2}}$, $\mathbf{2}$ and $\underline{\mathbf{1}} = \mathbf{2}^0 = \underline{\mathbf{2}}^0$ denote the two-element chain, the two and the one-element antichain, respectively. The disjoint union of m copies of a poset P will be denoted by mP with the convention that $0P$ means a one-element chain.

Note that $(\underline{\mathbf{1}} \oplus \mathbf{2}) \oplus_c (\mathbf{2} \oplus \underline{\mathbf{1}}) = HF\mathbf{K}(1)$, where $HF\mathbf{K}(1)$ denotes the poset of the Priestley dual of the free Kleene algebra on one generator.

Considering just the underlying ordered set of Priestley duals of the free algebras on n generators and the distributive lattice reducts of the same algebras for the subvarieties of $\mathbf{S}$ under consideration, we can state the two following results:

Theorem 3.3. *Consider $\mathbf{S}_j$ with $j \in \{4, 5, 6\}$. Then the following are descriptions of the partial order reducts of the respective Priestley spaces:*

a) $HF\mathbf{S}_4(n) = 2^n\binom{n}{0}\,(\underline{\mathbf{1}} \oplus_c \mathbf{2}^0 \oplus_c \underline{\mathbf{1}}) \,\dot{\cup}\, \left(\dot{\bigcup}_{1\le i\le n} 2^{n-i}\binom{n}{i}\,(\underline{\mathbf{1}} \oplus \mathbf{2}^i \oplus \underline{\mathbf{1}})\right)$;

b) $HF\mathbf{S}_5(n) = \dot{\bigcup}_{0\le i\le n} 2^{n-i}\binom{n}{i}\, HF\mathbf{K}(i)$;

c) $HF\mathbf{S}_6(n) = \dot{\bigcup}_{0\le i\le n} 2^{n-i}\binom{n}{i}\,(\underline{\mathbf{2}}^2)^i$.

Proof. These results follow from the knowledge that we have obtained of the translation process between natural dual and Priestley dual. In particular, notice that $n = 1$ gives a description of the order quotients of the quasi-orders given in figures 3, 4 and 5.

To make clear the combinatorial calculations that we have done to obtain these results, we take the case of the variety $\mathbf{S}_4$, so we concentrate on figure 3. For the other varieties the reasoning is analogous.

We suppose on a first level the elements of the n-fold power of the component P_{14}, on a second level, under the first one, the elements of the n-fold power of the component P_{24} and in a third level, under the second one, the elements of the n-fold power of the component P_{04}. For each n-tuple of the first level we must consider the following two cases:

Case 1) The n-tuple of P_{14}^n has no coordinates equal to a_1, so they belong to $\{0_1, 1_1\}$. Clearly there are 2^n of these n-tuples and each one is tied by a loop with exactly one n-tuple of the component P_{24}^n. This last one is tied by a loop with exactly one n-tuple of the component P_{04}^n. These three n-tuples collapse into a point and give the parcel $2^n\binom{n}{0}(\underline{\mathbf{1}} \oplus_c \mathbf{2}^0 \oplus_c \underline{\mathbf{1}})$ of $HF\mathbf{S}_4(n)$.

Case 2) The n-tuple of P_{14}^n has exactly s coordinates equal to a_1, with $s \in \{1, \dots, n\}$, while the other ones belong to $\{0_1, 1_1\}$. The number of these n-tuples is $2^{n-s}\binom{n}{s}$. To facilitate the explanation we will suppose that $s < n$ and that the coordinates equal to a_1 fill in the first s positions, with the other ones being equal to 0_1.

Above this n-tuple we must consider the antichain formed by the n-tuples of P_{24}^n which have a_2 or b_2 in the first s coordinates and have the other ones equal to 0_2. Clearly this antichain has 2^s elements.

Consider on the third level the n-tuple that has a_0 on the first s positions and has the other ones equal to 0_0. Since the order is the product order we obtain $\underline{\mathbf{1}} \oplus \mathbf{2}^s \oplus \underline{\mathbf{1}}$.

This case gives the parcel $2^{n-s}\binom{n}{s}(\underline{\mathbf{1}} \oplus \mathbf{2}^s \oplus \underline{\mathbf{1}})$ of $HF\mathbf{S}_4(n)$. □

Denoting by $^\wedge F\mathbf{B}(n)_\perp$ the distributive lattice reduct of the free Boolean algebra on n generators with a new zero and a new unit adjoined, by $F\mathbf{D}(n)$ the $\{0,1\}$-free distributive lattice on n generators and by $F\mathbf{K}(n)$ the distributive lattice reduct of the free Kleene algebra on n generators, we have the following theorem:

Theorem 3.4. *Consider* $\mathbf{S}_j$, *with* $j \in \{4,5,6\}$, *subvarieties of* $\mathbf{S}$. *Then the following are descriptions of the lattice reducts of the respective free algebras on* n *generators:*

a) $F\mathbf{S}_4(n) = F\mathbf{D}(0)^{2^n} \times \prod_{i=1}^n {}^\wedge F\mathbf{B}(i)_\perp^{2^{n-i}\binom{n}{i}}$;
b) $F\mathbf{S}_5(n) = \prod_{i=0}^n F\mathbf{K}(i)^{2^{n-i}\binom{n}{i}}$;
c) $F\mathbf{S}_6(n) = \prod_{i=0}^n F\mathbf{D}(2i)^{2^{n-i}\binom{n}{i}}$.

PROOF. The result follows from the previous theorem applying Priestley duality and remembering that $\mathcal{O}(\underline{\mathbf{1}} \oplus \mathbf{2}^i \oplus \underline{\mathbf{1}}) = {}^\wedge F\mathbf{B}(i)_\perp$ and that $\mathcal{O}(\underline{\mathbf{2}}^{2i}) = F\mathbf{D}(2i)$. □

The next corollary is now immediate.

Corollary 3.5. *Consider* $\mathbf{S}_j$ *with* $j \in \{4,5,6\}$. *The free spectra of these varieties, are given by:*

a) $|F\mathbf{S}_4(n)| = 2^{2^n} \prod_{i=1}^{n} (2^{2^i}+2)^{2^{n-i}\binom{n}{i}}$;
b) $|F\mathbf{S}_5(n)| = \prod_{i=0}^{n} |F\mathbf{K}(i)|^{2^{n-i}\binom{n}{i}}$;
c) $|F\mathbf{S}_6(n)| = \prod_{i=0}^{n} |F\mathbf{D}(2i)|^{2^{n-i}\binom{n}{i}}$.

Analogously we can deduce the following two theorems describing the underlying posets of Priestley duals and the distributive lattice reducts of the free algebras on n generators for the subvarieties of $\mathbf{R}$, respectively:

Theorem 3.6. *Consider* $\mathbf{R}_j$, *with* $j \in \{3,4,5\}$, *and* $\mathbf{R}_3 \vee \mathbf{R}_4$ *subvarieties of* $\mathbf{R}$, *then the following are descriptions of the partial order reducts of the respective Priestley spaces:*

a) $HF\mathbf{R}_3(n) = 2^n(\underline{\mathbf{2}}^0) \,\dot\cup\, (3^n - 2^n)\,(\underline{\mathbf{2}}^0 \oplus \underline{\mathbf{2}}^0)$;
b) $HF\mathbf{R}_4(n) = 2^n(\underline{\mathbf{2}}^n \oplus_c \underline{\mathbf{2}}^n)$;
c) $HF(\mathbf{R}_3 \vee \mathbf{R}_4)(n) = 2^n(\underline{\mathbf{2}}^n \oplus_c \underline{\mathbf{2}}^n) \,\dot\cup\, (3^n - 2^n)\,(\underline{\mathbf{2}}^0 \oplus \underline{\mathbf{2}}^0)$;
d) $HF\mathbf{R}_5(n) = 2^n\binom{n}{0}\,(\underline{\mathbf{2}}^n \oplus_c \underline{\mathbf{2}}^n) \,\dot\cup\, \left(\dot\bigcup_{1\le i\le n} 2^{n-i}\binom{n}{i}\,(\underline{\mathbf{2}}^{n-i} \oplus \underline{\mathbf{2}}^{n-i})\right)$.

Proof. These results follow from the knowledge that we have obtained of the translation process between natural dual and Priestley dual. In particular notice that $n=1$ gives a description of the order quotients of the quasi-orders given in figures 6, 7, 8 and 9.

To make clear the combinatorial calculations that we have done to obtain these results, we take the case of the variety $\mathbf{R}_3$, so we concentrate on figure 6. For the other varieties the reasoning is analogous.

We suppose on a first level the elements of the n-fold power of the component P_{03} and on a second level, under the first one, the elements of the n-fold power of the component P_{13}. For each n-tuple of the first level we must consider the following two cases:

Case 1) The n-tuple of P_{03}^n has no coordinates equal to a_0. Clearly there are 2^n of these n-tuples.. Under this n-tuple we must consider the n-tuple that has no coordinates equal to a_1 and such that each coordinate in the n-tuple of the top level is tied by a loop with the corresponding coordinate below. These pairs of n-tuples collapse into a point and give the parcel $2^n(\underline{\mathbf{2}}^0)$ of $HF\mathbf{R}_3(n)$.

Case 2) The n-tuple of P_{03}^n has s coordinates equal to a_0, with $s \in \{1,\dots,n\}$, while the other ones belong to $\{0_0, 1_0\}$. The number of these n-tuples is $3^n - 2^n$.

To facilitate the explanation let's suppose that $s < n$ and that the coordinates equal to a_0 fill in the first s positions, with the other ones being equal to 0_0.

Consider on the second level the n-tuple that has a_1 filling in the first s positions and that has the other ones equal to 0_1.

These two n-tuples form a two element chain, since the order is the product order, and give the parcel $(3^n - 2^n)(\underline{\mathbf{2}}^0 \oplus \underline{\mathbf{2}}^0)$ of $HF\mathbf{R}_3(n)$. □

Let $M_n = \mathcal{O}(\underline{\mathbf{2}}^n \oplus_c \underline{\mathbf{2}}^n)$, $N_s = \mathcal{O}(\underline{\mathbf{2}}^s \oplus \underline{\mathbf{2}}^s)$ with $s \in \{0, ..., n-1\}$. Then according to the previous Remark we have $M_n = F\mathbf{D}(n) \oplus_{sc} F\mathbf{D}(n)$ and $N_s = F\mathbf{D}(s) \oplus_c F\mathbf{D}(s)$.

As a consequence of Theorem 3.6, applying Priestley duality, we have the next theorem:

Theorem 3.7. *Consider $\mathbf{R}_j$ with $j \in \{3, 4, 5\}$ and $\mathbf{R}_3 \vee \mathbf{R}_4$ subvarieties of $\mathbf{R}$. Then the following are descriptions of the lattice reducts of the respective free algebras on n generators:*

a) $F\mathbf{R}_3(n) = F\mathbf{D}(0)^{2^n} \times F\mathbf{D}(1)^{3^n - 2^n}$;
b) $F\mathbf{R}_4(n) = M_n^{2^n}$;
c) $F(\mathbf{R}_3 \vee \mathbf{R}_4)(n) = M_n^{2^n} \times F\mathbf{D}(1)^{3^n - 2^n}$;
d) $F\mathbf{R}_5(n) = M_n^{2^n} \times \prod_{1 \le i \le n} N_{n-i}^{2^{n-i}\binom{n}{i}}$.

As a consequence of the Theorem 3.7 we easily conclude the following corollary:

Corollary 3.8. *Consider $\mathbf{R}_j$, with $j \in \{3, 4, 5\}$, and $\mathbf{R}_3 \vee \mathbf{R}_4$ subvarieties of $\mathbf{R}$. The free spectra of these varieties, are given by:*

a) $|F\mathbf{R}_3(n)| = 2^{2^n} 3^{3^n - 2^n}$;
b) $|F\mathbf{R}_4(n)| = (2|F\mathbf{D}(n)| - 2)^{2^n}$;
c) $|F(\mathbf{R}_3 \vee \mathbf{R}_4)(n)| = (2|F\mathbf{D}(n)| - 2)^{2^n} 3^{3^n - 2^n}$;
d) $|F\mathbf{R}_5(n)| = (2|F\mathbf{D}(n)| - 2)^{2^n} \prod_{1 \le i \le n} (2|F\mathbf{D}(i)| - 1)^{2^{n-i}\binom{n}{i}}$.

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