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FINITE GROUPS IN WHICH THE ZEROS OF EVERY NONLINEAR IRREDUCIBLE CHARACTER ARE CONJUGATE MODULO ITS KERNEL

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ABSTRACT. In this note we classify the groups G in which the zeros of every nonlinear irreducible character χ are conjugate in $G/\ker(\chi)$. Our proof depends on the classification of finite simple groups. We prove a related result for monolithic characters (see the corollary below). Some open questions are posed and discussed.

Let $\mathrm{Irr}(G)$ be the set of irreducible characters of a finite group G (we consider only finite groups), $\mathrm{Irr}_1(G)$ the set of nonlinear characters in $\mathrm{Irr}(G)$. For $\chi \in \mathrm{Irr}_1(G)$, let $\mathrm{T}(\chi) = \{x \in G \mid \chi(x) = 0\}$. The elements of $\mathrm{T}(\chi)$ are called *zeros* of χ. By Burnside's Theorem (see [I, Theorem 3.15] or [K, Corollary 23.1.5]), $\mathrm{T}(\chi) \neq \emptyset$ for every $\chi \in \mathrm{Irr}_1(G)$. Obviously, $\mathrm{T}(\chi)^x = \mathrm{T}(\chi)$ for $x \in G$, i.e., $\mathrm{T}(\chi)$ is a union of conjugacy classes of G (= G-classes). For further information on the sets $\mathrm{T}(\chi)$ and related subgroups see [K], Chapter 23.

E.M. Zhmud [Z1], [Z2] treated some properties of finite groups G possessing a faithful irreducible character χ such that $\mathrm{T}(\chi)$ is a G-class. The set of groups satisfying the Zhmud condition, is very big, and it is impossible to classify all such groups. In the other extreme, S.C. Gagola [G] studied the groups having an irreducible character vanishing on all but two classes. For further information on

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zeros of characters see [Ga], [Z3], [Z4]. Note that induced characters have many zeros, and we make use of this fact in what follows.

For $X \subseteq G$ and $N \trianglelefteq G$, let $XN/N = \{xN \mid x \in X\}$ be the subset in G/N. A subset X is invariant in G (or G-invariant) if $X^g = X$ for all $g \in G$. If X is G-invariant, then XN/N is G/N-invariant. In particular, if $\chi \in \mathrm{Irr}_1(G)$, then by the above, $\mathrm{T}(\chi)\ker(\chi)/\ker(\chi)$ is a nonempty (since $\mathrm{T}(\chi) \cap \ker(\chi)$ is empty) $G/\ker(\chi)$-invariant subset.

Definition 1. A group G is said to be a *CZ-group* if $\mathrm{T}(\chi)$ is a conjugacy class of G for every $\chi \in \mathrm{Irr}_1(G)$. A group G is said to be a *CZK-group* if $\mathrm{T}(\chi)\ker(\chi)/\ker(\chi)$ is a conjugacy class in $G/\ker(\chi)$ for every $\chi \in \mathrm{Irr}_1(G)$.

By definition, abelian groups are CZ-groups and CZ-groups are CZK-groups. Both the properties are inherited by epimorphic images.

Note that if $x \in \mathrm{T}(\chi)$, $z \in \ker(\chi)$, then $xz \in \mathrm{T}(\chi)$. Indeed, if D is a representation of G with character χ, then $D(xz) = D(x)D(z) = D(x)$, and so $\chi(xz) = \mathrm{tr}(D(x)) = \chi(x) = 0$. Therefore, $\mathrm{T}(\chi)$ is a union of cosets of $\ker(\chi)$, and so $\mathrm{T}(\chi)\ker(\chi)/\ker(\chi) = \mathrm{T}(\chi)/\ker(\chi)$.

Obviously, G is a CZ-group if and only if the character table of G has a minimal possible number (namely, $|\mathrm{Irr}_1(G)|$) zero entries. As a corollary of the main theorem, we obtain that a subgroups of CZ-groups are also CZ-groups. It is surprising that the symmetric group S_4 is the only CZK-group that is not a CZ-group. Note that S_4 has subgroups (namely, A_4 and Sylow 2-subgroups) that are not CZK-groups.

The proof of the main theorem in solvable case is based essentially on a corollary of the Isaacs-Passman Theorem [IP] on groups all of whose nonlinear irreducible characters have prime degrees (see Lemma 3 and Corollary 4 below). To prove the solvability of CZK-groups, we make use of the classification of finite simple groups and its consequence, due to Willems (see Lemma 1(a)).

Let $\{1\} < N \triangleleft G$, $\phi \in \mathrm{Irr}_1(N)$ and χ an extension of ϕ to G. Since ϕ is G-invariant, it follows that $T(\phi)$ is G-invariant and $\mathrm{T}(\phi) \subseteq \mathrm{T}(\chi)$. In particular, if $\mathrm{T}(\chi)$ is a G-class, then $\mathrm{T}(\chi) = \mathrm{T}(\phi)$. We make use of this remark in the proof of the theorem.

In the proof of the theorem we make use of the following

Lemma 1. *(a) ([W1], [W2]) Every simple group of Lie type possesses an irreducible character χ such that $|G|/\chi(1)$ is odd ($\chi \in Irr(G)$ is said to be of p-defect 0 if $p \nmid |G|/\chi(1)$).*

(b) A group G, containing a nilpotent subgroup of index 2, is supersolvable.

(c) (Burnside; see also [N]*) A group G admitting a fixed-point-free automorphism of order* 3 *is nilpotent (of class at most* 2*).*

Lemma 1(b) follows easily from [BZ, Exercise 3.19].

For $H < G$, set $H_G = \bigcap_{x\in G} H^x$, $D_H = G - \bigcup_{x\in H} H^x$. It is known that H_G is the maximal normal subgroup of G contained in H and D_H a nonempty G-invariant subset.

Lemma 2. *Let H be a nontrivial subgroup of a solvable group G such that D_H is a G-class. Then:*

(a) If $H \lhd G$, then $|G : H| = 2$ and G is a Frobenius group with kernel H.

(b) If H is nonnormal maximal subgroup of G, then G/H_G is a Frobenius group with kernel P/H_G of order p^α and complement H/H_G of order $p^\alpha - 1$, where p is a prime. If, in addition, G is a CZK-group, then $G/H_G \cong S_3$, the symmetric group of degree 3.

(c) If G is a nilpotent CZK-group, it is abelian.

Proof. (a) Let $H \lhd G$. Then $D_H = G - H$ is a G-class, and so $(G/H)^\#$ is a conjugacy class so that $|G/H| = 2$. If $x \in G - H$, then $|G : C_G(x)| = |G - H| = \frac{1}{2}|G|$, and we obtain a Frobenius group with kernel H of index 2.

(b) Suppose H is nonnormal maximal subgroup of G. It suffices to consider the case when $H_G = \{1\}$. Let P be a minimal normal subgroup of G. Then $N_G(P \cap H) \geq \langle P, H\rangle > H$, and so $P \cap H = \{1\}$, $G = P \cdot H$, a semidirect product. Set $|P| = p^\alpha$. Since $P^\# \subseteq D_H$ and D_H is a G-class by assumption, it follows that $D_H = P^\#$ and $|D_H \cup \{1\}| = |P| = |G : H|$. On the other hand, it is easy to check that $|D_H| \geq |G : H| - 1$ with equality if and only if $H \cap H^x = \{1\}$ for all $x \in G - H$. Therefore, $H \cap H^x = \{1\}$ for all $x \in G - H$, i.e., G is a Frobenius group with complement H and kernel P. Since $P^\#$ is a G-class and P is elementary abelian, it follows that $|H| = |P| - 1 = p^\alpha - 1$. Let, in addition, G be a CZK-group. Every faithful irreducible character of G vanishes outside P by [I], Theorem 6.34, and so $G - P$ is a G-class. By (a), $|G : P| = 2$ so $p^\alpha - 1 = 2$, $p^\alpha = 3$ and $G \cong S_3$.

(c) is a corollary of (a) because a nonlinear irreducible character χ of G always vanishes outside some proper normal subgroup (since G is an M-group) and $G/ker(\chi)$ is not a Frobenius group. □

Lemma 3. [IP] *Let $cd(G) = \{\chi(1) \mid \chi \in Irr(G)\} = \{1, p, q\}$, where p, q are distinct primes. Then G has one of the following normal series:*

(a) $G > F > Z(F) = Z(G)$, where $|G : F| = p$, $G/Z(G)$ is a Frobenius group whose kernel $F/Z(G)$ of order q^2 is a minimal normal subgroup.

(b) $G > F > M = Z(G) \times R$, where $|G : F| = p$, $|F : M| = q$, G/M and F are nonabelian, R is elementary abelian of order r^m for a prime r, F/M acts irreducibly on R, $\frac{r^m - 1}{r^{m/p} - 1} = q$.

Corollary 4. *Let $cd(G) = \{1, 2, 3\}$ and $|G : G'| = 2$. Then $G \cong S_4$.*

PROOF. By assumption (in the notation of Lemma 3), $F = G'$, $p = 2$, $q = 3$. Obviously, G is a group of Lemma 3(b). Then $r^{m/2} + 1 = q = 3$, and so $r = 2$, $m = 2$, $|G/Z(G)| = 24$. Since Sylow subgroups of $G/Z(G)$ are not normal, it follows that $G/Z(G) \cong S_4$. By assumption, $Z(G) < G'$, and so G is an epimorphic image of a covering group of S_4. Since covering groups of S_4 have irreducible character of degree 4, we get $Z(G) = \{1\}$, completing the proof. □

Our principal result is the following

Theorem 5. *A nonabelian group G is a CZK-group if and only if it is either a Frobenius group with kernel of index 2 or S_4.*

PROOF. It follows from the description of irreducible characters of Frobenius groups (see [I], Theorem 6.34) that a Frobenius group with kernel of index 2 is a CZK-group (moreover, it is a CZ-group). It is easy to check that S_4 is a CZK-group (however it is not a CZ-group).

Let G be a CZK-group. Suppose that the theorem has proved for all CZK-groups of order $< |G|$. In what follows we assume that G is not abelian.

(i) We claim that G is not simple (in the case under consideration, G is a CZ-group). Assume that this is false. To obtain a contradiction, we make use of the classification of finite simple groups. By [Atlas], the sporadic simple groups are not CZK-groups. Therefore, by the classification, it remains to show that the simple groups of Lie type and the alternating groups A_n of degree $n > 4$ are not CZK-groups.

Assume that G is a simple group of Lie type. Then by Lemma 1(a), there exists a character $\chi \in \mathrm{Irr}(G)$ of 2-defect 0. By [I], Theorem 8.17, χ vanishes on all elements of even order. Since, by assumption, $T(\chi)$ is a conjugacy class, all elements of even order in G have the same order, and so are involutions. This means that a Sylow 2-subgroup S of G is elementary abelian and $C_G(x) = S$ for every $x \in S^{\#}$. Hence by Brauer-Suzuki-Wall Theorem (see [HB], Theorem 11.2.7), $G \cong L_2(2^n)$, $n > 1$. The group $G = L_2(2^n)$ $(n \geq 2)$ has an irreducible character χ of degree $2^n + 1$ (Schur [S]; see also [D], §38). Note that G has a cyclic Hall subgroup Z of order $2^n + 1$. Since $(\chi(1), |G|/\chi(1)) = 1$ and $T(\chi)$ is a conjugacy class, it follows that χ vanishes on $Z^{\#}$ and all its conjugates by [I], Theorem 8.17, and so $2^n + 1$ is a prime number. Set $T = \bigcup_{x \in G}(Z^{\#})^x$. Obviously, T is G-invariant subset, $T = \mathrm{T}(\chi)$ (since G is a CZK-group). Since $|\mathrm{N}_G(Z) : Z| = 2$ and Z is a TI-subgroup of G, we have $|T| = |Z^{\#}| \cdot |G : \mathrm{N}_G(Z)| = 2^{2n-1}(2^n - 1)$; however this number does not divide $|G| = 2^n(2^{2n} - 1)$ so T is not a G-class. It follows that $\mathrm{L}_2(2^n)$ is not a CZK-group.

Assume that $G = \mathrm{A}_n$, the alternating group of degree $n > 4$. For $n \leq 7$ the result follows from the character tables of A_n (see [Atlas]). In what follows we assume that $n > 7$. Define a function $\pi : \mathrm{A}_n \to \mathbb{N} \cup \{0\}$ as follows: if $g \in G$, then $\pi(g)$ is the number of points fixed by g. Since $G = \mathrm{A}_n$ is 2-transitive, we have $\pi = 1_G + \chi$, where 1_G is the principal character of G and $\chi \in \mathrm{Irr}(G)$.

Let $n = 2m$, $(m \geq 4)$ be even. Consider the following permutations in G:

$$a = (1, 2, \ldots, 2m - 1), \ b = (1, 2)(3, 4)(5, \ldots, 2m - 1).$$

Then $\chi(a) = 0 = \chi(b)$, but a and b are not conjugate in $G = \mathrm{A}_n$, so that A_{2m} is not a CZK-group.

Let $n = 2m + 1$, $(m \geq 4)$ be odd. Consider the following permutations in $G = \mathrm{A}_{2m+1}$:

$$a = (1, 2)(3, \ldots, 2m), \ b = ((1, 2, 3, 4)(5, \ldots, 2m).$$

As in the previous paragraph, a and b are nonconjugate zeros of χ, and so $G = \mathrm{A}_{2m+1}$ is not a CZK-group.

(ii) We claim that $G' < G$. Indeed, if M is a maximal normal subgroup of G, then G/M is a simple CZK-group. By (i), G/M is abelian, and so $G' \leq M < G$, as desired.

(iii) Suppose that G has a proper normal subgroup M such that $\lambda^G = \chi \in \mathrm{Irr}(G)$ for some $\lambda \in \mathrm{Irr}(M)$; then χ is nonlinear. Since $M \lhd G$, χ vanishes outside M; in particular, $\ker(\chi) < M$. Therefore, $G/\ker(\chi) - M/\ker(\chi) = \mathrm{T}(\chi)/\ker(\chi)$ (since $G/\ker(\chi) - M/\ker(\chi)$ is $G/\ker(\chi)$-invariant and $\mathrm{T}(\chi)/\ker(\chi)$ is a $G/\ker(\chi)$-class by assumption). In that case, $(G/M)^{\#}$ is a G/M-class so that $|G : M| = 2$. By Lemma 2(a), $G/\ker(\chi)$ is a Frobenius group with kernel $M/\ker(\chi)$ (of index 2).

A. Let G be solvable. We will use induction on $|G|$ to prove the theorem in this case.

(iv) We claim that if G has an abelian subgroup A of index 2, then G is a Frobenius group with kernel A. By Lemma 1(b), G is supersolvable. By [I], Lemma 12.12, $|G| = 2|G'||\mathrm{Z}(G)|$. If $\mathrm{Z}(G) = \{1\}$, then $A = G'$. In the case under consideration, A is of odd order, and every involution from $G - A$ inverts

A; it follows that G is a Frobenius group with kernel A. Assume that $\mathrm{Z}(G) > \{1\}$. Since the intersection of kernels of the nonlinear irreducible characters of a nonabelian group is $\{1\}$ (see, for example, [BZ], Theorem 4.35), there exists $\chi \in \mathrm{Irr}_1(G) - \mathrm{Irr}(G/\mathrm{Z}(G))$. If $\lambda \in \mathrm{Irr}(\chi_A)$, then $\chi = \lambda^G$ and $\ker(\chi) < A$. Then $\mathrm{T}(\chi)/\ker(\chi) = G/\ker(\chi) - A/\ker(\chi)$ is a $G/\ker(\chi)$-class, and, by Lemma 2(a), $G/\ker(\chi)$ is a Frobenius group, which is impossible in view of $\mathrm{Z}(G/\ker(\chi)) > \{1\}$ (by the choice of χ).

(v) We will prove by induction on $|G|$ that $|G : G'| = 2$. We may assume that G' is a minimal normal subgroup of G. By Lemma 2(c), G is not nilpotent. Therefore, by [H], Satz 3.3.11, $G' \not\leq \Phi(G)$ ($\Phi(G)$ is the Frattini subgroup of G), and so $G = H \cdot G'$, where H is maximal in G; obviously, $H \cap G' = \{1\}$ and H is abelian. Let $H_G = \{1\}$; then G is a Frobenius group with kernel G'. Since all faithful irreducible characters of G vanish off G' (see [I], Theorem 6.34), $G - G'$ is a G-class, and we get $|G : G'| = 2$ by Lemma 2(a). Let $H_G > \{1\}$. Then $|G : G' \times H_G| = 2$ by induction, contrary to (iv) (since $H_G \leq \mathrm{Z}(G)$ and $G' \times H_G$ is abelian of index 2 in G). This completes the proof of (v).

(vi) We claim that if G has a nilpotent subgroup A of index 2, then A is abelian. Assume that G is a counterexample of minimal order. By (v), $A = G'$. By Lemma 1(b), G is supersolvable. By induction, A is a nonabelian p-group, p is a prime, $|A'| = p$. Since G/A' is a Frobenius group by (iv) (in particular, $p > 2$), $A' \leq \mathrm{Z}(A)$ and G is not a Frobenius group (otherwise, A is abelian by Burnside), we get $A' = \mathrm{Z}(G)$. By induction, A' is the only minimal normal subgroup of G. By Fitting's Lemma (applied to $\mathrm{Z}(A)$), we get $A' = \mathrm{Z}(A)$. If $x, y \in A$, then $[x, y^p] = [x, y]^p = 1$ (since the nilpotence class of A is 2) so $y^p \in \mathrm{Z}(A) = A'$. It follows that A/A' is elementary abelian so A is extraspecial. Let $\theta \in \mathrm{Irr}_1(A)$. Then θ^G is faithful, vanishes outside A'; therefore, since $G - A$ is not a conjugacy class of G, $\theta^G = \chi_1 + \chi_2$, where $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ are two distinct extensions of θ (by Clifford theory and Lemma 2(a)). Then $\mathrm{T}(\chi_1) = \mathrm{T}(\theta)(= A - A')$ (see the remark preceding Lemma 1). If $x \in A - A' = \mathrm{T}(\chi_1)$, then $2p = |G : \mathrm{C}_G(x)| = |\mathrm{T}(\chi_1)| = |A - A'|$. Setting $|A| = p^{1+2m}$, $m \in \mathbb{N}$, we get $2p = p^{2m+1} - p$, which is impossible. Thus, A is abelian.

In what follows, we will assume that G' is not nilpotent; then $G'' > \{1\}$ and $G'' \not\leq \Phi(G)$ by [H], Satz 3.3.5.

(vii) We will prove that if G'' is the unique minimal normal subgroup of G, then $G \cong \mathrm{S}_4$. Set $|G''| = p^\alpha$, $|G : G''| = 2a$, where $a > 1$ is odd; then G/G'' is a Frobenius group with kernel of order a (see (iv)). Since $G'' \not\leq \Phi(G)$ we get $G = H \cdot G''$, where H is maximal in G and $H \cap G'' = \{1\}$. By assumption,

$C_G(G'') = G''$. Let $H = X \cdot A$, where $|X| = 2$, $|A| = a$, A is the abelian kernel of a Frobenius group H. Assume that $G' = A \cdot G''$ is not a Frobenius group. Then $yz = zy$ for some $y \in A^\#$ and $z \in (G'')^\#$. Since $\langle y \rangle \lhd H$ and H is maximal in G, it follows that $\langle y \rangle \lhd G$, contrary to the uniqueness of G''. Thus, $G' = A \cdot G''$ is a Frobenius group (in particular, A is cyclic). Let a nonprincipal $\mu \in \mathrm{Irr}(G'')$. Then $\theta = \mu^{G'} \in \mathrm{Irr}(G')$ by [I], Theorem 6.34. Since $G - G'$ is not a G-class (see Lemma 2(a)) and θ^G vanishes outside G', it follows that $\theta^G = \chi_1 + \chi_2$, where χ_1, χ_2 are distinct extensions of θ to G. Since $(\chi_1)_{G'} = \theta$, $\mathrm{T}(\chi_1)$ is a G-class and $\mathrm{T}(\theta)$ is a G-invariant subset (since θ is a G-invariant character of $G' \lhd G$), it follows that $\mathrm{T}(\chi_1) = \mathrm{T}(\theta)$. Note that $\mathrm{T}(\theta) = G' - G''$ is the set of size $ap^\alpha - p^\alpha = (a-1)p^\alpha$. If $z \in G' - G''$, then $|G : C_G(z)| = 2p^\alpha$. Hence $(a-1)p^\alpha = 2p^\alpha$, and so $a = 3$. In particular, $G/G'' \cong \mathrm{S}_3$. Assume that $\mathrm{Irr}(G)$ has a character χ of degree 6. If $\mu \in \mathrm{Irr}(\chi_{G''})$, then $\chi = \mu^G$ (since $|G : G''| = 6$ and μ is linear) and χ is faithful. In the case considered, χ vanishes outside G''. This is impossible since $G - G''$ is not a G-class in view of $G/G'' \cong \mathrm{S}_3$. Thus, $\mathrm{cd}(G) = \{1, 2, 3\}$ by [I], Theorem 6.15. By (v) and Corollary 4, $G \cong \mathrm{S}_4$.

(viii) We claim that if G'' is a minimal normal subgroup of G, then $G \cong \mathrm{S}_4$. By (vii) we may assume that G has another minimal normal subgroup R. By (v), $R < G'$. Moreover, $R \times G'' < G'$, by (iv) and (v). By induction, $G/R \cong \mathrm{S}_4$, and so $|G''| = 4$. We have $|G : R \times G''| = 6$. As in (vii), $\mathrm{Irr}(G)$ has no character of degree 6 (if such a character exists, it is faithful, and then $G - R \times G''$ is a G-class by assumption, which is a contradiction). By Ito's Theorem ([I], Theorem 6.15), $\mathrm{cd}(G) = \{1, 2, 3\}$. By Corollary 4 and (v), $G \cong \mathrm{S}_4$ (in particular, R does not exist).

(ix) We claim that if $G'' > \{1\}$ is abelian, then $G \cong \mathrm{S}_4$. As before, we will use induction on G. By (viii), we may assume that G'' is not a minimal normal subgroup of G. Let R be a minimal normal subgroup of G contained in G''. By induction, $G/R \cong \mathrm{S}_4$. It follows that G'' is an (abelian) 2-subgroup of index 6 in G. As in the proof of (vii) and (viii), $\mathrm{cd}(G) = \{1, 2, 3\}$. By Corollary 4 and (v), $G \cong \mathrm{S}_4$ (in particular, R does not exist).

(x) We claim that if $G'' > \{1\}$ is nilpotent, then $G \cong \mathrm{S}_4$. In view of (ix), we may assume that G'' is nonabelian. By (ix), $|G''/G'''| = 4$, and so G'' is a 2-group of maximal class by Taussky's Theorem (see [H], Satz 3.11.9). By (vi), G' is not nilpotent. Therefore, G'' is the ordinary quaternion group (if P is a 2-group of maximal class such that $\mathrm{Aut}(P)$ is not a 2-group, then P is the ordinary quaternion group). In that case, G is a covering group of S_4 (by Schur's description of covering groups of the symmetric groups [S]; see also [Su], (3.2.21)).

Then G has a faithful irreducible character χ of degree 4. Since $\mathrm{cd}(G') = \{1,2,3\}$, it follows, by Clifford's Theorem, that $\chi_{G'} = \phi_1 + \phi_2$, where $\phi_1, \phi_2 \in \mathrm{Irr}(G')$ and $\phi_1^G = \chi$. Then χ vanishes outside G' so G is a Frobenius group with kernel G' (Lemma 2(a)), which is not the case.

(xi) We claim that $G''' = \{1\}$ (in particular, if $G'' > \{1\}$, then $G \cong \mathrm{S}_4$). Assume that this is false. Without loss of generality, we may assume that $N = G'''$ is a minimal normal subgroup of G. By (x), G'' is not nilpotent, and so $N \not\leq \Phi(G)$ by [H], Satz 3.3.5. Therefore, $G = H \cdot N$, where $H \cap N = \{1\}$ and H is maximal in G. Since G'' is not nilpotent, $H_G = \{1\}$, and so $\mathrm{C}_G(N) = N$. By (ix), $H \cong G/N \cong \mathrm{S}_4$. Set $|N| = p^\alpha$. We have $p > 2$ (since G'' is not nilpotent). In particular, $\alpha > 1$. We have $4 \in \mathrm{cd}(G'')$ (otherwise, by [A], G'' has an abelian subgroup of index 2, and then $\mathrm{C}_G(N) > N$, which is not the case). Let $\phi \in \mathrm{Irr}(G'')$, $\phi(1) = 4$. If $\mathrm{Irr}(G)$ has a character χ of degree 24, then $\mathrm{T}(\chi) = G - N$ is a G-class (since N is normal abelian of index 24 in G), contrary to Lemma 2(a). Let us consider the following two cases.

(1xi) Suppose that $\theta = \phi^{G'} \in \mathrm{Irr}(G')$. Since $24 \notin \mathrm{cd}(G)$, θ is G-invariant by Clifford theory, and so $\theta^G = \chi_1 + \chi_2$, where $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ are distinct (faithful) extensions of θ to G. As above, $\mathrm{T}(\theta) = \mathrm{T}(\chi_1)$ (see the remark, preceding Lemma 1). But θ vanishes on $G' - N$ (since $|G' : N| = 12 = \theta(1)$ and N is abelian), and this set is not a G-class since $G'/N \cong \mathrm{A}_4$, and we obtain a contradiction.

(2xi) Let $\phi^{G'} \notin \mathrm{Irr}(G')$. Then $\phi^{G'} = \theta + \theta_1 + \theta_2$, where $\theta, \theta_1, \theta_2 \in \mathrm{Irr}(G')$ are distinct extensions of ϕ to G' (by Clifford theory). Since $N \not\leq \ker(\theta)$, it follows that $\theta^G = \chi_1 + \chi_2$, where $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ are (faithful) distinct extensions of θ to G (see Lemma 2(a)). We have $(\chi_1)_{G''} = \phi$, and so $\mathrm{T}(\chi_1) = \mathrm{T}(\phi)$ (since ϕ is G-invariant, the set $\mathrm{T}(\phi)$ is invariant in G). Since G is a CZK-group, $\mathrm{T}(\phi)$ is a G-class, and, by [I], Theorem 8.17, it consists of elements of even order in G'', which are, consequently, involutions. But this is not true: G'' is not a Frobenius group (since its Sylow 2-subgroup is nonnormal abelian of type $(2,2)$), and so G'' has an element of order $2p$. This contradiction completes the proof of (xi).

Thus, the theorem is proved in the solvable case. It remains to prove that G is solvable.

B. We claim that G is solvable. Suppose that G is a counterexample of minimal order. Then G is not simple (by (i)) and has only one minimal normal subgroup, say R; R is a direct product of isomorphic nonabelian simple groups, G/R is solvable. By (ii) and (v), $|G : G'| = 2$. Let a nonprincipal $\phi \in \mathrm{Irr}(R)$. Assume that $\phi^x \neq \phi$ for some $x \in G$. Then the inertia subgroup $I = \mathrm{I}_G(\phi)$ is a proper subgroup of G. Obviously, $R \leq I$. If $\theta \in \mathrm{Irr}(\phi^I)$, then $\theta^G = \chi \in \mathrm{Irr}(G)$ by

[I], Theorem 6.11(a). Since R is the only minimal normal subgroup of G and $R \not\leq \ker(\chi)$, χ is faithful. The induced character χ vanishes on $D_I = G - \bigcup_{x \in G} I^x$, and so $\mathrm{T}(\chi) = D_I$ (since D_I is a nonempty invariant subset of G and $\mathrm{T}(\chi)$ is a G-class by assumption). If $I \triangleleft G$, then G is a Frobenius group with kernel I, $|G : I| = 2$, by Lemma 2(a). In that case, G is solvable, which is not the case. Assume that $I \ntriangleleft G$. Let $I \leq H < G$, where H is maximal in G. Since $D_H = G - \bigcup_{x \in G} H^x$ is a nonempty G-invariant subset and $D_H \subseteq \mathrm{T}(\chi)$, it follows that $D_H = \mathrm{T}(\chi)$ (since $\mathrm{T}(\chi)$ is a G-class). By the induction hypothesis, G/R is solvable. Therefore, by Lemma 2(b), $G/H_G \cong \mathrm{S}_3$ (if $H \triangleleft G$, we obtain a Frobenius group with kernel H of index 2 by Lemma 2(a), which is not the case: G is nonsolvable). In particular, $|G : H| = 3$. Let H, H_1, H_2 be all G-conjugates of H. Then $H \cap H_1 = H \cap H_2 = H_1 \cap H_2 = H_G$, and so $|H \cup H_1 \cup H_2| = 3|H| - 2|H_G| = \frac{2}{3}|G|$. We obtain $|D_H| = \frac{1}{3}|G|$. Therefore, if $x \in D_H$, then $|G : \mathrm{C}_G(x)| = |D_H| = \frac{1}{3}|G|$ (since D_H is a G-class), and so $\mathrm{C}_G(x) = \langle x \rangle$ is of order 3. Since $x \notin H_G$, it follows that x induces a fixed-point-free automorphism of H_G of order $o(x) = 3$. By Lemma 1(c), H_G is nilpotent. Since $R \leq H_G$, it follows that R is solvable, contrary to the assumption. Thus, all irreducible characters of R are G-invariant. By the Brauer Permutation Lemma ([I], Theorem 6.32), every R-class is a G-class. Therefore, R is simple. It follows that R is a nonabelian simple CZK-group (in fact, if a nonprincipal $\phi \in \mathrm{Irr}(R)$ and $\chi \in \mathrm{Irr}(\phi^G)$, then $\chi_R = e\phi$; it follows that $\mathrm{T}(\phi) = T(\chi)$ is a G-class, and so an R-class), contrary to (i). This completes the proof of the theorem. □

In particular, a nonabelian group is a CZ-group if and only if it is a Frobenius group with kernel of index 2. (According to the report of D. Chillag, he also classified CZ-groups.)

A character χ of G is said to be *monolithic* if $\chi \in \mathrm{Irr}(G)$ and $G/\ker(\chi)$ is a monolith. If $N \triangleleft G$ and χ is a monolithic character of G/N, then χ (considered as a character of G) is also a monolithic character of G. We consider the principal character 1_G of G to be monolithic by definition. As a rule, the set of monolithic characters of G is a proper subset of $\mathrm{Irr}(G)$. As an easy consequence of the theorem we will prove the following

Corollary 6. *If $T(\chi)$ is a conjugacy class for every nonlinear monolithic character of a nonabelian group G, then G is a CZ-group.*

PROOF. Let M be a maximal normal subgroup of G. Since all irreducible characters of G/M are monolithic, it is a CZ-group. It follows from the theorem that G/M is abelian. In particular, $G' < G$. Moreover, this reasoning shows that if

$N \triangleleft G$, then $(G/N)' < G/N$. Suppose that the corollary is proved for all groups of order $< |G|$. Let R be a minimal normal subgroup of G. By the induction hypothesis, G/R is solvable.

Assume that G/R is nonabelian. Let H/R be a normal subgroup of G/R such that G/H is nonabelian but every proper epimorphic image of G/H is abelian. All nonlinear irreducible characters of G/H are monolithic (see [I], Theorem 12.3). Therefore by the theorem, G/H is a Frobenius group with kernel L/H of index 2 (by the above, G/H is not nilpotent). Let λ be a nonprincipal character of L/H. Then $\lambda^G = \chi \in \mathrm{Irr}_1(G)$ (see [I], Theorem 6.34), and χ vanishes outside L. By what we have said above, the character χ is monolithic. Therefore, $G - L$ is a G-class, by assumption. By Lemma 2(a), G is a Frobenius group with kernel L of index 2.

Assume that G is not solvable. Then G/R is solvable and R is not solvable. By the result of the previous paragraph, G/R is abelian. Since this is true for every choice of R, it follows that $R = G'$. In that case, G is a monolith and all its nonlinear irreducible characters are monolithic, i.e., G is a CZ-group, contrary to the theorem. □

Question 1. *Classify the groups G such that the character table of G has $|Irr_1(G)|+1$ zero entries (A_4, S_4 and A_5 satisfy this condition).*

Question 2. *Study the nonsolvable groups G such that $T(\chi)$ is a conjugacy class whenever $\chi \in Irr_1(G)$ and $\chi(1)$ is even ($L_2(2^n)$ satisfies this condition, but $Aut(L_2(2^3))$ does not satisfy by* [I], *Theorem 8.17: it has elements of order 6).*

Probably, $\mathrm{L}_2(2^n)$ are the only simple groups satisfying Problem 2 (see the reasoning in part (i) of the proof of the theorem). We do not know nonsolvable groups G such that $\mathrm{T}(\chi)$ is a conjugacy class for all $\chi \in \mathrm{Irr}_1(G)$ of odd degree.

Question 3. *Classify the groups G such that $T(\chi)$ is a conjugacy class for all but one nonlinear irreducible characters χ of G (examples: $SL(2,3)$ and, by* [I], *Theorem 3.15, all the groups of Question 1).*

Question 4. *Let G be a nonabelian group. For $\chi \in Irr_1(G)$, let $z(\chi)$ be $k(\chi) - 1$, where $T(\chi)$ is a union of $k(\chi)$ conjugacy classes. Set $z(G) = \sum_{\chi \in Irr_1(G)} z(\chi)$. Classify the simple groups G with small $z(G)$.*

Question 5. *Classify the groups G such that $T(\chi)/\ker(\chi)$ is a conjugacy class for all nonlinear monolithic characters χ of G. It is easy to show that all such G are solvable.*

Let $\chi \in \mathrm{Irr}(G)$. Set $\mathrm{Z}(\chi) = \{x \in G \mid |\chi(x)| = \chi(1)\}$. The set $\mathrm{Z}(\chi)$ is a normal subgroup of G (the *quasikernel* of χ). It is easy to show that if $\chi \in \mathrm{Irr}_1(G)$ then $\mathrm{T}(\chi)\mathrm{Z}(\chi) = \mathrm{T}(\chi)$. A group G is said to be *CZQ-group* if it is abelian or $\mathrm{T}(\chi)/\mathrm{Z}(\chi)$ is a $G/\mathrm{Z}(\chi)$-class for every $\chi \in \mathrm{Irr}_1(G)$. The property CZQ is inherited by epimorphic images.

Question 6. *Classify CZQ-groups.*

As in part B of the proof of the theorem, we can show that CZQ-groups are solvable. If $G/\mathrm{Z}(G)$ is a CZK-group then G is not necessary a CZQ-group (indeed, if G is a covering group of the symmetric group S_4 of degree 4, then $G/\mathrm{Z}(G) \cong \mathrm{S}_4$ is a CZK-group but G is not a CZQ-group. If a nonnilpotent group G of order 12 has a cyclic subgroup of order 4, then G is a CZQ-group. Probably, the derived length of a CZQ-group G is at most two, unless $G = \mathrm{S}_4 \times \mathrm{Z}(G)$.

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