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A NOTE ON QUOTIENTS FORMED BY UNIT GROUPS OF SEMILOCAL RINGS

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1. INTRODUCTION

It is well known that for a proper field extension $k \subset K$ the group $K^\times/k^\times$ is not finitely generated if k is infinite ([6, Theorem 4.3.11]). In this paper we generalize this to the case where k and K are semilocal noetherian rings and K is a finitely generated k-module.

Our interest in such quotients comes from the theory of class groups. Let R be a noetherian integral domain whose integral closure $\bar{R}$ is a finitely generated R-module and let S be the monoid of non zero divisors of $\bar{R}/R$. Then the rings $\bar{R}_S$, R_S are semilocal and there is an exact sequence ([5])

$$1 \longrightarrow \bar{R}^\times/R^\times \longrightarrow \bar{R}_S^\times/R_S^\times \longrightarrow \mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(\bar{R}) \longrightarrow \mathrm{Cl}(\bar{R}_S) \longrightarrow 0 \quad .$$

Thus, in order to study the singular part of $\mathrm{Cl}(R)$ (i. e. the part coming from non normal points in $\mathrm{Spec}(R)$) one is naturally led to consider the quotient group $\bar{R}_S^\times/R_S^\times$.

All rings in this paper are assumed to be commutative and to possess an unit element.

2. THE RESULT

Let R be a ring. We let $R^\times$ be its group of units. For a R-module M we denote by $\mathrm{Ass}_R(M)$ the set of prime ideals associated to M.

Theorem 2.1. *Let $R_1 \subset R_2$ be an extension of rings such that R_1 is noetherian and semilocal and such that R_2 is a finitely generated R_1-module. Then the following assertions are equivalent:*

1. *Each $\mathfrak{p} \in Ass_{R_1}(R_2/R_1)$ has a finite residue class field.*
2. *R_2/R_1 is finite.*

3. $R_2^\times/R_1^\times$ *is finite.*
4. $R_2^\times/R_1^\times$ *is finitely generated.*

Remark. For related results see [2].

For the proof of this theorem we need some lemmata which handle special cases of the theorem.

Lemma 2.2. *Let k be an infinite field and A a finite dimensional k-algebra such that $k \neq A$. Then $A^\times/k^\times$ is not finitely generated.*

PROOF. This follows from [4, Lemma 1.6]. □

In the following let $R_1 \subset R_2$ be as in the theorem. Let $\mathfrak{M}$ be the set of maximal ideals of R_1. We denote by $J_1 = \bigcap_{\mathfrak{m}\in\mathfrak{M}} \mathfrak{m}$ the radical of R_1 and set $J_2 = J_1 R_2$. Then we have, $J_2 \cap R_1 = J_1$, $1 + J_i \subset R_i^\times$ and an exact sequence

$$1 \longrightarrow \frac{1+J_2}{1+J_1} \longrightarrow R_2^\times/R_1^\times \longrightarrow \frac{(R_2/J_2)^\times}{(R_1/J_1)^\times} \longrightarrow 1 \quad . \tag{1}$$

For $l \geq 0$ let U_l be the image of $1 + J_1^l J_2$ in $(1+J_2)/(1+J_1)$. Further set $M = J_2/J_1$.

Lemma 2.3. *For all $l \geq 0$ we have $U_l/U_{l+1} \cong J_1^l M/J_1^{l+1}M$.*

PROOF. Let $x \in J_1$ and $y \in J_1^l J_2$. Then we have

$$1 + x + y = (1+x)(1+(1+x)^{-1}y) \in (1+J_1)(1+J_1^l J_2) \quad .$$

Hence we obtain $(1+J_1)(1+J_1^l J_2) = 1 + J_1 + J_1^l J_2$. Define a map

$$\varphi\colon 1 + J_1 + J_1^l J_2 \longrightarrow \frac{J_1 + J_1^l J_2}{J_1 + J_1^{l+1} J_2}$$

by $\varphi(1+x) = x + J_1 + J_1^{l+1}J_2$. Since $(J_1 + J_1^l J_2)^2 \subset J_1 + J_1^{l+1}J_2$, φ is a homomorphism. It is surjective and has kernel $1 + J_1 + J_1^{l+1}J_2$. Hence we obtain an isomorphism

$$\begin{aligned} U_l/U_{l+1} &= \frac{(1+J_1)(1+J_1^l J_2)}{(1+J_1)(1+J_1^{l+1}J_2)} = \frac{1+J_1+J_1^l J_2}{1+J_1+J_1^{l+1}J_2} \\ &\cong \frac{J_1 + J_1^l J_2}{J_1 + J_1^{l+1}J_2} = J_1^l M/J_1^{l+1}M \quad . \end{aligned}$$

□

Lemma 2.4. *The following assertions are equivalent:*

1. $R_2^\times/R_1^\times$ *is finite.*

2. *R_2/R_1 is finite.*
3. *Each $\mathfrak{p} \in Ass_{R_1}(R_2/R_1)$ has a finite residue class field.*

PROOF. The exact sequence (1) shows that $R_2^\times/R_1^\times$ is finite if and only if

$$\frac{1+J_2}{1+J_1} \quad \text{and} \quad \frac{(R_2/J_2)^\times}{(R_1/J_1)^\times}$$

are both finite. By Lemma 2 $(1+J_2)/(1+J_1)$ and J_2/J_1 possess filtrations with isomorphic quotients. Hence $(1+J_2)/(1+J_1)$ is finite if and only if J_2/J_1 is finite. By the Chinese Remainder Theorem there are isomorphisms

$$\frac{R_2/J_2}{R_1/J_1} \cong \prod_{\mathfrak{m}\in\mathfrak{M}} \frac{R_2/\mathfrak{m}R_2}{R_1/\mathfrak{m}} \quad \text{and} \quad \frac{(R_2/J_2)^\times}{(R_1/J_1)^\times} \cong \prod_{\mathfrak{m}\in\mathfrak{M}} \frac{(R_2/\mathfrak{m}R_2)^\times}{(R_1/\mathfrak{m})^\times} \quad .$$

By Lemma 2.2 the second product will be finite if and only if $(R_2/J_2)/(R_1/J_1)$ is finite. Hence $R_2^\times/R_1^\times$ is finite $\iff$ J_2/J_1 and $(R_2/J_2)/(R_1/J_1)$ are finite $\iff$ R_2/R_1 is finite. The equivalence of 2 and 3 is obvious. □

Lemma 2.5. *Suppose that R_1 is an infinite integral domain and $Ass_{R_1}(R_2/R_1) = \{0\}$. Then $R_2^\times/R_1^\times$ is not finitely generated.*

PROOF. We assume first that there is some maximal ideal $\mathfrak{m}$ of R_1 having an infinite residue class field. Then $(R_1)_\mathfrak{m} \neq (R_2)_\mathfrak{m}$ (since $\mathrm{Ass}_{R_1}(R_2/R_1) = \{0\}$). By Nakayama we obtain $R_1/\mathfrak{m} \neq R_2/\mathfrak{m}R_2$. By Lemma 2.2 $(R_2/\mathfrak{m}R_2)^\times/(R_1/\mathfrak{m})^\times$ is not finitely generated. The projection $R_2 \to R_2/\mathfrak{m}R_2$ induces a surjective homomorphism

$$R_2^\times/R_1^\times \to \frac{(R_2/\mathfrak{m}R_2)^\times}{(R_1/\mathfrak{m})^\times}$$

(here we use the elementary fact that $R^\times \to (R/I)^\times$ is surjective for a semilocal ring and an ideal I of R, see for example [1, Lemma 2]). Hence $R_2^\times/R_1^\times$ is not finitely generated.

From now on we assume that the maximal ideals of R_1 (and hence these of R_2) have a finite residue class field. In particular $\dim R_1 \geq 1$ (since R_1 is assumed to be infinite). We consider two cases.

1. $\dim R_1 = 1$. Let $\mathfrak{N}$ be the nilradical of R_2. First we suppose $\mathfrak{N} \neq 0$. Let $k \geq 1$ be the smallest integer such that $\mathfrak{N}^{k+1} = 0$. Mapping x to $1+x$ gives an embedding of groups $\mathfrak{N}^k \cong 1+\mathfrak{N}^k \subset R_2^\times/R_1^\times$. Suppose that $\mathfrak{N}^k$ is a finitely generated abelian group. Since it is a torsion free R_1-module ($\mathrm{Ass}_{R_1}(R_2/R_1) = \{0\}$) this implies that R_1 is finitely generated, too. In particular R_1 is integral

over its prime ring. Since R_1 is semilocal this is only possible if the prime ring and hence also R_1 is a finite field. This contradicts our assumption $\dim R_1 = 1$.

Next assume $\mathfrak{N} = 0$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals of R_2. Then we have $0 = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r$. Suppose $r \geq 2$. We have the following inclusions:

$$R_1 \subset R_2 \subset \prod_{i=1}^{r} R_2/\mathfrak{p}_i =: \tilde{R} \quad .$$

Since $(R_2)_{\mathfrak{p}_i} = \tilde{R}_{\mathfrak{p}_i}$ we have $\mathfrak{p}_i \notin \operatorname{Ass}_{R_2}(\tilde{R}/R_2)$. By Lemma 2.4, $\tilde{R}^\times/R_2^\times$ is finite. Hence it suffices to show that $\tilde{R}^\times/R_1^\times$ is not finitely generated. Note that $\mathfrak{p}_i \cap R_1 = 0$. Hence (embedding $R_1^\times$ diagonally) we have an inclusion

$$(R_1^\times)^r/R_1^\times \subset \tilde{R}^\times/R_1^\times \quad .$$

But $(R_1^\times)^r/R_1^\times \cong (R_1^\times)^{r-1}$. Now R_1 contains one of the following rings: $\mathbb{Z}_{p\mathbb{Z}}$, (p prime) or $k[T]$ where k is a finite field and T an indeterminate. Hence $R_1^\times$ is not finitely generated.

It remains to handle the case $r = 1$. Then R_2 is an integral domain, too. Let K_i be the quotient field of R_i and $\bar{R}_i$ the integral closure of R_i. We have $K_1 \neq K_2$ and $R_2 \cap K_1 = R_1$ (since $\operatorname{Ass}_{R_1}(R_2/R_1) = \{0\}$). If K_2 is purely inseparable over K_1 then $R_2^\times/R_1^\times$ is a torsion group and hence by Lemma 2.4 it is not finitely generated. So we may assume that K_2 is not purely inseparable over K_1. For an abelian group G let $r(G)$ be its torsion free rank. By [3, Proposition 3.6] $r(K_2^\times/K_1^\times) = \infty$ (note that since $\dim R_2 = 1$, K_2 cannot be algebraic over a finite field). By the Theorem of Krull-Akizuki $\bar{R}_2$ is a principal ideal domain with only (up to associates) finitely many prime elements. Hence $r(K_2^\times/\bar{R}_2^\times) < \infty$. From the exact sequence

$$1 \to \bar{R}_2^\times/\bar{R}_1^\times \longrightarrow K_2^\times/K_1^\times \longrightarrow K_2^\times/\bar{R}_2^\times K_1^\times \longrightarrow 1$$

we deduce $r(\bar{R}_2^\times/\bar{R}_1^\times) = \infty$. Let T be a subring of $\bar{R}_2$ containing R_2 such that T is a finitely generated R_2-module. By Lemma 2.4, $T^\times/R_2^\times$ is finite. Since $\bar{R}_2$ is an union of such rings T, $\bar{R}_2^\times/R_2^\times$ is a torsion group. Using the exact sequence

$$1 \longrightarrow R_2^\times/R_1^\times \longrightarrow \bar{R}_2^\times/\bar{R}_1^\times \longrightarrow \bar{R}_2^\times/R_2^\times\bar{R}_1^\times \longrightarrow 1$$

we see that $r(R_2^\times/R_1^\times) = \infty$.

2. $\dim R_1 \geq 2$. We suppose that $R_2^\times/R_1^\times$ is finitely generated, say by n elements. As above let U_l ($l \geq 0$) be the image of $1 + J_1^l J_2$ in $(1 + J_2)/(1 + J_1)$. Then U_l/U_{l+1} can be generated by n elements, too. Set $d = \prod_{\mathfrak{m} \in \mathfrak{M}} \operatorname{char} R_1/\mathfrak{m}$.

By Lemma 2.3 d annihilates U_l/U_{l+1}. Denoting the length of a module M over a ring R by $\ell_R(M)$ we obtain for all $l \geq 0$:

$$\ell_{\mathbb{Z}}(U_l/U_{l+1}) \leq \ell_{\mathbb{Z}}((\mathbb{Z}/d\mathbb{Z})^n) < \infty \quad .$$

On the other hand we have by Lemma 2.3 ($M = J_2/J_1$):

$$\ell_{\mathbb{Z}}(U_l/U_{l+1}) = \ell_{\mathbb{Z}}(J_1^l M/J_1^{l+1}M) \geq \ell_{R_1}(J_1^l M/J_1^{l+1}M) \quad .$$

But it is well known that the function $l \mapsto \ell_{R_1}(J_1^l M/J_1^{l+1}M)$ is a polynomial function (for large l) of degree $\dim M - 1$. Since $\mathrm{Ass}_{R_1}(R_2/R_1) = \{0\}$ we have $\dim M - 1 = \dim R_1 - 1 \geq 1$. Hence that function cannot be bounded. This contradiction finishes the proof of the Lemma. $\square$

We come now to the proof of the Theorem. $1 \iff 2 \iff 3$ by Lemma 2.4. $3 \Rightarrow 4$ is clear. So it remains to show $4 \Rightarrow 1$. Let $\mathfrak{p} \in \mathrm{Ass}_{R_1}(R_2/R_1)$ be a prime ideal having an infinite residue class field. We show that $R_2^\times/R_1^\times$ is not finitely generated.

By replacing $\mathfrak{p}$ with some minimal member of $\mathrm{Ass}_{R_1}(R_2/R_1)$ contained in $\mathfrak{p}$ we may suppose that $\mathfrak{p}$ is already minimal. Let

$$\tilde{R} = \{r \in R_2 \mid sr \in R_1 \text{ for some } s \notin \mathfrak{p}\}$$

be the $\mathfrak{p}$-primary component of R_1 in R_2. Obviously $\tilde{R}$ is a subring of R_2 and we have $\mathrm{Ass}_{R_1}(R_2/\tilde{R}) = \{\mathfrak{p}\}$. From $(R_1)_\mathfrak{p} = \tilde{R}_\mathfrak{p}$ we deduce that there is only one prime ideal $\tilde{\mathfrak{p}}$ of $\tilde{R}$ lying above $\mathfrak{p}$. From $\mathrm{Ass}_{R_1}(R_2/\tilde{R}) \supset \{\mathfrak{q} \cap R_1 \mid \mathfrak{q} \in \mathrm{Ass}_{\tilde{R}}(R_2/\tilde{R})\}$ we obtain $\mathrm{Ass}_{\tilde{R}}(R_2/\tilde{R}) = \{\tilde{\mathfrak{p}}\}$. Hence replacing R_1 by $\tilde{R}$ we may assume that $\mathfrak{p}$ is the only prime ideal associated to R_2/R_1. Let $x \in R_2$ be such that $\mathfrak{p} = \{r \in R_1 \mid rx \in R_1\}$. Then we have $\mathfrak{p}x \subset R_1 \cap \mathfrak{p}R_2 = \mathfrak{p}$,which implies $\mathfrak{p}R_1[x] \subset R_1$, i. e. $\mathfrak{p}$ is the conductor of $R_1 \subset R_1[x]$. Therefore we obtain an isomorphism

$$R_1[x]^\times/R_1^\times \cong \frac{(R_1[x]/\mathfrak{p})^\times}{(R_1/\mathfrak{p})^\times} \quad .$$

By Lemma 2.5, $R_1[x]^\times/R_1^\times$ is not finitely generated.

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